Math 295: Exam 2
Fall – 2018
10/16/2018
80 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 10 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	15	
3	10	
4	10	
5	20	
6	15	
7	10	
8	10	
Total:	100	

1. (10 points) Find the given derivatives. You do not need to simplify your answer.

(a) $\dfrac{d}{dx}\left(x^2e^{6x}+\dfrac{1}{\sqrt{x}}-\log_3(x)\right)=2xe^{6x}+6x^2e^{6x}-\dfrac{1}{2\sqrt{x^3}}-\dfrac{1}{x\ln 3}$

(b) $\dfrac{d}{dx}\left(\dfrac{x^2-1}{x^3+2x+1}\right)=\dfrac{(x^3+2x+1)(2x)-(x^2-1)(3x^2+2)}{(x^3+2x+1)^2}$

(c) $\dfrac{d}{dx}\ \ln(\sin(2^x))=\dfrac{1}{\sin 2^x}\cdot\cos 2^x\cdot 2^x\ln 2$

2. (15 points) Find the given derivatives. You do not need to simplify your answer.

(a) $\frac{d}{dx}\left(\sec(e^x)(x^3+4)^6\sec^{-1}(3x)\right) =$

$$\sec(e^x)\tan(e^x)e^x \cdot (x^3+3)^6\sec^{-1}(3x) + 6(x^3+4)^5(3x^2)\cdot\sec(e^x)\sec^{-1}(3x) + \frac{1}{|3x|\sqrt{(3x)^2-1}}\cdot 3\cdot\sec(e^x)(x^3+4)^6$$

(b) $\frac{d}{dx}\left(\frac{\sqrt{x}+1}{\cos(x^2)}\right) = \frac{\cos(x^2)\cdot\frac{1}{2\sqrt{x}} - (-\sin(x^2)\cdot 2x)\cdot(\sqrt{x}+1)}{\cos^2(x^2)}$

(c) $\frac{d}{dx}\left(\cot^3(5^{x\csc(7x)})\right) =$

$$3\cot^2(5^{x\csc(7x)})\cdot\big(-\csc^2(5^{x\csc(7x)})\big)\cdot\big(5^{x\csc(7x)}\ln 5\big)\cdot\big(\csc(7x) - 7x\csc(7x)\cot(7x)\big)$$

3. (10 points) Prove that $\dfrac{d}{dx}\,(\arctan x) = \dfrac{1}{1+x^2}$.

$$y = \arctan x$$

$$\tan y = \tan(\arctan x)$$

$$\tan y = x$$

$$\frac{d}{dx}\ \tan y = \frac{d}{dx}\ x$$

$$\sec^2 y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Now $\sin^2 x + \cos^2 x = 1$ *so that* $\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$. *But this is exactly* $\tan^2 x + 1 = \sec^2 x$.

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\tan^2 y + 1}$$

$$\frac{dy}{dx} = \frac{1}{x^2 + 1}$$

4. (10 points) Find the following derivative:

$$\frac{d}{dx}\,\big(\tan(3x)\big)^{xe^{x^2}+1}$$

$$y = \big(\tan(3x)\big)^{xe^{x^2}+1}$$

$$\ln y = \ln\big(\tan(3x)\big)^{xe^{x^2}+1}$$

$$\ln y = (xe^{x^2}+1)\ln\tan(3x)$$

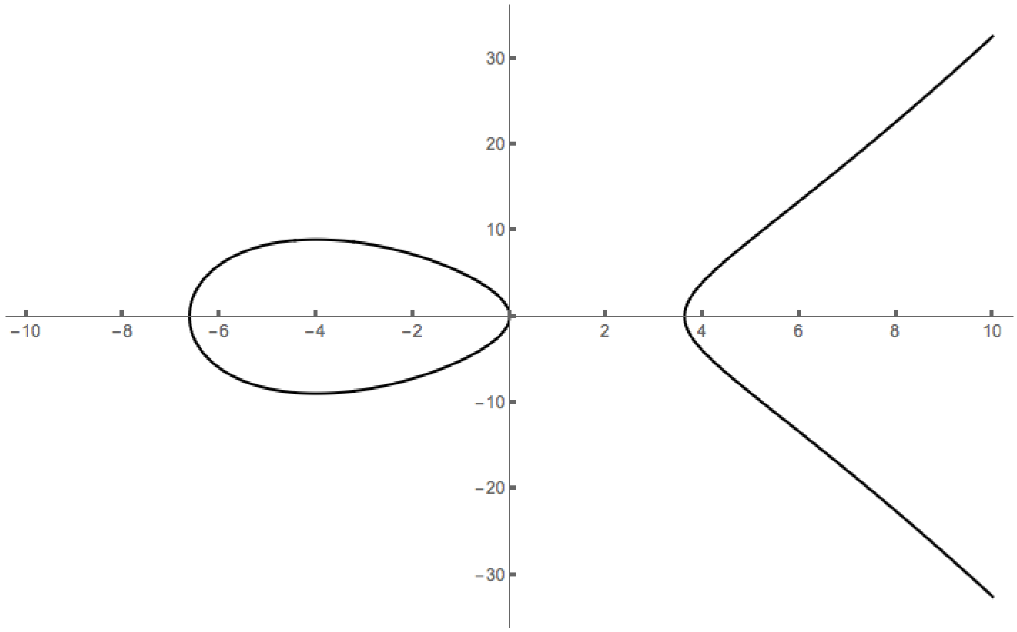
$$\frac{d}{dx}\ln y = \frac{d}{dx}\Big[(xe^{x^2}+1)\ln\tan(3x)\Big]$$

$$\frac{1}{y}\cdot y' = (e^{x^2} + xe^{x^2}\cdot 2x)\ln\tan(3x) + (xe^{x^2}+1)\cdot\frac{1}{\tan(3x)}\cdot\sec^2(3x)\cdot 3$$

$$y' = y\left[(e^{x^2}+2x^2e^{x^2})\ln\tan(3x) + \frac{3(xe^{x^2}+1)\sec^2(3x)}{\tan(3x)}\right]$$

$$y' = \big(\tan(3x)\big)^{xe^{x^2}+1}\left[(e^{x^2}+2x^2e^{x^2})\ln\tan(3x) + \frac{3(xe^{x^2}+1)\sec^2(3x)}{\tan(3x)}\right]$$

5. (20 points) An elliptic curve is a curve given by an equation $y^2 = f(x)$, where $f(x)$ is a cubic polynomial in x. Consider the elliptic curve given by $y^2 = x^3 + 3x^2 - 24x$.



(a) Find $\frac{dy}{dx}$ for this curve.

$$y^2 = x^3 + 3x^2 - 24x$$
$$\frac{d}{dx}\, y^2 = \frac{d}{dx}\left(x^3 + 3x^2 - 24x\right)$$
$$2y\, y' = 3x^2 + 6x - 24$$
$$\frac{dy}{dx} = \frac{3x^2 + 6x - 24}{2y}$$

(b) Find $\frac{d^2y}{dx^2}$ for this curve.

$$\frac{dy}{dx} = \frac{3x^2 + 6x - 24}{2y}$$
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3x^2 + 6x - 24}{2y}\right)$$
$$\frac{d^2y}{dx^2} = \frac{2y(6x - 5) - 2y'\left(3x^2 + 6x - 24\right)}{4y^2}$$
$$\frac{d^2y}{dx^2} = \frac{2y(6x - 5) - \dfrac{3x^2 + 6x - 24}{2y}\left(3x^2 + 6x - 24\right)}{4y^2}$$

(c) Find the tangent line to the curve at the point $(4, -4)$.

$$\frac{dy}{dx} = \frac{3x^2+6x-24}{2y}\bigg|_{\substack{x=4,\\ y=-4}} = \frac{3(4^2)+6(4)-24}{2(-4)} = \frac{3(16)+24-24}{-8} = 3(-2) = -6$$

$$y = -4 + (-6)(x-4)$$

$$y = -4 - 6x + 24$$

$$y = 20 - 6x$$

(d) Find the points on the curve where the tangent line is horizontal. Show your work. Then use the graph to approximate the points where the curve has a vertical tangent (no work necessary for this part).

$$\begin{aligned} 3x^2+6x-24 &= 0 \\ x^2+2x-8 &= 0 \\ (x+4)(x-2) &= 0 \\ x &= -4 \text{ or } 2 \end{aligned}$$

$$\begin{aligned} x=-4: &\quad y^2 = (-4)^3 + 3(-4)^2 - 24(-4) = 80 \Longrightarrow y = \pm\sqrt{80} = \pm 4\sqrt{5} \\ x=2: &\quad y^2 = 2^3 + 3(2^2) - 24(2) = -28 \Longrightarrow y^2 = -28 \quad \textit{Impossible.} \end{aligned}$$

Therefore, the horizontal tangent points are $(-4, \pm 4\sqrt{5})$.

From the graph, we see the vertical tangents occur at the points $(-6.5, 0), (0, 0), (3.5, 0)$.

6. (15 points) Complete the following:

(a) Find the linearization of $f(x) = x^3 + x - 1$ at $x = 2$.

$$f(x) = x^3 + x - 1$$
$$f(2) = 2^3 + 2 - 1 = 9$$

$$f'(x) = 3x^2 + 1$$
$$f'(2) = 3(2^2) + 1 = 13$$

$$\mathcal{L}(x) = 9 + 13(x - 2) = 13x - 17$$

(b) Approximate $\sqrt{85}$.

$$f(x) = \sqrt{x}$$
$$f(81) = 9$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$
$$f'(81) = \frac{1}{2\sqrt{81}} = \frac{1}{18}$$

$$\mathcal{L}(x) = 9 + \frac{1}{18}(x - 81)$$

$$\sqrt{85} \approx \mathcal{L}(85) = 9 + \frac{1}{18}(85 - 81) = 9 + \frac{4}{18} = 9\frac{4}{18} = \frac{166}{18} = \frac{83}{9}$$

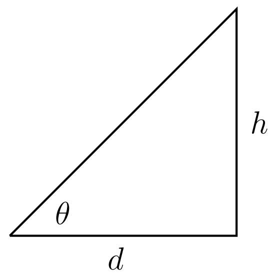
(c) The energy in stored in a capacitor obeys the law $E = \frac{1}{2}CV^2$, where V is the voltage and C is a *fixed* capacitance depending on the capacitor. A scientist is performing experiments with a capacitor where $C = 3$ F. Estimate the error in the measurement of energy stored in the capacitor if the scientist measures the voltage to be 10 V with an error of ± 0.2 V.

$$E = \frac{1}{2}C\,V^2 = \frac{1}{2} \cdot 3 \cdot V^2 = \frac{3}{2}V^2$$

$$dE = 3V\,dV = 3(10)0.2 = 6J$$

7. (10 points) An observer with a camera stands 200 ft away from the spot where a hot air ballon is rising straight up into the sky.

(a) When the angle of the camera with the ground is $\frac{\pi}{3}$ and the observer increases this angle at a rate of $\frac{1}{5}$ rad/min to keep the ballon centered in the frame. What rate is the balloon rising at that moment?

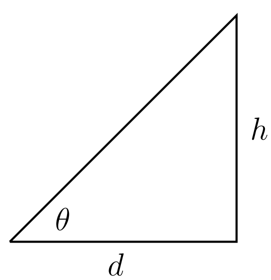


$$\theta = \frac{\pi}{3}$$
$$\theta' = \frac{1}{5}$$
$$d = 200$$
Want: h'

$$\tan\theta = \frac{h}{d}$$
$$\tan\theta = \frac{h}{200}$$
$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{h}{200}$$
$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{200}\frac{dh}{dt}$$

$$\frac{dh}{dt} = 200\sec^2\theta\,\frac{d\theta}{dt}$$
$$\frac{dh}{dt} = 200(\sec(\pi/3))^2 \cdot \frac{1}{5}$$
$$\frac{dh}{dt} = 200 \cdot (2)^2 \cdot \frac{1}{5}$$
$$\frac{dh}{dt} = 200 \cdot 4 \cdot \frac{1}{5}$$
$$\frac{dh}{dt} = 160 \textit{ ft/min}$$

(b) At what rate is the angle between the camera's line of sight of the balloon and the ground changing when the balloon is 1,000 ft above the ground if the ballon is rising at a constant rate of 5 ft/s.



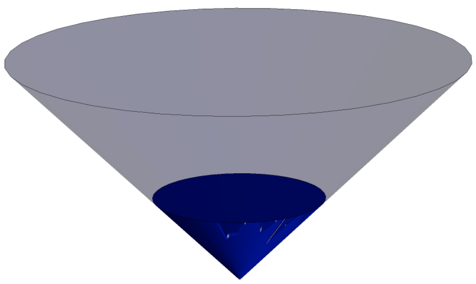
$$h = 1000$$
$$h' = 5$$
$$d = 200$$
Want: θ'

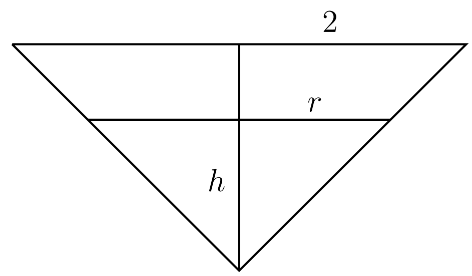
$$\tan\theta = \frac{h}{d}$$
$$\theta = \tan^{-1}\left(\frac{h}{d}\right)$$
$$\theta = \tan^{-1}\left(\frac{h}{200}\right)$$
$$\frac{d}{dt}\theta = \frac{d}{dt}\tan^{-1}\left(\frac{h}{200}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{1+(h/200)^2} \cdot \frac{dh/dt}{200}$$
$$\frac{d\theta}{dt} = \frac{1}{1+(1000/200)^2} \cdot \frac{5}{200}$$
$$\frac{d\theta}{dt} = \frac{1}{1+5^2} \cdot \frac{1}{40}$$
$$\frac{d\theta}{dt} = \frac{1}{1040} \textit{ rad/min}$$

8. (10 points) In order to walk around and easily water plants, a conical watering pail is covered in a grid of holes. Water flows through the holes at a rate proportional to the surface area the water touches, i.e. kA ft^3/s, where k is a constant and A is the surface area. If the pail is 4 ft across and 3 ft tall and is currently filled to a level of $h = 0.25$ ft, find $\frac{dh}{dt}$, the rate at which the water level is changing. Recall that the volume and surface area of a cone are given in terms of its height h and radius r by the following equations:

$$V = \frac{1}{3}\pi r^2 h$$
$$A = \pi r\sqrt{h^2 + r^2}$$





$$\frac{r}{h} = \frac{2}{3} \Rightarrow r = \frac{2}{3}\,h \quad \textit{and if } h = \frac{1}{4} \Rightarrow r = \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{6}$$

$$V = \frac{1}{3}\,\pi r^2 h$$
$$V = \frac{1}{3}\,\pi\left(\frac{2}{3}\,h\right)^2 h$$
$$V = \frac{1}{3}\,\pi\frac{4}{9}\,h^3$$
$$V = \frac{4\pi}{27}\,h^3$$
$$\frac{dV}{dt} = \frac{4\pi}{9}\,h^2\,\frac{dh}{dt}$$
$$kA = \frac{4\pi}{9}\,h^2\,\frac{dh}{dt}$$
$$k\pi r\sqrt{h^2 + r^2} = \frac{4\pi}{9}\,h^2\,\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{9kr\sqrt{h^2 + r^2}}{4h^2}$$
$$\frac{dh}{dt} = \frac{9k(1/6)\sqrt{(1/4)^2 + (1/6)^2}}{4(1/4)^2}$$
$$\frac{dh}{dt} = \frac{\sqrt{13}}{2}\,k$$