Derivatives

Problem 1: Differentiate the following:

(a) $5x^4 \sin^2(3x)$

(b) $e^{x^2} \log_5(1-x)$

(c) $2^x \cot x$

(d) $(x^2+5x+3)^5$

(e) $(1-x)\sec x$

(f) $\log_{1/2}(x)\csc(x)$

Problem 2: Differentiate the following:

(a) $\dfrac{\cos^3 x}{1-\sqrt[3]{x^2}}$

(b) $\dfrac{\cos^{-1}(2x)}{x+\sin\sqrt{x}}$

(c) $\dfrac{(5x+1)^2}{(4x-1)^3}$

(d) $\dfrac{\tan^2(\pi x)}{1+\log_2(2x-1)}$

(e) $\dfrac{5^{2x}}{3x-\sec x}$

(f) $\dfrac{2\sqrt[3]{x}+1}{\sec^{-1} x}$

Problem 3: Differentiate the following:

(a) $2^{2x}\log_7(2x)\cot^6(1-x)$

(b) $\dfrac{x^8\sin(3x)}{x-x^6}$

(c) $(x^3+x+1)^2 e^{\cos x}\sec^2 x$

(d) $\dfrac{\csc^{-1}(3x)}{(x+5)^2}$

(e) $7^{1/x^3}\sin\sqrt[5]{x}\,\arcsin x$

(f) $\dfrac{x\csc x+1}{xe^{-1/x}}$

Problem 4: Differentiate the following:

(a) $\dfrac{x^{3/2}-\sin x\cos x}{e^x\cot x}$

(b) $\dfrac{\log(\sin(2x))}{\tan(2x)-\operatorname{arccsc}^{-1}(x)}$

(c) $\dfrac{\cos(e^x)\,2^{\csc^2(1-x)}}{\ln(\tan(x))\,\cos(x^5)}$

(d) $\dfrac{1-\sqrt{x}-x\cot(\sqrt[3]{\pi x})}{1-\sec^{-1}(x)}$

(e) $\dfrac{\log_6(\tan^2(1-\sqrt{x}))}{x^2-x\tan^{-1}(5^{\ln(x^2+1)})}$

(f) $\dfrac{\sin(2x)\csc^{-1}(3x)-\tan(\pi^2 x)\sec^2(\cos(\log_5(\sqrt{x})))}{x\arctan(\sqrt{\pi x})+\operatorname{arccot}(x)\cos^{-1}\left(\log_5\left(\sin\left(\frac{2^x+1}{1-2^x}\right)\right)\right)}$

Problem 5: Let $f(x) := |x|$ and $g(x) := x+|x|$.

(a) Show that $f(x)$ is not differentiable at $x=0$.

(b) Plot $f'(x)$.

(c) Find and plot $g'(x)$.

Tangent Lines

Problem 6: Find the tangent line to the given function $f(x)$ at $x = a$.

(a) $f(x) = 3x^2 + 2x - 1$, $a = -1$

(b) $f(x) = (x^2 + 1)\cos(x^2 - 1)$, $a = 1$

(c) $f(x) = \dfrac{x+1}{1-x}$, $a = 3$

(d) $f(x) = (2x + 1)^3$, $a = -1$

(e) $f(x) = (x - 5)^2(x - 2)^3$, $a = 4$

(f) $f(x) = x\sin(x - 3)\log_5(2x - 1)$, $a = 3$

Problem 7: Find the equations of the tangent line to the curve $y = 1 - x^2$ which is parallel to the line $2x - 4y = 8$.

Problem 8: Find the *points* (if any) on the curve $y = x^3 - 6x^2 + 9x + 1$ where the tangent line to the curve are parallel to the x-axis.

Problem 9: Find a and b such that the line $y = 3x + 1$ is tangent to the curve $y = ax^3 + bx + 5$ at $x = 1$.

Implicit Differentiation

Problem 10: Find $\dfrac{dy}{dx}$ in the following:

(a) $y^2 + x = \sqrt[3]{x} - \sqrt{y}$

(b) $xy + xy^2 = 3 - y$

(c) $x - x\sin(xy) = y^2$

(d) $\dfrac{x}{1+y^2} = x^3$

(e) $x\tan^{-1}(y^2) = \dfrac{1}{\sqrt[3]{x}}$

(f) $e^{xy-y} = \log_2(xy)$

Problem 11: Find $\dfrac{dy}{dx}$ in the following:

(a) $\sqrt{x + xy} = \dfrac{y}{x}$

(b) $y\cos(x) - x\sin(y) = 5$

(c) $\dfrac{x-y}{y+x} = \dfrac{2y-x}{x+2y}$

(d) $x\operatorname{arcsec}(y) + y = 2 - x$

(e) $y5^{x-y^2} = x\log_4(y)$

(f) $\dfrac{\sin(x)\sin(y)}{e^y\tan(x)} = \dfrac{\log_6(xy)}{\cot(y-x)}$

Problem 12: Find the tangent line to the curve at the given point.

(a) $x^2 + y^2 = 5$, $(\sqrt{5/2}, -\sqrt{5/2})$

(b) $x^2y^2 - 2y = 2 - y$, $(1, 2)$

(c) $e^{xy^2} = y - x$, $(-1, 0)$

(d) $x^2y = \cos(y^2 - 1)$, $(-1, 1)$

(e) $\arctan(xy) = \dfrac{5\pi y - x + 5}{x - 5y}$

(f) $2(x^2 + y^2 - 3x)^2 = x^2 + y^2$, $(1, 1)$

Problem 13: Find the *points* where the curve given by $(x^2+y^2)^2 = x^2-y^2$ has horizontal or vertical tangent lines.

Problem 14: Find $\dfrac{d^2y}{dx^2}$ in the following:

(a) $x^2+y^2=4$

(b) $xy^2+y-\sqrt{x}=5$

(c) $y-y^2=xy$

(d) $\dfrac{x}{1+y}=\dfrac{y}{1+x}$

(e) $x\sin(xy)=y^2$

(f) $y\tan^{-1}(x)=-\dfrac{\sin e^x}{\cos(\ln y)}$

Logarithmic Differentiation

Problem 15: Use logarithmic differentiation to find the derivatives of the following functions:

(a) $f(x)=\dfrac{x+1}{x-1}$.

(b) $g(x)=\dfrac{(x+1)(x+3)}{x-4}$

(c) $h(x)=\dfrac{(2x+1)^2(x+1)}{(x-6)(x^2+1)}$

(d) $j(x)=\dfrac{x(x^2+x+1)\sin x}{(x+7)(2x+1)^2}$

(e) $k(x)=\dfrac{(3x^2+1)^5(x+4)(x-7)^3}{(1-2x)^6\sec^2 x}$

(f) $\ell(x)=\dfrac{(5x^2+1)^7(1-x)^2\sin^2\sqrt{x}}{(3x+1)(x+1)\cot^2(\ln x)}$

Problem 16: Differentiate the following:

(a) x^x

(b) $(2x)^{1-x}$

(c) $(\sin x)^x$

(d) $\left(\ln(1-x^2)\right)^{\sqrt{1-\cos x}}$

(e) $\left(\operatorname{arccsc}^{-1}(5x)\right)^{(2x+1)^4}$

(f) $\left(\cot^2\ln_2 x\right)^{\sec^2(1-x)}$

Problem 17: Prove that the following derivatives are correct:

(a) $\dfrac{d}{dx}\arcsin x=\dfrac{1}{\sqrt{1-x^2}}$

(b) $\dfrac{d}{dx}\arccos x=\dfrac{-1}{\sqrt{1-x^2}}$

(c) $\dfrac{d}{dx}\arctan x=\dfrac{1}{1+x^2}$

(d) $\dfrac{d}{dx}\operatorname{arccsc} x=\dfrac{-1}{|x|\sqrt{x^2-1}}$

(e) $\dfrac{d}{dx}\operatorname{arcsec} x=\dfrac{1}{|x|\sqrt{x^2-1}}$

(f) $\dfrac{d}{dx}\operatorname{arccot} x=\dfrac{-1}{1+x^2}$

Linear Approximation

Problem 18: Find the linearization of $f(x) = x^2$ at $x = 1$. Find the values of x for which this linear approximation estimates $f(x)$ with an error of at most 0.1.

Problem 19: Find the linearization of the given functions $f(x)$ at $x = a$:

(a) $f(x) = \dfrac{x}{\sqrt{x+1}}$, $a = 3$

(b) $f(x) = x\cos x$, $a = \pi$

(c) $f(x) = \sqrt[3]{x}$, $a = -27$

(d) $f(x) = 2x^2 - 3x + 6$, $a = -1$

(e) $f(x) = \dfrac{x+1}{1-x}$, $a = 2$

(f) $f(x) = x^2e^x - e^{x^2} + 4$, $a = 0$

Problem 20: Complete the following:

(a) The radius of a sphere is increased from 12 cm to 12.2 cm, estimate the change in volume.

(b) The dimension of a box are measured with an accuracy of at least 0.5 in. If one measured that the box was 10 in × 5 in × 3 in, what is the approximate uncertainty in the measurement of the volume?

(c) The radius of a disk is measured to be 5 m± 0.1 m. Estimate the maximum error in the approximate error of the disk.

Problem 21: Use linearization to approximate the following:

(a) $\sqrt{51}$

(b) $\ln(1.07)$

(c) $(0.95)^4$

(d) $e^{0.2}$

(e) $\sin(33°)$

(f) $\arctan(1.1)$

Related Rates

Problem 22: If the length of one side of a rectangle is increasing at a rate of 2 cm/s and the other is decreasing at a rate of 2.5 cm/s, what is the rate of change of the area of the rectangle when the increasing side has length 4 cm and the other 6 cm? What is the rate of change of the ratio of the lengths of the sides?

Problem 23: Two cars leave the same destination, at the same time. One car travels north and the other west. If the northbound moves at 57 mph and the westbound car moves at 68 mph, at what rate is the distance between the cars changing 3 hours after they begin their trip?

Problem 24: A spotlight shines on top of a wall 13 m in height. If a person 1.7 m in height walks towards the building at a speed of 1.4 m/s, how quickly is the length their shadow changing when they are 5 m from the wall?

Problem 25: A baseball diamond is a square with side length of 90 ft. A batter hits the ball, and a player on second base begins running towards third base at a pace of 18 ft/s. At what rate is their distance to home plate changing when they are halfway to third base?

Problem 26: If $R(w) = \sqrt{1 + \sin w}$, $w(a) = \dfrac{1-a}{1+a}$, and $a(t) = \tan^{-1} e^t$, find $\dfrac{dR}{dt}$ in terms of w, a, and t.

Problem 27: A weather ballon floating above the Earth travels parallel to the ground at a rate of 15 m/s. A ground station telescope is observing the ballon. At what rate is the angle the telescope makes with the ground changing in order to keep the ballon in view when the ballon is 1000 m from the station, assuming the ballon is 3,000 m above the ground?

Problem 28: A kite is being dragged out by the wind. If the kite is 50 ft above the ground and is moving horizontally at a rate of 3 ft/s, at what rate is the angle between the string and horizontal changing when 100 ft of string has been let out?

Problem 29: A 13 m ladder is being pushed up a wall by its base at a rate of 0.5 m/s. When the ladder is 5 m from the base of the wall, at what rate is the height that the ladder reaches up the wall changing?

Problem 30: A triangular trough is 6 ft long, 3 ft long across at the top, and 2 ft deep. If water is being poured into the trough at a rate of 2 ft^3/min, how fast is the water level rising when the rough is filled to half its depth?

Problem 31: A motion detector sits in the middle of a square room with wall lengths of 30 m, rotating as it scans the room. Assume the detector is 10 m from the center of each wall. If the detector makes 5 scans of the full room each minute, how fast does the scan move along the wall when the detector is pointed at a spot 5 m from the center of a wall?

Problem 32: A clocktower has a giant clock with hand lengths 5 ft and 3 ft, respectively. If a student walks by the clock at 12:19 pm, what would they measure the rate of change of the distance between the tips of the clock hands to be?

Problem 33: A particle is moving along a curve given by $\dfrac{xy^2}{1+x^2} = \dfrac{9}{2x}$. If the particle's y-coordinate is decreasing at a rate of 3 units/s when the particle is at $(-1, 3)$, what is the rate of change of the particle's x-coordinate at this point? Is the particle moving to the left or to the right?

Problem 34: If s is the length of the *diagonal* of a square, at what rate is s changing if the sides are increasing in length at a rate of 2 cm/s?

Problem 35: A storage tank is 20 ft long, and the ends of the tank are isosceles triangles. The tank is 10 ft wide and 7 ft deep. What is the rate of change of the water depth if water is being poured in at 5 ft^3/min and the tank is filled to a depth 4 ft below its top?

Problem 36: If the velocity of a particle is given by $v(T) = \cos^2(e^T)$, where T is the temperature of the particle, and the temperature at time t, $T(t)$, is given by $T(t) = \ln_3(\sec^{-1}(t^2))$, find $\dfrac{dv}{dt}$ in terms of T and t.

Problem 37: A conical tank is being filled with syrup at a rate of 0.1 m^3/min. If the top of the conical tank had radius 2 m and height 8 m, find the rate of the change of the depth the syrup filling the container when the tank is filled to a level 2 m below the top of the tank. Redo this problem if the tank were instead a cylinder with the given dimensions.

Problem 38: You watch a car pass by you on the highway. Let D denote the distance between you and the car. What is $\frac{dD}{dt}$ at the moment when the car passes you?

Problem 39: A mechanic uses a pulley system to drag a car. If the pulley stands on top of a pole 5 m in height and is attached to the car at a height of 1 m above the ground, find the speed of the car, i.e. the rate of change of the distance between the car and pole, if the mechanic pulls the rope at a rate of P m/s.

Problem 40: A particle moves around in a particle accelerator which has the shape of of an ellipse given by $x^2 + 4y^2 = 9$ km. When is $\frac{dx}{dt} > 0$? Explain. When is $\frac{dy}{dt} < 0$? Explain. What is the rate of change of x when the particle passes through the point $(\sqrt{5}, -1)$? Find $\frac{dy}{dt}$ when the particle is at the largest and smallest possible y values.

Problem 41: A particle is traveling around a curve given by $y = f(x)$. Let $\theta(x)$ denote the angle the line connecting the origin and the point $(x, f(x))$ on the curve makes with the x-axis. Find $\theta'(x)$. Redo the problem if instead the line connected the point $(x, f(x))$ with the point $(1, -3)$, assuming the angle is instead measured from the line $y = -3$.

Problem 42: If two cars leave a house at the same time, traveling in straight lines with angle θ between them, at a constant velocity v, find the rate of change of the distance between them. Is the case when $\theta = \pi$ familiar? What would the rate of change be if the cars were traveling at a velocity v_1 and v_2, respectively?

Problem 43: The wheels of a train are driven by a piston system: a wheel of radius r connected at a point P on the edge of the wheel to a piston (steel block) by a rod of length L. The wheel spins causing the rod to drag the piston back and forth along a long rod passing through the center of the piston and the center of the wheel Let x denote the distance from the point where the rod meets the piston and the center of the wheel. Show that

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta.$$

Use this to show that the velocity of the piston moving along the rod is given by

$$\frac{dx}{dt} = -\frac{r^2\theta'\sin(2\theta)}{2(x - r\cos\theta)} - r\theta'\sin\theta,$$

recalling that $\sin(2\theta) = 2\sin\theta\cos\theta$.