Math 295: Exam 3
Fall – 2018
11/16/2018
50 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 10 pages (including this cover page) and 6 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	15	
2	15	
3	15	
4	20	
5	15	
6	20	
Total:	100	

1. (15 points) Compute the following limits. Be sure to to show all your work and justify your answer completely.

(a) $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2} \overset{\text{L.H.}}{=} \lim\limits_{x\to 0} \dfrac{e^x - 1}{2x} \overset{\text{L.H.}}{=} \lim\limits_{x\to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

(b) $\lim\limits_{x\to\infty} \dfrac{\ln(1+e^{6x})}{5x} \overset{\text{L.H.}}{=} \lim\limits_{x\to\infty} \dfrac{\frac{1}{1+e^{6x}} \cdot 6e^{6x}}{5} = \lim\limits_{x\to\infty} \dfrac{6e^{6x}}{5(1+e^{6x})} \overset{\text{L.H.}}{=} \lim\limits_{x\to\infty} \dfrac{36e^{6x}}{5(6e^{6x})} = \lim\limits_{x\to\infty} \dfrac{36}{5(6)} = \dfrac{6}{5}$

(c) $\lim\limits_{x\to 0^+} \sqrt[3]{x}\,\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{x^{-1/3}} \overset{\text{L.H.}}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{3}x^{-4/3}} = \lim\limits_{x\to 0^+} -\dfrac{3x^{4/3}}{x} = \lim\limits_{x\to 0^+} -3x^{1/3} = 0$

(d) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{xe^{4x}}\right)=\lim_{x\to 0}\left(\frac{e^{4x}}{xe^{4x}}-\frac{1}{xe^{4x}}\right)=\lim_{x\to 0}\frac{e^{4x}-1}{xe^{4x}}\overset{L.H.}{=}\lim_{x\to 0}\frac{4e^{4x}}{e^{4x}+4xe^{4x}}=\frac{4}{1+0}=4$

(e) $\lim_{x\to 0}(1+2x)^{3/x}$

Let $L=\lim_{x\to 0}(1+2x)^{3/x}$. *Then* $\ln L=\lim_{x\to 0}\ln(1+2x)^{3/x}=\lim_{x\to 0}\frac{3}{x}\ \ln(1+2x)$.

$$\lim_{x\to 0}\frac{3}{x}\ \ln(1+2x)=\lim_{x\to 0}\frac{3\ln(1+2x)}{x}\overset{L.H.}{=}\lim_{x\to 0}\frac{3\,\frac{1}{1+2x}\cdot 2}{1}=\lim_{x\to 0}\frac{6}{1+2x}=6.$$

Therefore, $\ln L=6$ *so that* $L=e^6$.

2. (15 points) l'Hôpital's Rule is a useful tool but can fail to compute a limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ for the following reasons:

   A. $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist.

   B. $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is not an indeterminate form.

   C. The problem 'loops', i.e. $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ is essentially the same limit as the initial limit.

   D. $f(x)$ or $g(x)$ are not differentiable.

   For each of the following limits, indicate one of the reasons above why l'Hôpital's Rule does not apply to the limit.

   ___*C*___ $\lim_{x\to\infty} \frac{3x+5}{\sqrt{2x^2+1}}$

   ___*B*___ $\lim_{x\to\infty} \frac{x\sin x}{x^2+x\cos x}$

   ___*C*___ $\lim_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}}$

   ___*A*___ $\lim_{x\to\infty} \frac{2x}{3x+\sin x}$

   Choose one of the limits above and compute it below. Be sure to justify your answer completely.

$$\lim_{x\to\infty} \frac{3x+5}{\sqrt{2x^2+1}} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{3+\frac{5}{x}}{\frac{\sqrt{2x^2+1}}{x}} = \lim_{x\to\infty} \frac{3+\frac{5}{x}}{\sqrt{\frac{2x^2+1}{x^2}}} = \lim_{x\to\infty} \frac{3+\frac{5}{x}}{\sqrt{2+\frac{1}{x^2}}} = \frac{3+0}{\sqrt{2+0}} = \frac{3}{\sqrt{2}}$$

$$\lim_{x\to\infty} \frac{x\sin x}{x^2+x\cos x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\frac{\sin x}{x}}{1+\frac{\cos x}{x}} = \frac{0}{1+0} = 0$$

$$\lim_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to\infty} \frac{1-\frac{1}{e^{2x}}}{1+\frac{1}{e^{2x}}} = \frac{1-0}{1+0} = 1$$

$$\lim_{x\to\infty} \frac{2x}{3x+\sin x} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{2}{3+\frac{\sin x}{x}} = \frac{2}{3+0} = \frac{2}{3}$$

3. (15 points) Complete each of the following parts. Be sure to show all your work, and justify your answers completely.

   (a) Verify that $f(x) = x^3 + x - 1$ satisfies the hypotheses of the Mean Value Theorem on $[0, 3]$. Find all numbers c satisfying the conclusions of the Mean Value Theorem on this interval.

   Observe $f(x) = x^3 + x - 1$ is continuous on $[0, 3]$. Because $f'(x) = 3x^2 + 1$ is defined on $[0, 3]$, $f(x)$ is differentiable on $(0, 3)$. By the Mean Value Theorem, there exists $c \in (0, 3)$ such that $f(3) - f(0) = f'(c)(3 - 0)$. But then we have

$$\begin{aligned} f(3) - f(0) &= f'(c)(3 - 0) \\ 29 - (-1) &= 3(3c^2 + 1) \\ 30 &= 3(3c^2 + 1) \\ 10 &= 3c^2 + 1 \\ 9 &= 3c^2 \\ c^2 &= 3 \\ c &= \pm\sqrt{3} \end{aligned}$$

   Therefore, $c = \sqrt{3}$ satisfies the hypothesis of the Mean Value Theorem on $[0, 3]$.

   (b) Use the Mean Value Theorem to prove that if $f'(x) = 0$ for all $x \in [a, b]$, then $f(x)$ is constant on $[a, b]$. [Hint: Show $f(x_0) = f(a)$ for all $a \leq x_0 \leq b$.]

   Since $f'(x)$ exists on $[a, b]$, $f(x)$ is differentiable (hence continuous) on $[a, b]$. Therefore, $f(x)$ satisfies the Mean Value Theorem on $[a, x_0]$ for any $x_0 \leq b$. Then by the Mean Value Theorem, $f(x_0) - f(a) = f'(c)(x_0 - a)$ for some $c \in [a, x_0]$. But $f'(x) = 0$ for all $x \in [a, b]$. Therefore,

$$\begin{aligned} f(x_0) - f(a) &= f'(c)(x_0 - a) \\ f(x_0) - f(a) &= 0 \\ f(x_0) &= f(a) \end{aligned}$$

   This shows $f(x) = f(a)$ for all $x \in [a, b]$. But then $f(x)$ is constant on $[a, b]$.

   Note that this also shows that if $f'(x) = g'(x)$ on $[a, b]$, then $f(x) = g(x) + C$ for some constant C. The function $H(x) := f(x) - g(x)$ has derivative 0 because $H'(x) = f'(x) - g'(x) = 0$. But then $H'(x) = 0$ so that $H(x)$ is constant, i.e. $H(x) = C$. But then $f(x) - g(x) = C$ so that $f(x) = g(x) + C$.

4. (20 points) A function $f(x)$ and its derivatives $f'(x)$ and $f''(x)$ are given below:

$$f(x) = \left(x - \frac{7}{4}\right) x^{4/3} \qquad f'(x) = \frac{7}{3}x^{1/3}(x-1) \qquad f''(x) = \frac{7(4x-1)}{9x^{2/3}}$$

Answer the following questions. You may show your work on the next page. Be sure to indicate the part.

(a) What are the intervals on which $f(x)$ is increasing?

$$(-\infty, 0) \cup (1, \infty)$$

(b) What are the intervals on which $f(x)$ is decreasing?

$$(0, 1)$$

(c) Find all critical values for $f(x)$. Classify these critical values as local maxima, local minima, or neither. Be sure to use the First or Second Derivative Test to justify your answer.

$x = 0$ is a local maxima and $x = 1$ is a local minima

(d) What are the intervals on which $f(x)$ is concave?

$$(-\infty, 0) \cup (0, \tfrac{1}{4})$$

(e) What are the intervals on which $f(x)$ is convex?

$$(\tfrac{1}{4}, \infty)$$

(f) Find the x-values of any points of inflection on $f(x)$.

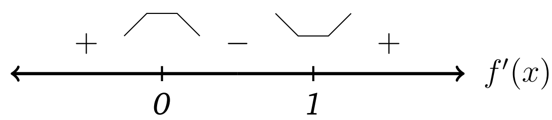
$$x = \tfrac{1}{4}$$

(g) Find the absolute minimum and absolute maximum *values* of $f(x)$ on $[-1, \frac{7}{4}]$.

$-\dfrac{11}{4}$ is the absolute minimum while 0 is the absolute maximum.

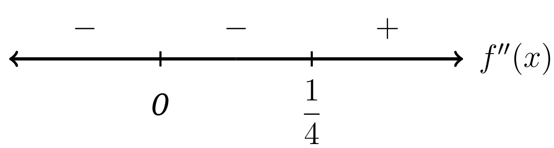
$$f(x) = \left(x - \frac{7}{4}\right) x^{4/3} \qquad f'(x) = \frac{7}{3}x^{1/3}(x-1) \qquad f''(x) = \frac{7(4x-1)}{9x^{2/3}}$$

Setting $f'(x) = 0$*, we obtain* $x = 0$ *and* $x = 1$.



Therefore, $x = 0$ *is a maximum and* $x = 1$ *is a minimum.*

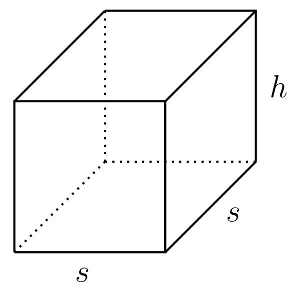
Setting $f''(x) = 0$*, we obtain* $x = \frac{1}{4}$. *Notice also that* $f''(x)$ *is undefined at* $x = 0$.



Since $f''(x)$ *switches sign at* $x = \frac{1}{4}$ *and* $f(x)$ *is defined at* $x = \frac{1}{4}$, $f(x)$ *has a point of inflection at* $x = \frac{1}{4}$.

$$f(-1) = -\frac{11}{4}$$
$$f(0) = 0$$
$$f(1) = -\frac{3}{4}$$
$$f\left(\frac{7}{4}\right) = 0$$

5. (15 points) A rectangular box has a square bottom and an open top. If only 2,700 cm^2 of material is available to construct the box, what dimensions maximize the volume of the box? Be sure to draw a picture and justify completely that these dimensions are optimal.



We want to optimize $V = lwh = s \cdot s \cdot h = s^2 h$. *But we know that*

$$\text{Surface Area} = 2,700 \text{ cm}^2$$
$$\text{Surface Area} = 4sh + s^2$$

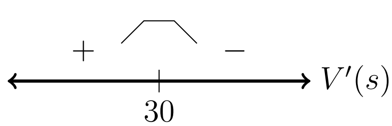
But then we have

$$4sh + s^2 = 2700$$
$$4sh = 2700 - s^2$$
$$h = \frac{2700 - s^2}{4s}$$

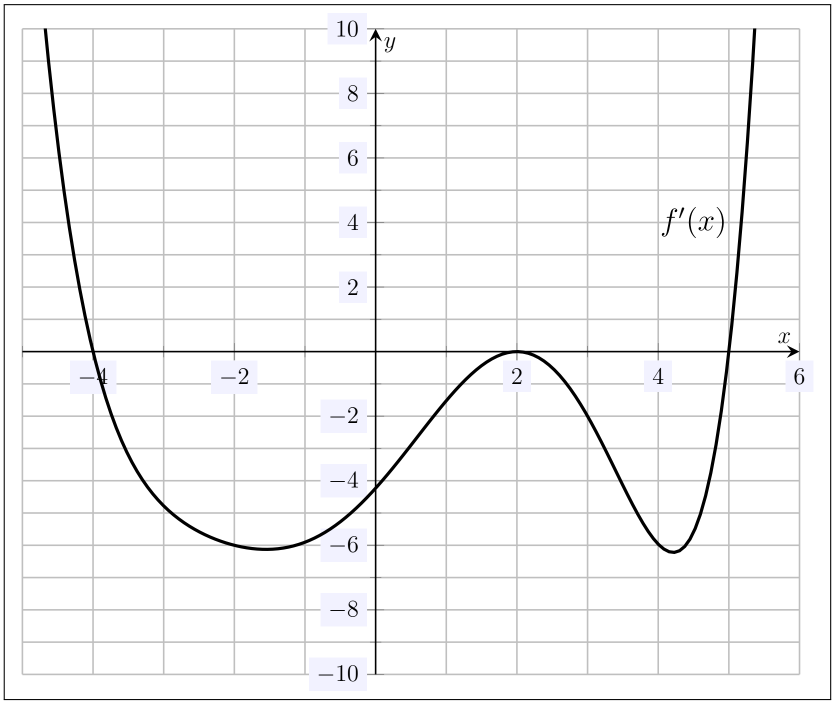
But then $V = s^2 h = s^2 \left(\frac{2700 - s^2}{4s} \right) = \frac{2700s - s^3}{4}$. *Clearly,* $s \in [0, \sqrt{2700}]$, *where in each case the box has* $V = 0$. *We have* $V' = \frac{2700 - 3s^2}{4}$. *Now setting* $V' = 0$, *we have*

$$\frac{2700 - 3s^2}{4} = 0$$
$$2700 - 3s^2 = 0$$
$$3s^2 = 2700$$
$$s^2 = 900$$
$$s = \pm\sqrt{900} = \pm\sqrt{9 \cdot 100}$$
$$s = \pm 30$$

But then we must have $s = 30$ *cm*^{2}. *Then* $h = \frac{2700 - 900}{120} = \frac{800}{120} = 15$. *The dimensions then are* $30 \times 30 \times 15$. *We confirm this is a maximum,*



6. (20 points) The graph of $f'(x)$ for some function $f(x)$ is plotted in the figure below. Based on this graph, complete the questions below.



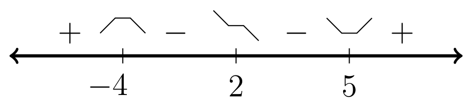
(a) Find all the intervals on which $f(x)$ is increasing.

$$(-\infty, -4) \cup (5, \infty)$$

(b) Find all the intervals on which $f(x)$ is decreasing.

$$(-4, 2) \cup (2, 5)$$

(c) Find all critical points for $f(x)$. Using the First or Second Derivative Test, classify these x-values as locations of maximums, minimums, or neither.



Therefore, $x = -4$ *is a maxima,* $x = 5$ *is a minimum, and* $x = 2$ *is neither.*

(d) Find the intervals on which $f(x)$ is concave.

$$(-\infty, -1.5) \cup (2, 4.2)$$

(e) Find the intervals on which $f(x)$ is convex.

$$(-1.5, 2) \cup (4.2, \infty)$$

(f) Find the x-values of any points of inflection on $f(x)$.

$x = -1.5$*,* $x = 2$*, and* $x = 4.2$