Math 295: Exam 1 **Name:** *Caleb M^cWhorter — Solutions*
Fall – 2018
09/18/2018
80 Minutes

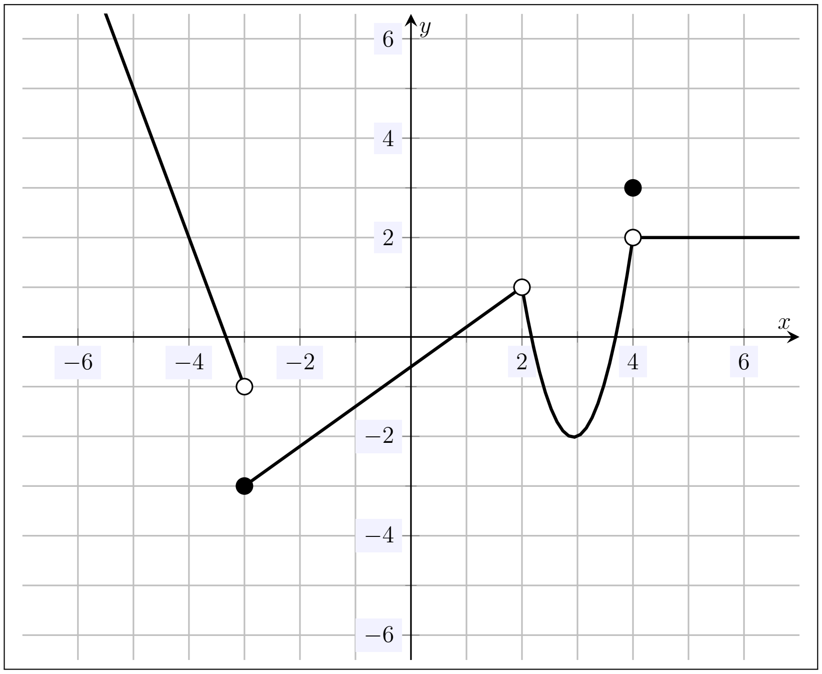
Write your name on the appropriate line on the exam cover sheet. This exam contains 10 pages (including this cover page) and 9 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	15	
6	10	
7	10	
8	10	
9	15	
Total:	100	

1. (10 points) Mark the following statements True or False:

		True	False
(a)	If $\lim_{x\to -8^+} f(x) = 3$ and $\lim_{x\to -8^-} f(x) = 3$, then $\lim_{x\to -8} f(x) = 3$.	✓	
(b)	All continuous functions are differentiable.		✓
(c)	The function $\dfrac{x-2}{x+1}$ is continuous on $[0,1]$.	✓	
(d)	If $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a^+} g(x)$ and $\lim_{x\to a^-} g(x)$ exist.	✓	
(e)	If $h(-1) = 5$, then $\lim_{x\to -1} h(x) = 5$.		✓
(f)	The function $\varphi(x) = \dfrac{\sin x}{x}$ has a removable discontinuity at $x = 0$.	✓	
(g)	All differentiable functions are continuous.	✓	
(h)	The graphs of $f(x) = \dfrac{(x+1)(x+2)}{x+1}$ and $g(x) = x + 2$ are identical.		✓
(i)	The function $\psi(x) = \dfrac{x^2 \cos x + 3}{x^2 + 1}$ is everywhere continuous.	✓	
(j)	The function $\lvert x - 7\rvert$ is differentiable at $x = 7$.		✓

2. (10 points) For the function $f(x)$, whose graph is shown in the figure below, compute the following limits. If the limit does not exist, write 'DNE.'



(a) $\lim_{x\to -3^-} f(x) = -1$

(b) $\lim_{x\to -3^+} f(x) = -3$

(c) $\lim_{x\to -3} f(x) = \text{DNE}$

(d) $\lim_{x\to 2^-} f(x) = 1$

(e) $\lim_{x\to 2^+} f(x) = 1$

(f) $\lim_{x\to 2} f(x) = 1$

(g) $\lim_{x\to -\infty} f(x) = \infty$

(h) $\lim_{x\to \infty} f(x) = 2$

(i) Is $f(x)$ continuous at $x = 4$? Be sure to justify your answer using the definition of continuity.

$f(x)$ is not continuous at $x = 4$ because $f(4) = 3$ but $\lim_{x\to 4} f(x) = 2$.

3. (10 points) In each of the following, determine the limit, if it exists.

(a) $\lim_{x\to 2} \frac{x^2-7x+10}{|x-2|} = \text{DNE}$

$$\frac{x^2-7x+10}{|x-2|} = \frac{(x-2)(x-5)}{|x-2|} = \begin{cases} \frac{(x-2)(x-5)}{x-2} = x-5, & x>2 \\ \frac{(x-2)(x-5)}{-(x-2)} = -(x-5), & x<2 \end{cases}$$

Now $\lim_{x\to 2^+}(x-5) = -3$ *but* $\lim_{x\to 2^-}[-(x-5)] = 3$*, so the limit does not exist.*

(b) $\lim_{x\to 1} \frac{x^2+2x+3}{x+1} = \frac{1^2+2(1)+3}{1+1} = \frac{6}{2} = 3$

(c) $\lim_{x\to 0} \frac{x}{3-\sqrt{9+x}} = \lim_{x\to 0} \frac{x}{3-\sqrt{9+x}} \cdot \frac{3+\sqrt{9+x}}{3+\sqrt{9+x}}$

$$= \lim_{x\to 0} \frac{x(3+\sqrt{9+x})}{9-(9+x)}$$

$$= \lim_{x\to 0} \frac{x(3+\sqrt{9+x})}{-x}$$

$$= \lim_{x\to 0}[-(3+\sqrt{9+x})] = -6$$

4. (10 points) In each of the following, determine the limit, if it exists.

(a) $\displaystyle\lim_{x\to-3}\frac{x^2-x-12}{2x^2+7x+3}=\lim_{x\to-3}\frac{(x-4)\cancel{(x+3)}}{(2x+1)\cancel{(x+3)}}=\lim_{x\to-3}\frac{x-4}{2x+1}=\frac{-3-4}{2(-3)+1}=\frac{7}{5}$

(b) $\displaystyle\lim_{x\to3}\frac{x+1}{|x^2-9|}=\infty$

(c) $\displaystyle\lim_{x\to0}\frac{(x+4)^2-16}{x}=\lim_{x\to0}\frac{x^2+8x+16-16}{x}=\lim_{x\to0}\frac{x^2+8x}{x}=\lim_{x\to0}(x+8)=8$

5. (15 points) Complete the following parts:

(a) Prove there is a solution to $2x^4 + 2x - 1 = 7x - 2$ on the interval $[0, 1]$.

Observe $2x^4+2x-1 = 7x-2$ *if and only if* $2x^4-5x+2 = 0$. *Let* $f(x) = 2x^4-5x+1$. *Now* $f(0) = 1$ *and* $f(1) = -2$. *Since* $f(x)$ *is a polynomial, it is continuous. As* $f(1) < 0 < f(0)$, *by the Intermediate Value Theorem, there exists* $c \in (0, 1)$ *such that* $f(c) = 0$, *i.e.* $2c^4 - 5c + 1 = 0$. *But that means* $2c^4 + 2c - 1 = 7c - 2$ *so that the equation has a solution.*

(b) Use the Squeeze Theorem to prove $\lim_{x\to 0} 3x^2 \sin\left(\frac{1}{\sqrt{|x|}}\right) = 0$.

$$\begin{aligned} -1 \leq \sin\left(\frac{1}{\sqrt{|x|}}\right) &\leq 1 \\ -1 \cdot 3x^2 \leq 3x^2 \sin\left(\frac{1}{\sqrt{|x|}}\right) &\leq 1 \cdot 3x^2 \\ -3x^2 \leq 3x^2 \sin\left(\frac{1}{\sqrt{|x|}}\right) &\leq 3x^2 \end{aligned}$$

Now $\lim_{x\to 0}(-3x^2) = \lim_{x\to 0} 3x^2 = 0$. *Therefore by the Squeeze Theorem, we have* $\lim_{x\to 0} 3x^2 \sin\left(\frac{1}{\sqrt{|x|}}\right) = 0$.

6. (10 points) Use the definition of the derivative to compute the derivative of the function $f(x) = \dfrac{2x+1}{x-1}$ at $x = 2$.

$$f'(2) = \left.\frac{df}{dx}\right|_{x=2} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{2(2+h)+1}{(2+h)-1} - \dfrac{2(2)+1}{2-1}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{2h+5}{h+1} - 5}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(2h+5) - 5(h+1)}{h+1}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{-3h}{h+1}}{h}$$

$$= \lim_{h\to 0} \frac{-3h}{h(h+1)}$$

$$= \lim_{h\to 0} \frac{-3}{h+1}$$

$$= -3$$

7. (10 points) Compute the following limits—no work is necessary.

(a) $\lim_{x\to\infty} \frac{2x-1}{3x+4} = \frac{2}{3}$

(b) $\lim_{x\to\infty} \frac{4x^2+3x-5}{5x^3-9} = 0$

(c) $\lim_{x\to\infty} \frac{x^2+4x-1}{x+3} = \infty$

(d) $\lim_{x\to-\infty} \frac{\sin x}{x} = 0$

(e) $\lim_{x\to\infty} \frac{x+6}{e^x} = 0$

(f) $\lim_{x\to\infty} \frac{x}{6+\ln x} = \infty$

(g) $\lim_{x\to-\infty} \frac{x^3+6}{x^2+x+1} = -\infty$

(h) $\lim_{x\to\infty} \frac{1-5x^2}{x^2+4} = -5$

(i) $\lim_{x\to\infty} \frac{x-3}{\sqrt{x+1}} = \infty$

(j) $\lim_{x\to\infty} \frac{x^2+3x-1}{(x+3)^2} = 1$

8. (10 points) Compute the following limit. Show all steps of your computation.

$$
\begin{aligned}
\lim_{x\to-\infty} \frac{x+1}{\sqrt{4x^2+3}} &= \lim_{x\to-\infty} \frac{x+1}{\sqrt{4x^2+3}} \cdot \frac{1/x}{1/x} \\
&= \lim_{x\to-\infty} \frac{\dfrac{x}{x} + \dfrac{1}{x}}{\dfrac{\sqrt{4x^2+3}}{x}} \\
&= \lim_{x\to-\infty} \frac{1+1/x}{\dfrac{\sqrt{4x^2+3}}{-\sqrt{x^2}}} \\
&= \lim_{x\to-\infty} \frac{1+1/x}{-\sqrt{\dfrac{4x^2+3}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{1+1/x}{-\sqrt{4+3/x^2}} \\
&= \frac{1+0}{-\sqrt{4+0}} \\
&= -\frac{1}{2}
\end{aligned}
$$

9. (15 points) Determine the values of a and b that make the function $f(x)$ continuous. Justify completely that $f(x)$ is continuous.

$$f(x) = \begin{cases} 3x-1, & x \leq -1 \\ x^2+ax+b, & -1 < x < 2 \\ 2x+1, & x \geq 2 \end{cases}$$

The functions $3x-1$ and $2x+1$ are everywhere continuous because they are polynomials. Furthermore, the polynomial x^2+ax+b is a polynomial for all a, b; hence, it is continuous. Therefore, $f(x)$ is continuous for all $x \neq -1, 2$. For $f(x)$ to be continuous at $x=-1$ and $x=2$, we need $f(-1) = \lim_{x\to -1} f(x)$ and $f(2) = \lim_{x\to 2} f(x)$. Now $f(-1) = 3(-1)-1 = -4$ and $f(2) = 2(2)+1 = 5$. However,

$$\begin{aligned} \lim_{x\to -1^+} f(x) &= \lim_{x\to -1^-} (x^2+ax+b) = b-a+1 \\ \lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} (3x-1) = -4 \\ \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (x^2+ax+b) = 2a+b+4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (2x+1) = 5 \end{aligned}$$

This gives a system of equations

$$\begin{aligned} b-a+1 &= -4 \\ 2a+b+4 &= 5 \end{aligned}$$

Subtracting the first equation from the second yields $3a+3=9$. But then $3a=6$ so that $a=2$. Then $5 = 2a+b+4 = 2(2)+b+4 = b+8$. This gives $b=-3$. But then the values of a and b that make $f(x)$ continuous are

$$\boxed{\begin{aligned} a &= 2 \\ b &= -3 \end{aligned}}$$