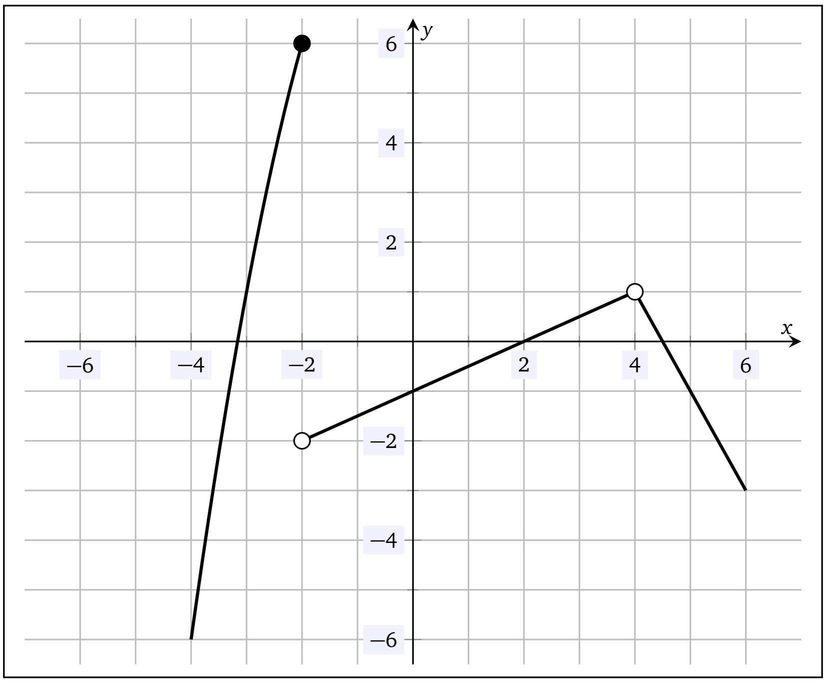
(Left/Right) Limits:

Problem 1: True or False: If $\lim_{x \to a} f(x)$ exist, then $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$. Explain your answer.

Problem 2: True or False: If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ exists. Explain your answer.

Problem 3: True or False: If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x)$, then $f(a) = \lim_{x \to a} f(x)$. Explain your answer.

Problem 4: Use the graph of $f(x)$ below to evaluate the following:



(a) $\lim_{x \to 2^+} f(x)$

(b) $\lim_{x \to 2^-} f(x)$

(c) $\lim_{x \to 2} f(x)$

(d) $f(2)$

(e) $\lim_{x \to -2^-} f(x)$

(f) $\lim_{x \to -2^+} f(x)$

(g) $\lim_{x \to -2} f(x)$

(h) $f(-2)$

(i) $\lim_{x \to 4^-} f(x)$

(j) $\lim_{x \to 4^+} f(x)$

(k) $\lim_{x \to 4} f(x)$

(l) $f(4)$

Problem 5: Let $f(x) = [\![x]\!]$ denote the largest integer n such that $n \leq x$. For example, $[\![1.5]\!] = 1$, $[\![2]\!] = 2$, $[\![-1]\!] = -1$, $[\![-2.2]\!] = -3$, and $[\![0]\!] = 0$. This function is used in Computer Science since $[\![x]\!]$ gives the 'integer part' of x.

(a) Graph the function $f(x) = [\![x]\!]$

(b) Determine $\lim\limits_{x\to 3.2^+} f(x)$, $\lim\limits_{x\to 3.2^-} f(x)$, and $\lim\limits_{x\to 3.2} f(x)$.

(c) Determine $\lim\limits_{x\to 5^+} f(x)$, $\lim\limits_{x\to 5^-} f(x)$, and $\lim\limits_{x\to 5} f(x)$.

(d) Using the previous parts for what values a does $\lim\limits_{x\to a} f(x)$ exist?

Problem 6: Sketch the graph of a function, $f(x)$, satisfying the following:

(i) $\lim\limits_{x\to 2^+} f(x) = -3$

(ii) $\lim\limits_{x\to 2^-} f(x) = 5$

(iii) $\lim\limits_{x\to -2^-} f(x) = \infty$

(iv) $\lim\limits_{x\to -2^+} f(x) = -\infty$

Problem 7: Sketch the graph of a function, $g(x)$, satisfying the following:

(i) $\lim\limits_{x\to 0} g(x) = 1$.

(ii) $\lim\limits_{x\to \infty} g(x) = -1$

(iii) $\lim\limits_{x\to -\infty} g(x) = -\infty$

(iv) $\lim\limits_{x\to 3^+} g(x) = 3$

(v) $\lim\limits_{x\to 3^+} g(x) = -2$

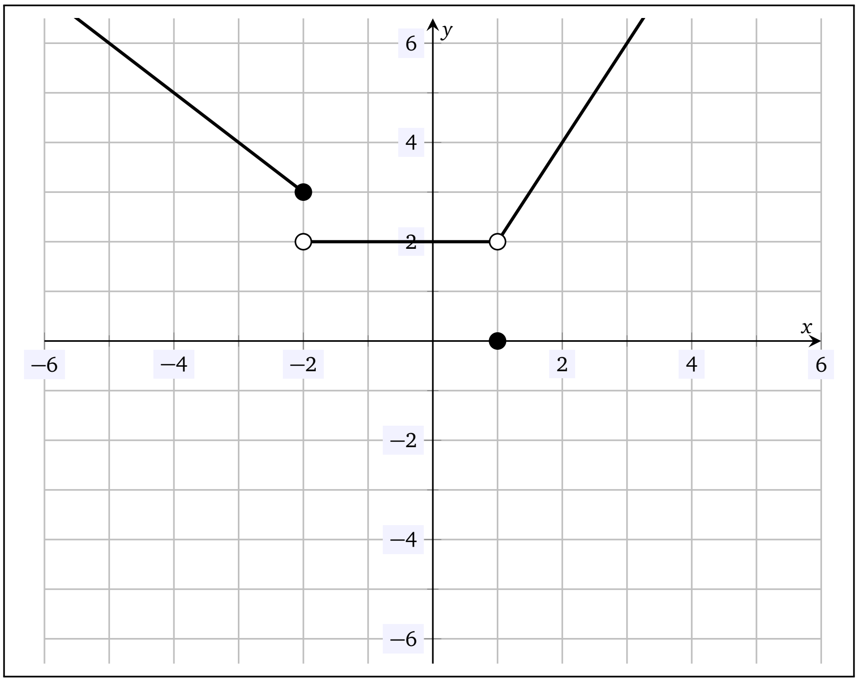
Problem 8: If $\lim\limits_{x\to a} \big(f(x)+g(x)\big)$ exists, must $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist? Prove or give a counterexample. If $\lim\limits_{x\to a} \big(f(x)g(x)\big)$ exists, must $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist? Prove or give a counterexample.

Problem 9: In Special Relativity, the energy of a particle moving at a velocity v is given by

$$E(v) = \frac{mc^2}{\sqrt{1 - v^2/c^2}},$$

where c is the speed of light and m is the mass of the particle. What happens if $v = 0$? What happens as v approaches c? What does this limit imply? Is this something you already knew from Science?

Problem 10: Use the graph of $f(x)$ below to evaluate the following:



(a) $\lim_{x \to -2^-} f(x)$

(b) $\lim_{x \to -2^+} f(x)$

(c) $\lim_{x \to -2} f(x)$

(d) $f(-2)$

(e) $\lim_{x \to 1^-} f(x)$

(f) $\lim_{x \to 1^+} f(x)$

(g) $\lim_{x \to 1} f(x)$

(h) $f(1)$

(i) $\lim_{x \to -\infty} f(x)$

(j) $\lim_{x \to \infty} f(x)$

Computing Limits

Problem 11: Using the fact that $\lim_{x \to 1} f(x) = 3$, $\lim_{x \to 1} g(x) = -2$, and $\lim_{x \to 1} h(x) = 0$, determine—*if possible*—the following limits. Be sure to justify each step!

(a) $\lim_{x \to 1} (f(x) - g(x))$

(b) $\lim_{x \to 1} (2f(x) - 5g(x))$

(c) $\lim_{x \to 1} \left(f(x)^2 - g(x)\right)$

(d) $\lim_{x \to 1} \sqrt{f(x)^2 + g(x)^4}$

(e) $\lim_{x \to 1} \dfrac{2f(x)}{g(x)}$

(f) $\lim_{x \to 1} \dfrac{f(x) - g(x)}{2f(x) + 1}$

(g) $\lim_{x \to 1} \dfrac{h(x)}{f(x)}$

(h) $\lim_{x \to 1} \dfrac{f(x)}{h(x)}$

(i) $\lim_{x \to 1} \dfrac{2f(x) + 3g(x)}{h(x)}$

Problem 12: Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 2} \frac{x^2+3x+2}{x-3}$

(b) $\displaystyle\lim_{x\to 0} \frac{x^2-x+1}{x+1}$

(c) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^2-4x+3}$

(d) $\displaystyle\lim_{x\to -2} x\sin x$

(e) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

(f) $\displaystyle\lim_{x\to -2} \frac{\frac{1}{2}+\frac{1}{x}}{2+x}$

(g) $\displaystyle\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

Problem 13: Evaluate the following limits:

(a) $\displaystyle\lim_{t\to 5} \frac{t^2-7t+6}{t-5}$

(b) $\displaystyle\lim_{t\to 1} \frac{t^2-4t+3}{t-1}$

(c) $\displaystyle\lim_{t\to 1} \frac{t^2-2t-1}{t^4-1}$

(d) $\displaystyle\lim_{t\to 0} \left(\frac{5}{t}-\frac{5}{t^2+t}\right)$

Problem 14: Evaluate the following limits:

(a) $\displaystyle\lim_{w\to 0} \frac{w}{|w|}$

(b) $\displaystyle\lim_{w\to -2} \frac{2w+4}{|w+2|}$

(c) $\displaystyle\lim_{w\to 6} \frac{|w-5|-1}{w-6}$

(d) $\displaystyle\lim_{w\to 3} \frac{w^2+w-12}{|w-3|}$

(e) $\displaystyle\lim_{w\to 2} \left(3w^3-|w-2|\right)$

Problem 15: If $\displaystyle\lim_{x\to -4} \frac{f(x)-9}{x+4}=6$, what is $\displaystyle\lim_{x\to -4} f(x)$?

Problem 16: Evaluate the following limits:

(a) $\displaystyle\lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h}$

(b) $\displaystyle\lim_{x\to -3} \frac{x^2+7x+12}{x^2+2x-3}$

(c) $\displaystyle\lim_{h\to 0} \frac{(1+h)^3-1}{h}$

(d) $\displaystyle\lim_{x\to 0} \frac{x}{1-\sqrt{1+x}}$

(e) $\displaystyle\lim_{w\to 0} \frac{\cos w}{\sin w}$

(f) $\displaystyle\lim_{\theta\to 0} \frac{\theta}{\tan\theta}$

(g) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x}$ [Hint: Multiply by $\frac{1+\cos x}{1+\cos x}$.]

(h) $\displaystyle\lim_{x\to 1} \frac{x^2+3x-5}{x^2-2x-1}$

Limits with Infinity

Problem 17: Find the following limits (if they exist):

(a) $\lim\limits_{x\to\infty} 1/x$

(b) $\lim\limits_{x\to\infty} \sin x$

(c) $\lim\limits_{x\to\infty} \dfrac{x-1}{x+2}$

(d) $\lim\limits_{x\to\infty} \dfrac{2x^2+5x+7}{3x^2-x-7}$

(e) $\lim\limits_{x\to\infty} \dfrac{x^5-x+2}{x^3+16x^2+14x+2}$

(f) $\lim\limits_{x\to\infty} \dfrac{x^2-2x+8}{x^4+4x+1}$

Problem 18: Find the following limits (if they exist):

(a) $\lim\limits_{x\to\infty} \dfrac{6}{x^2+4}$

(b) $\lim\limits_{x\to\infty} 2^{-x}$

(c) $\lim\limits_{x\to\infty} \ln(x+6)$

(d) $\lim\limits_{x\to\infty} \ln\left(\dfrac{2x+1}{3x-2}\right)$

(e) $\lim\limits_{x\to\infty} \cos(1/x)$

(f) $\lim\limits_{x\to\infty} x\sin(1/x)$

Problem 19: Compute the following limits, if they exist:

(a) $\lim\limits_{x\to-\infty} \dfrac{2x^3-x+3}{x^2+7x-1}$

(b) $\lim\limits_{x\to\infty} \dfrac{5x^4+x+1}{13x^4-9x^2+6}$

(c) $\lim\limits_{x\to\infty} \dfrac{x+7}{x^3+1}$

(d) $\lim\limits_{x\to-\infty} \dfrac{x+1}{x-3}$

Problem 20: Find the x-intercepts, y-intercepts, vertical asymptotes, and horizontal asymptotes of the following function. If there are discontinuities, identify them. If there are removable discontinuities, identify the point.

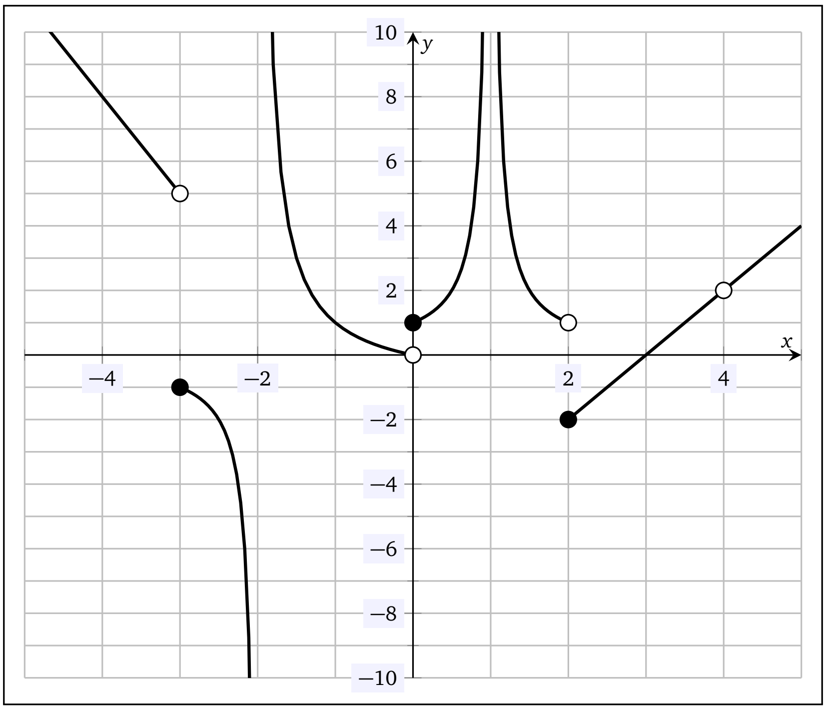
$$\frac{(x+2)(x-3)(x+7)(2x-3)}{(x-3)(2x+1)(x-2)(x-7)}$$

Problem 21: Compute $\lim\limits_{x\to\infty}\left(\sqrt{9x^2+1}-3x\right)$.

Problem 22: Find the limit of $\sqrt{2x^2+4x-1}-\sqrt{2x^2+8x+7}$ as x tends to infinity.

Continuity

Problem 23: For the following plot, find the values of x for which the function is discontinuous and identify the type of discontinuity.



Problem 24: Explain why the following functions are discontinuous:

(a) $f(x) = \sin(1/x)$

(b) $h(x) = \dfrac{1}{2-x}$

(c) $r(x) = \begin{cases} 2x+3, & x < 1 \\ x-7, & x \geq 1 \end{cases}$

(d) $s(x) = \begin{cases} -2x, & x < 0 \\ 4x, & x > 0 \end{cases}$

Problem 25: Explain why the function $g(x) = x\sin(1/x)$ is discontinuous on the interval $[-1, 1]$. What type of discontinuity does $g(x)$ have on this interval? If possible, 'repair' the discontinuity.

Problem 26: Find the values of x at which the following function is continuous. Explain your reasoning.

$$f(x) = \begin{cases} -2-x, & -1 \leq x \\ -1, & -1 < x \leq 0 \\ \sqrt{x}, & 0 < x < 1 \\ 2-x, & 1 \leq x < 2 \\ (x-2)^2, & 2 \leq x \end{cases}$$

Problem 27: Find the intervals on which the following functions are continuous:

(a) $f(x) = 2x+3$

(b) $g(x) = \dfrac{1}{6-5x}$

(c) $h(x) = \dfrac{x-7}{x+6}$

(d) $r(x) = \dfrac{\sin x}{x^2+2x+3}$

(e) $s(x) = \sin(\cos(x^2+1))$

(f) $t(x) = \dfrac{x\sin(1-x)}{\sqrt{x^2+2}}$

Problem 28: Find the values for x for which the following functions are discontinuous and demonstrate they are correct by graphing the function.

(a) $f(x) = \dfrac{5}{2+2\cos x}$

(b) $g(x) = \tan\sqrt{x}$

(c) $h(x) = \ln|x^2-4|$

Problem: Find values for b and c that make the following function continuous:

$$f(x) = \begin{cases} x^2+3x-1, & x \le -1 \\ x^3+bx^2+cx+2, & -1 < x < 2 \\ 2|x+1|, & 2 \le x \end{cases}$$

Problem 29: Show that the following function is everywhere continuous.

$$f(x) = \begin{cases} \dfrac{\sin(x-3)}{x-3}, & x \ne 3 \\ 1, & x = 3 \end{cases}$$

Problem 30: Show that the following function is everywhere continuous.

$$f(x) = \begin{cases} x^2\sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

Intermediate Value Theorem & Squeeze Theorem

Problem 31: Use the Squeeze Theorem to prove the following:

(a) $\displaystyle\lim_{x\to 0} x^2\sin^2\left(\frac{1}{x}\right) = 0$

(b) $\displaystyle\lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right) = 0$

(c) $\displaystyle\lim_{x\to 0} x^2 e^{\sin 1/x} = 0$

(d) $\displaystyle\lim_{x\to\infty} \frac{2+\sin x}{x-3} = 0$

Problem 32: Use the Intermediate Value Theorem to show that there is a root for the function in the given interval.

(a) $f(x) = x^3 - x - 1$ on $(1, 2)$

(b) $g(x) = x\sin^2 x - 1$ on $(0, 2)$

(c) $h(x) = \ln(4x+1) - x + 2$ on $(4, 6)$

Problem 33: Use the Intermediate Value Theorem to show that there is a solution to the given equation.

(a) $\sin x = x$

(b) $4x^2 - 4 = 2x$

(c) $e^x = 10 - \sqrt{x}$

(d) $\pi x^{15} + e^2 x^{13} - 5x^4 + \sqrt[3]{2} = e^\pi x^{12} + \pi^e x^3 + 6x - 1729$

Problem 34: Let $f(x) = x^2 + x\sin x - 3$. Prove there is a number $c \in \mathbb{R}$ so that $f(c) = \sqrt[3]{\pi}$. Is this value unique? Use WolframAlpha to approximate these values.

Derivative Definition

Problem 35: Use the definition of the derivative to find the derivative of $f(x) = 2x + 1$.

Problem 36: Use the definition of the derivative to find the derivative of $g(x) = x - x^2$.

Problem 37: Use the definition of the derivative to find the derivative of $h(x) = \dfrac{1}{x}$.

Problem 38: Use the definition of the derivative to find the derivative of the given function at the given x value:

(a) $f(x) = 1 - 2x$, $x = 1$

(b) $g(x) = x^2 + x + 1$, $x = 0$

(c) $h(x) = \dfrac{1}{x}$, $x = 4$

(d) $j(x) = \sqrt{x}$, $x = 1$

Problem 39: The following represents the derivative of some function f at some value a. Find such an f and a:

$$\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$$

Problem 40: The following represents the derivative of some function f at some value a. Find such an f and a:

$$\lim_{h \to 0} \frac{\sqrt{9+h}-3}{h}$$

Problem 41: The following represents the derivative of some function f at some value a. Find such an f and a:

$$\lim_{h \to 0} \frac{\frac{1}{(h-3)^2} - \frac{1}{9}}{h}$$