Limits

Definition

- $\lim_{x \to a} f(x)$: Means what happens to the values of $f(x)$ when x is 'close' to a. So $\lim_{x \to a} f(x) = L$ means that when x is 'close' to a, the values of $f(x)$ are 'close' to L.

But there are also left/right limits.

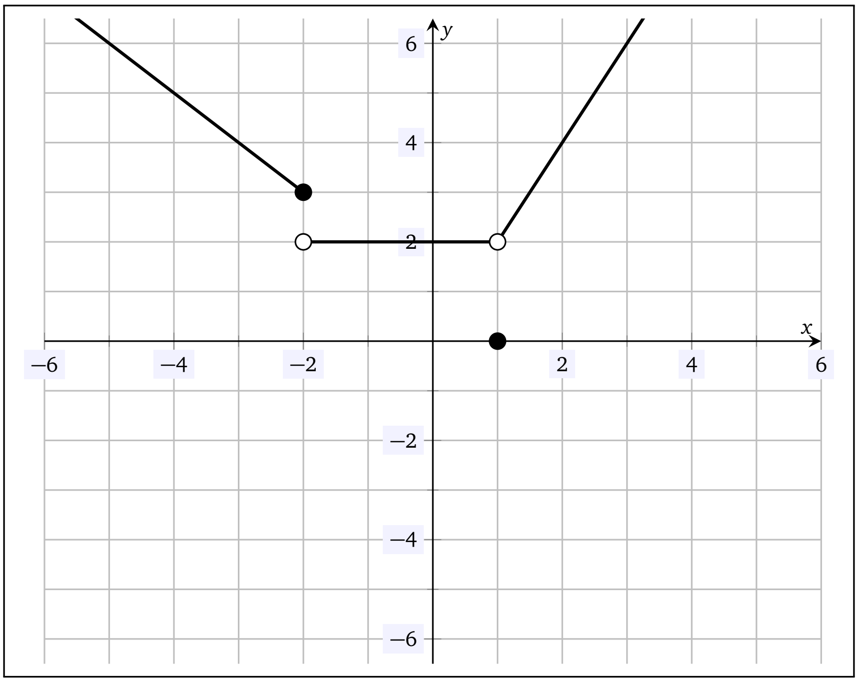
- $\lim_{x \to a^-} f(x)$: Means what happens to the values of $f(x)$ when x is 'close' to a but 'from the left', i.e. x is always less than a. So $\lim_{x \to a^-} f(x) = L$ means that when x is 'close' to a on the left, the values of $f(x)$ are 'close' to L.
- $\lim_{x \to a^+} f(x)$: Means what happens to the values of $f(x)$ when x is 'close' to a but 'from the right', i.e. x is always greater than a. So $\lim_{x \to a^+} f(x) = L$ means that when x is 'close' to a on the right, the values of $f(x)$ are 'close' to L.

Because $\lim_{x \to a} f(x) = L$ means that when x is 'close' to a, no matter x is to the left or right of a, the values of $f(x)$ are 'close' to L, this implies if $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$ and that if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$ then $\lim_{x \to a} f(x) = L$.

Graphical Limits

We can compute limits graphically.

Example. Use the graph of $f(x)$ below to evaluate the following:



(a) $\lim_{x\to -2^-} f(x) = 3$

(b) $\lim_{x\to -2^+} f(x) = 2$

(c) $\lim_{x\to -2} f(x) = \text{D.N.E}$

(d) $f(-2) = 3$

(e) $\lim_{x\to 1^-} f(x) = 2$

(f) $\lim_{x\to 1^+} f(x) = 2$

(g) $\lim_{x\to 1} f(x) = 2$

(h) $f(1) = 0$

Computing Limits

We also compute limits algebraically, using a number of tricks:

- Plug-in: If one can 'plug-in' the limiting value, then the limit is that value (since the functions we give you are often continuous).

  Example. $\lim_{x\to 0} \frac{\sin x}{x+1} = \frac{\sin 0}{0+1} = \frac{0}{1} = 0$

  Example. $\lim_{x\to 1} \frac{x^2+2x+1}{x-3} = \frac{1+2+1}{-2} = \frac{4}{-2} = -2$

  Example. $\lim_{x\to 2^+} \frac{x^2+3}{x-2} = \frac{4+3}{+0} = \infty$

- Expanding/Factoring/Combining: When one obtains $\frac{0}{0}$, often expanding or factoring will cancel the terms causing the 'problem.' [Note: When allowed, one could use l'Hôpital's rule, see that section.]

  Example. $\lim_{x\to 0} \frac{(3+x)^2-9}{x} = \lim_{x\to 0} \frac{(x^2+6x+9)-9}{x} = \lim_{x\to 0} \frac{x^2+6x}{x} = \lim_{x\to 0}(x+6) = 6$

  Example. $\lim_{x\to 1} \frac{x^2+3x-4}{x^2-5x+4} = \lim_{x\to 1} \frac{(x+4)\cancel{(x-1)}}{(x-4)\cancel{(x-1)}} = \lim_{x\to 1} \frac{x+4}{x-4} = -\frac{5}{3}$

  Example. $\lim_{x\to 5} \frac{\frac{1}{5}-\frac{1}{x}}{5-x} = \lim_{x\to 5} \frac{\frac{x-5}{5x}}{5-x} = \lim_{x\to 5} \frac{x-5}{(5-x)(5x)} = \lim_{x\to 5} \frac{\cancel{x-5}}{-\cancel{(x-5)}(5x)} = \lim_{x\to 5} \frac{-1}{5x} = -\frac{1}{25}$

- Conjugate: When doing limits involving roots, often multiplication by 1 using the conjugate is the trick, i.e. change the sign on the radical or constant and multiply by 1 using this form.

  Example. $\lim_{x\to 0} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 0} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}$

  Example. $\lim_{h\to 0} \frac{3-\sqrt{h+9}}{h} \cdot \frac{3+\sqrt{h+9}}{3+\sqrt{h+9}} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(3+\sqrt{h+9})} = \lim_{h\to 0} \frac{-1}{3+\sqrt{h+9}} = -\frac{1}{6}$

- Multiplication by One: In special limits, one can multiply by 1 in a special form (possible using an identity, such as $\sin^2 x + \cos^2 x = 1$) to compute the limit.

Example.

$$\lim_{x\to 0} \frac{\sin^2 x}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{\sin^2(\cos x + 1)}{\cos^2 x - 1} = \lim_{x\to 0} \frac{\cancel{\sin^2 x}(\cos x + 1)}{-\cancel{\sin^2 x}} = \lim_{x\to 0} -(\cos x + 1) = -2$$

- Absolute Value Limits: When computing limits with absolute values, often the trick is writing out the definition of the argument.

Example. $\lim_{x\to 8^-} \frac{|x-8|}{x-8}$

$$\frac{|x-8|}{x-8} = \begin{cases} \frac{x-8}{x-8} = 1, & x > 8 \\ \frac{-(x-8)}{x-8} = -1, & x < 8 \end{cases}$$

Therefore, $\lim_{x\to 8^-} \frac{|x-8|}{x-8} = \lim_{x\to 8^-} (-1) = -1.$

Example. $\lim_{x\to 2} \frac{x-2}{|x-2}$

$$\frac{x-2}{|x-2|} = \begin{cases} \frac{x-2}{x-2} = 1, & x > 2 \\ \frac{-(x-2)}{x-2} = -1, & x < 2 \end{cases}$$

$$\lim_{x\to 2^+} \frac{x-2}{|x-2|} = \lim_{x\to 2^+} 1 = 1$$

$$\lim_{x\to 2^-} \frac{x-2}{|x-2|} = \lim_{x\to 2^-} (-1) = -1$$

Since $\lim_{x\to 2^+} \frac{x-2}{|x-2|} \neq \lim_{x\to 2^-} \frac{x-2}{|x-2|}$, $\lim_{x\to 2} \frac{x-2}{|x-2}$ D.N.E..

- Special Limits: Some limits could involve previously memorized limits along with multiplication by one. The three most common special memorized limits are

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0 \qquad \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$$

Example. $\lim_{x\to 0} \frac{\sin(4x)}{x} = \lim_{x\to 0} \frac{\sin(4x)}{x} \cdot \frac{4}{4} = \lim_{x\to 0} \frac{\sin(4x)}{4x} \cdot 4 = 1 \cdot 4 = 4$

Example. $\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)} \cdot \frac{x}{x} = \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{x}{\sin(5x)} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{5x}{\sin(5x)} \cdot \frac{3}{5} = 1 \cdot 1 \cdot \frac{3}{5} = \frac{3}{5}$

Example.

$$\lim_{x\to 0} \frac{4x}{\tan(6x)} = \lim_{x\to 0} \frac{4x}{\frac{\sin 6x}{\cos 6x}} = \lim_{x\to 0} \frac{4x}{\sin 6x} \cdot \cos 6x = \lim_{x\to 0} \frac{6x}{\sin 6x} \cdot \frac{1}{6} \cdot 4\cos 6x = 1 \cdot \frac{1}{6} \cdot 4 = \frac{2}{3}$$

Example.

$$\lim_{x\to\infty} \left(1 - \frac{2}{x}\right)^{3x} = \lim_{x\to\infty} \left(1 + \frac{1}{\frac{x}{-2}}\right)^{3x} = \lim_{x\to\infty} \left(1 + \frac{1}{\frac{x}{-2}}\right)^{\frac{x}{-2}\cdot(-2)\cdot 3} = \left(\lim_{x\to\infty} \left(1 + \frac{1}{\frac{x}{-2}}\right)^{\frac{x}{-2}}\right)^{-6} = e^{-6}$$

- Squeeze Theorem: If $g(x) \le f(x) \le h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

  Example. Find $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$.

  We know that $-1 \le \sin y \le 1$, but then $-1 \le \sin\left(\frac{1}{x}\right) \le 1$. Then $-x \le x\sin\left(\frac{1}{x}\right) \le x$. Notice that $\lim_{x\to 0}(-x) = \lim_{x\to 0} x = 0$. Therefore by Squeeze Theorem, $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

  Example. If $3 \le f(x) \le x^2 - 2x + 3$ for all x, what is $\lim_{x\to 2} f(x)$?

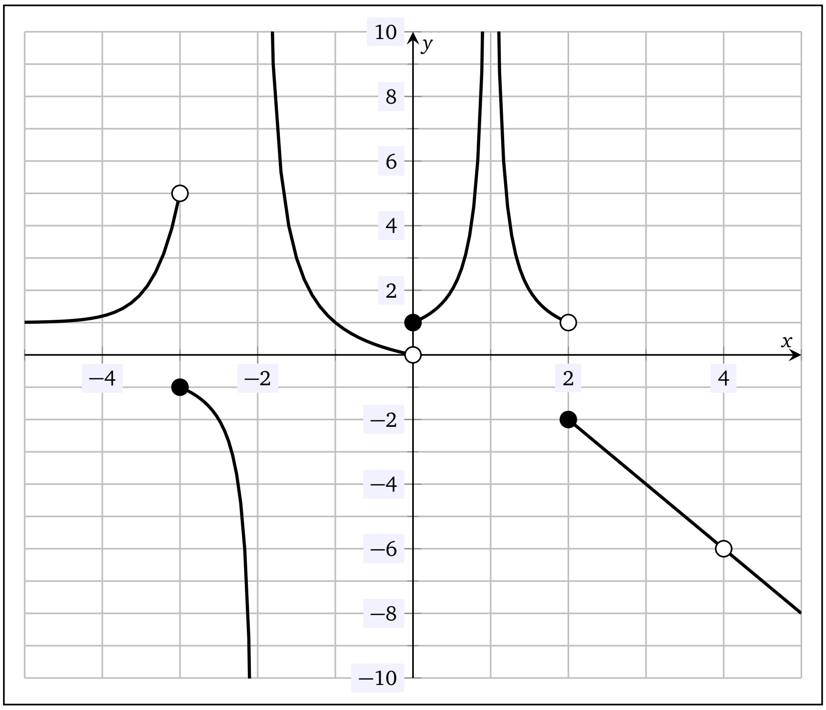
  We know that $\lim_{x\to 2} 3 = 3$ and $\lim_{x\to 2}(x^2 - 2x + 3) = (4 - 4 + 3) = 3$. Therefore by Squeeze Theorem, $\lim_{x\to 2} f(x) = 3$.

Limits with Infinity

- $\lim_{x\to\infty} f(x)$: Means what happens to the values of $f(x)$ when x gets larger and larger. So $\lim_{x\to\infty} f(x) = L$ means that when x is 'very large', the values of $f(x)$ are 'close' to L. Similarly, $\lim_{x\to-\infty} f(x) = L$ means that when x is 'very negative', the values of $f(x)$ are 'close' to L.

Again, we can do this graphically.

Example.



(a) $\lim_{x\to-2^-} f(x) = -\infty$

(b) $\lim_{x\to-2^+} f(x) = \infty$

(c) $\lim_{x\to-2} f(x) =$ D.N.E

(d) $\lim_{x\to 1^-} f(x) = \infty$

(e) $\lim\limits_{x\to 1^+} f(x) = \infty$

(f) $\lim\limits_{x\to 1} f(x) = \infty$

(g) $\lim\limits_{x\to -\infty} = 1$

(h) $\lim\limits_{x\to \infty} f(x) = -\infty$

We also compute these algebraically.

Example. $\lim\limits_{x\to\infty}(x^2+3) = \infty$

Example. $\lim\limits_{x\to-\infty} e^x = \lim\limits_{x\to\infty} e^{-x} = \lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$

There are special tricks with rational limits at infinity, i.e. limits at infinity of polynomials over polynomials, or functions that are 'close' to being rational. For these, you multiply by $\dfrac{1/x^{\text{deg den}}}{1/x^{\text{deg den}}}$. [Note: Often one can use l'Hôpital's with these, but one would have to use it many times. For ones with roots, l'Hôpital's rule will often fail because it 'loops.']

Example. $\lim\limits_{x\to\infty} \dfrac{x^2+2x+1}{3x^2-x+5} \cdot \dfrac{1/x^2}{1/x^2} = \lim\limits_{x\to\infty} \dfrac{1+\dfrac{2}{x}+\dfrac{1}{x^2}}{3-\dfrac{1}{x}+\dfrac{5}{x^2}} = \dfrac{1+0+0}{3-0+0} = \dfrac{1}{3}$

Example. $\lim\limits_{x\to\infty} \dfrac{x^4-2x+1}{-2x^5+3x^4+5} \cdot \dfrac{1/x^5}{1/x^5} = \lim\limits_{x\to\infty} \dfrac{\dfrac{1}{x}-\dfrac{2}{x^4}+\dfrac{1}{x^5}}{-2+\dfrac{3}{x}+\dfrac{5}{x^5}} = \dfrac{0-0+0}{-2+0+0} = 0$

Example.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x+1}{\sqrt{2x+1}} &= \lim_{x\to\infty} \frac{x+1}{\sqrt{2x+1}} \cdot \frac{1/\sqrt{x}}{1/\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x}+1/\sqrt{x}}{\dfrac{\sqrt{2x+1}}{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x}+1/\sqrt{x}}{\sqrt{\dfrac{2x+1}{x}}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x}+1/\sqrt{x}}{2+1/x} \\
&= \infty
\end{aligned}$$

Example.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+2x+2}{\sqrt{x^6+4x^2+1}} &= \lim_{x\to\infty} \frac{x^2+2x+2}{\sqrt{x^6+4x^2+1}} \cdot \frac{1/x^3}{1/x^3} \\
&= \lim_{x\to\infty} \frac{\dfrac{1}{x}+\dfrac{2}{x^2}+\dfrac{2}{x^3}}{\dfrac{\sqrt{x^6+4x^2+1}}{x^3}} \\
&= \lim_{x\to\infty} \frac{1/x+2/x^2+2/x^3}{\sqrt{\dfrac{x^6+4x^2+1}{x^6}}} \\
&= \lim_{x\to\infty} \frac{1/x+2/x^2+2/x^3}{\sqrt{1+4/x^4+1/x^6}} \\
&= \frac{0+0+0}{\sqrt{1+0+0}} \\
&= 0
\end{aligned}$$

These infinite limits are also used to find horizontal asymptotes. If $\lim_{x\to\infty} f(x) = L$, then $f(x)$ has horizontal asymptote $y = L$ and if $\lim_{x\to-\infty} f(x) = L$, then $f(x)$ has horizontal asymptote $y = L$. To find vertical asymptotes, look for where the function has an infinite limit. Usually this is where the denominator is 0—so long as the number is not also zero for otherwise this may be a removable discontinuity.

Example. Find the horizontal and vertical asymptotes of the curve given by $f(x) = \dfrac{x^2-4}{x^2+x-6}$.

For horizontal asymptotes, we look at $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

$$\lim_{x\to\infty} \frac{x^2-4}{x^2+x-6} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1-\dfrac{4}{x^2}}{1+\dfrac{1}{x}-\dfrac{6}{x^2}} = \frac{1-0}{1+0-0} = 1$$

$$\lim_{x\to-\infty} \frac{x^2-4}{x^2+x-6} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1-\dfrac{4}{x^2}}{1+\dfrac{1}{x}-\dfrac{6}{x^2}} = \frac{1-0}{1+0-0} = 1$$

So $y = 1$ is a horizontal asymptote. For vertical asymptotes, we look for values where the denominator is 0 but the numerator is not.

$$\frac{x^2-4}{x^2+x-6} = \frac{\cancel{(x-2)}(x+2)}{(x+3)\cancel{(x-2)}} = \frac{x+2}{x+3}$$

so that we have

$$\begin{aligned}
\lim_{x\to-3^+} \frac{x^2-4}{x^2+x-6} &= \lim_{x\to-3^+} \frac{x+2}{x+3} = \infty \\
\lim_{x\to-3^-} \frac{x^2-4}{x^2+x-6} &= \lim_{x\to-3^-} \frac{x+2}{x+3} = -\infty
\end{aligned}$$

so that $x = -3$ is a vertical asymptote (but $x = 2$ is not).

Continuity

- We say that $f(x)$ is continuous at $x = a$ if $f(a) = \lim_{x\to a} f(x)$. This says two important things (that one has to check):
  - (i) $f(x)$ is defined at a (because we can plug in $x = a$)
  - (ii) $\lim_{x\to a} f(x)$ exists, which means $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist and are equal.
- Constant functions, polynomials, exponential functions, and sin / cos are all continuous. Sums and products of continuous functions are continuous, and so are quotients of continuous functions continuous (wherever they are defined).

Example. Let a be constant, and define

$$f(x) = \begin{cases} x + a, & x \le 0 \\ \dfrac{\sin x}{x}, & x > 0 \end{cases}$$

What values of a make the function $f(x)$ continuous at all x-values?

The function $x + a$ is continuous for all values of a because it is a polynomial. The function $\dfrac{\sin x}{x}$ is continuous for $x > 0$ because it is a quotient of continuous functions and is defined for $x > 0$. So $f(x)$ is continuous for all values of x except for perhaps $x = 0$. But

$$\begin{aligned} f(0) &= 0 + a = a \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \\ \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} (x + a) = 0 + a = a \end{aligned}$$

For $f(x)$ to be continuous at $x = 0$, we need $f(0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$. But then $a = 1 = a$. So we must have $a = 1$.

Intermediate Value Theorem

- Intermediate Value Theorem (IVT): If $f(x)$ is continuous on $[a, b]$ and M is a number between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = M$.
- This can be used to show a function has a root, because if $f(x)$ is continuous and $f(a) > 0$ while $f(b) < 0$ (or vice versa) then $f(x)$ has a zero between $x = a$ and $x = b$.
- One only really needs to know to use this to find roots of a function. Why? If one wants to use IVT to prove that $f(x) = g(x)$ has a solution, you can just use IVT to show that $f(x) - g(x)$ has a root, say $x = c$, since then $f(c) - g(c) = 0$ which means $f(c) = g(c)$.

1. Explain why $f(x)$ is continuous.
2. Find values where $f(x) > 0$ and $f(x) < 0$.
3. Explain how IVT applies.

Example. Show that $x^5 - 2x^3 - 2 = 0$ has a root between $x = 0$ and $x = 2$.

We know that $f(x) = x^5 - 2x^3 - 2$ is continuous because it is a polynomial. Observe that $f(0) = -2 < 0$ and $f(2) = 14 > 0$. Therefore by IVT, there is a $c \in [0,2]$ so that $f(c) = 0$, i.e. $c^5 - 2c^3 - 2 = 0$; that is, $f(x)$ has a root between $x = 0$ and $x = 2$.

Differentiability

- $f'(a)$ gives the rate of change of $f(x)$ at $x = a$. The function $f'(x)$ gives the rate of change of $f(x)$ at various values x. Also, $f'(a)$ is the slope of the tangent line of $f(x)$ at $x = a$.
- If $f'(x) > 0$, then $f(x)$ is increasing; if $f'(x) < 0$, then $f(x)$ is decreasing. But if $f'(x) = 0$, then $f(x)$ is neither increasing nor decreasing.
- Values x where $f'(x) = 0$ are called *critical values* for $f(x)$.
- The second derivative is denoted $f''(x)$ or $\dfrac{d^2 f}{dx^2}$ and is the derivative of the derivative. Higher derivatives are defined and denoted similarly.
- $f''(x)$ gives the concavity of $f(x)$. If $f''(x) > 0$ then the function is concave up, while if $f''(x) < 0$ then $f(x)$ is concave down. If $f''(c) = 0$ and f'' 'changes sign' at $x = c$, then $(c, f(c))$ is a point of inflection for $f(x)$.

Derivative Definition

- The derivative is defined as

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Note that $f'(x)$ can also be denoted as $D_x f$.
- If a function is differentiable, then it is continuous. But if a function is continuous, then it is not necessarily differentiable. For example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$. An extreme example is $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin(\pi k^2 x)}{\pi k^2}$ (the Weierstrass function), which is everywhere continuous but nowhere differentiable.

Example. Use the limit definition of $f'(a)$ to compute $f'(1)$, where $f(x) = x^2 + 3x$.

We know $f(1) = 1 + 3 = 4$ so that

$$\begin{aligned}
f'(1) &:= \lim_{h \to 0} \frac{[(1+h)^2 + 3(1+h)] - 4}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 5h + 4 - 4}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 5h}{h} \\
&= \lim_{h \to 0} (h + 5) = 5
\end{aligned}$$

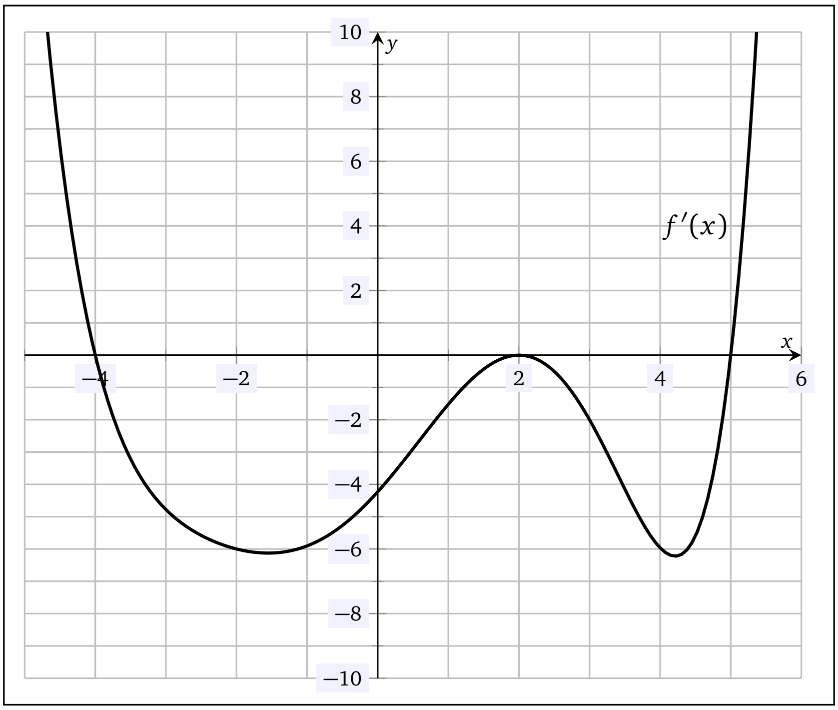
Example. Use the limit definition to find the derivative of $f(x) = x^2 + 3x - 2$.

$$\begin{aligned}
f'(x) &:= \lim_{h \to 0} \frac{[(x+h)^2 + 3(x+h) - 2] - (x^2 + 3x - 2)}{h} \\
&:= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + 3x + 3h - 2 - x^2 - 3x + 2}{h} \\
&:= \lim_{h \to 0} \frac{2hx + h^2 + 3h}{h} \\
&:= \lim_{h \to 0} (2x + h + 3) = 2x + 3
\end{aligned}$$

Graphical Derivatives

You should also know how to use graphical information about $f'(x)$ to make inferences about the function $f(x)$.

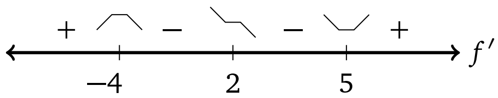
Example. The graph of $f'(x)$ for some function $f(x)$ is plotted in the figure below. Based on this graph, complete the questions below.



(a) Find all the intervals on which $f(x)$ is increasing: $f(x)$ is increasing when $f'(x) > 0$, so $f(x)$ is increasing on $(-\infty, -4) \cup (5, \infty)$.

(b) Find all the intervals on which $f(x)$ is decreasing: $f(x)$ is decreasing when $f'(x) < 0$, so $f(x)$ is decreasing on $(-4, 2) \cup (2, 5)$.

(c) Find all critical points for $f(x)$. Using the First or Second Derivative Test, classify these x-values as locations of maximums, minimums, or neither.



The critical values are where $f'(x) = 0$. We then classify these using the First Derivative Test. Therefore, $x = -4$ is a maxima, $x = 5$ is a minimum, and $x = 2$ is neither.

(d) Find the intervals on which $f(x)$ is concave down (or concave): $f(x)$ is concave down when $f''(x) < 0$, but then $f'(x)$ is decreasing. Therefore, $f(x)$ is concave down on $(-\infty, -1.5) \cup (2, 4.2)$.

(e) Find the intervals on which $f(x)$ is concave up (or convex): $f(x)$ is concave up when $f''(x) > 0$, but then $f'(x)$ is increasing. Therefore, $f(x)$ is concave down on $(-1.5, 2) \cup (4.2, \infty)$

(f) Find the x-values of any points of inflection on $f(x)$: $f(x)$ has a point of inflection when $f''(x) = 0$ and changes sign. But then $f'(x)$ has a maximum or minimum. Therefore, $x = -1.5$, $x = 2$, and $x = 4.2$ are the values.

Derivative Rules & Computing Derivatives

There are a number of derivatives you should have memorized:

(a) $\frac{d}{dx}$ constant $= 0$

(b) $\frac{d}{dx} x^n = nx^{n-1}$

(c) $\frac{d}{dx} \sin x = \cos x$

(d) $\frac{d}{dx} \cos x = -\sin x$

(e) $\frac{d}{dx} \tan x = \sec^2 x$

(f) $\frac{d}{dx} \sec x = \sec x \tan x$

(g) $\frac{d}{dx} \cot x = -\csc^2 x$

(h) $\frac{d}{dx} \csc x = -\csc x \cot x$

(i) $\frac{d}{dx} e^x = e^x$

(j) $\frac{d}{dx} a^x = a^x \ln a$

(k) $\frac{d}{dx} \ln x = \frac{1}{x}$

(l) $\frac{d}{dx} \log_b x = \frac{1}{x \ln b}$

(m) $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

(n) $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

(o) $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$

You combine these with the following rules:

- Product Rule: $\dfrac{d}{dx}\, f(x)g(x) = f'(x)g(x) + f(x)g'(x)$
- Quotient Rule: $\dfrac{d}{dx}\,\dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}$
- Chain Rule: $\dfrac{d}{dx}\, f(g(x)) = f'(g(x))\, g'(x)$

Example. $\dfrac{d}{dx}\, 2^x \cos^2(x) = 2^x \ln 2 \cdot \cos^2(x) + 2^x \cdot 2\cos(x) \cdot (-\sin x)$

Example. $\dfrac{d}{dx}\,\dfrac{e^{2x}}{x^2+1} = \dfrac{(x^2+1)\cdot 2e^{2x} - (2x)\cdot e^{2x}}{(x^2+1)^2}$

Example. $\dfrac{d}{dx}\, x^2 \arctan(3x) = 2x \cdot \arctan(3x) + x^2 \cdot \dfrac{1}{1+(3x)^2} \cdot 3$

Example. $\dfrac{d}{dx}\, \sin^3(\cos(2x)) = 3\sin^2(\cos(2x)) \cdot \cos(\cos(2x)) \cdot (-\sin(2x)) \cdot 2$

Example. $\dfrac{d}{dx}\,\dfrac{1}{\sqrt{x}} = \dfrac{d}{dx}\, x^{-1/2} = -\dfrac{1}{2}\, x^{-3/2}$

Tangent Line

- The tangent line of $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$.

Example. Find the tangent line of $f(x) = x^2 - 4$ at $x = 3$.

We know $f(3) = 5$. So we have the point $(3,5)$. Then $y - 5 = m(x - 3)$. We only need the slope m. But $m = f'(3)$. We have $f'(x) = 2x$ so that $f'(3) = 6$. Therefore, the tangent line is $y - 5 = 6(x - 3)$.

Logarithmic Differentiation

- Suppose we want $f'(x)$. Let $y = f(x)$, then we want $\dfrac{dy}{dx}$. But $\ln y = \ln f(x)$. Using implicit differentiation, we have $\dfrac{1}{y}\,\dfrac{dy}{dx} = (\ln f(x))'$ so that $\dfrac{dy}{dx} = y\,(\ln f(x))'$.

Example. Use logarithmic differentiation to find $f'(x)$ if $f(x) = \dfrac{x^6\sqrt{x+1}}{(x+2)^6(x+3)}$.

Using the properties of $\ln x$, we know that

$$\ln f(x) = 6\ln x + \frac{1}{2}\ln(x+1) - 6\ln(x+2) - \ln(x+3)$$

But then

$$\begin{aligned}
\frac{d}{dx}\ln f(x) &= \frac{d}{dx}\left(6\ln x + \frac{1}{2}\ln(x+1) - 6\ln(x+2) - \ln(x+3)\right)\\
\frac{1}{f(x)}f'(x) &= \frac{6}{x} + \frac{1}{2(x+1)} - \frac{6}{x+2} - \frac{1}{x+3}\\
f'(x) &= f(x)\left(\frac{6}{x} + \frac{1}{2(x+1)} - \frac{6}{x+2} - \frac{1}{x+3}\right)\\
f'(x) &= \frac{x^6\sqrt{x+1}}{(x+2)^6(x+3)}\left(\frac{6}{x} + \frac{1}{2(x+1)} - \frac{6}{x+2} - \frac{1}{x+3}\right)
\end{aligned}$$

Special Derivatives

- To compute derivatives of functions like $f(x)^{g(x)}$, e.g. $x^{\sin x}$, you need logarithmic differentiation.

Example. $\dfrac{d}{dx}x^{\sin x}$

Let $y = x^{\sin x}$. Then

$$\begin{aligned}
y &= x^{\sin x}\\
\ln y &= \ln x^{\sin x}\\
\ln y &= \sin x \ln x\\
\frac{d}{dx}\ \ln y &= \frac{d}{dx}\ \sin x \ln x\\
\frac{1}{y}\ \frac{dy}{dx} &= \cos x \ln x + \frac{\sin x}{x}\\
\frac{dy}{dx} &= y\left(\cos x \ln x + \frac{\sin x}{x}\right)\\
\frac{dy}{dx} &= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)
\end{aligned}$$

One can also memorize that derivatives of $f(x)^{g(x)}$ are like power and exponential rules for derivatives combined, i.e. take their sum:

$$\frac{d}{dx}\,f(x)^{g(x)} = \underbrace{g(x)f(x)^{g(x)-1}\cdot f'(x)}_{\text{`Power Rule'}} + \underbrace{f(x)^{g(x)}\ln f(x)\cdot g'(x)}_{\text{`Exponential Rule'}}$$

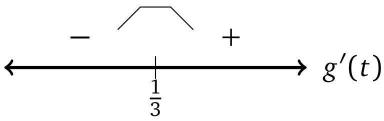
Example. $\dfrac{d}{dx}(\tan x)^{2x} = 2x(\tan x)^{2x-1}\cdot\sec^2 x + (\tan x)^{2x}\ln(\tan x)\cdot 2$

Finding Max/Mins on Intervals

- Find and classify the critical values for $f(x)$. Ignore those that are not in the interval you are considering. Evaluate $f(x)$ at the endpoints. Then compare the values at the endpoints with the max/mins you found before.

Example. Find the absolute maximum and minimum of the function $f(x) = 3t^2 - 2t + 1$ on the interval $[-2, 2]$.

We have $g(t) = 3t^2 - 2t + 1$ and $g'(t) = 6t - 2$. Setting $6t - 2 = 0$, we see the only critical value is $t = \frac{1}{3}$.



so that $t = 1/3$ is a minimum. Now $g(-2) = 17$, $g(2) = 9$, $g(1/3) = 2/3$. Therefore, the absolute minimum is $2/3$ and occurs at $x = 1/3$, and the absolute maximum is 17 and occurs at $x = -2$.

Linear Approximation

- The linearization of $f(x)$ at $x = a$ is simply the tangent line of $f(x)$ at $x = a$, i.e. the linearization $\mathscr{L}(x)$ for $f(x)$ at $x = a$ is $\mathscr{L}(x) := f(a) + f'(a)(x - a)$. This can be used to approximate the values of $f(x)$ 'near' $x = a$.

Example. Find the linearization of $f(x) = \sqrt{x}$ at $x = 82$ and use this to approximate the value of $\sqrt{85}$.

We have $f(x) = \sqrt{x}$ so $f(81) = \sqrt{81} = 9$. We know also $f'(x) = \dfrac{1}{2\sqrt{x}}$ so that $f'(81) = \dfrac{1}{2\sqrt{81}} = \dfrac{1}{18}$. Then we have $\mathscr{L}(x) = f(81) + f'(81)(x - 81) = 9 + \dfrac{1}{18}(x - 81)$.

To approximate $\sqrt{82}$, note that 82 is 'close' to 81. Then we have

$$\sqrt{82} = f(82) \approx \mathscr{L}(82) = 9 + \frac{1}{18}(82 - 81) = 9 + \frac{1}{18} = \frac{163}{18} \approx 9.0556.$$

Implicit Differentiation

- Implicit differentiation works 'the same' as normal differentiation, except when computing things like $\dfrac{d}{dx} y$, one needs to 'tack on' a $\dfrac{dy}{dx}$ term.

Example. Given $y^2 = x^3 + 3x^2 - 24x$, find $\dfrac{dy}{dx}$.

$$\begin{aligned} y^2 &= x^3 + 3x^2 - 24x \\ \frac{d}{dx}\, y^2 &= \frac{d}{dx}\left(x^3 + 3x^2 - 24x\right) \\ 2y\, \frac{dy}{dx} &= 3x^2 + 6x - 24 \\ \frac{dy}{dx} &= \frac{3x^2 + 6x - 24}{2y} \end{aligned}$$

Example. Find $\dfrac{dy}{dx}$ given $\sin(xy) = 4$.

$$\begin{aligned}
\sin(xy) &= 4 \\
\frac{d}{dx}\sin(xy) &= \frac{d}{dx}\,4 \\
\cos(xy)\left(y + x\,\frac{dy}{dx}\right) &= 0 \\
\frac{dy}{dx} &= -\frac{y}{x}
\end{aligned}$$

Example. Find the tangent line to the curve $y^4 = y^2 - x^2$ at the point $(\sqrt{3}/4, 1/2)$.

$$\begin{aligned}
y - \frac{1}{2} &= m\left(x - \frac{\sqrt{3}}{4}\right) \\
\frac{d}{dx}\,y^4 &= \frac{d}{dx}\,(y^2 - x^2) \\
4y^3\,\frac{dy}{dx} &= 2y\,\frac{dy}{dx} - 2x
\end{aligned}$$

But we have $x = \dfrac{\sqrt{3}}{4}, \quad y = \dfrac{1}{2}$ so

$$\begin{aligned}
4\left(\frac{1}{2}\right)^3\frac{dy}{dx} &= 2\left(\frac{1}{2}\right)\frac{dy}{dx} - 2\left(\frac{\sqrt{3}}{4}\right) \\
\frac{1}{2}\,\frac{dy}{dx} &= \frac{dy}{dx} - \frac{\sqrt{3}}{2} \\
\frac{dy}{dx} &= \sqrt{3}
\end{aligned}$$

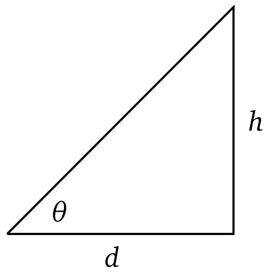
$$y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\sqrt{3}}{4}\right)$$

Related Rates

- This is an application of implicit differentiation.

1. Draw a picture.
2. Write down the desired quantity.
3. Write down all the knowns.
4. Write down an equation, call this (*), which relates the quantities of interest. [Plug in any values which are *always* constant.]
5. Implicit differentiate, plug in the known values, and solve.

Example. When the angle of the camera with the ground is $\frac{\pi}{3}$ and the observer increases this angle at a rate of $\frac{1}{5}$ rad/min to keep the ballon centered in the frame. What rate is the balloon rising at that moment?

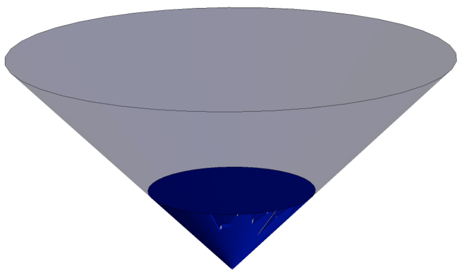


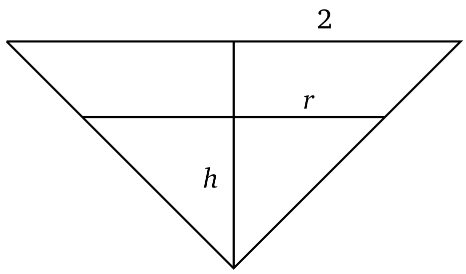
$$\theta = \frac{\pi}{3}$$
$$\frac{d\theta}{dt} = \frac{1}{5}$$
$$d = 200$$
$$\text{Want: } \frac{dh}{dt}$$

$$\tan\theta = \frac{h}{d}$$
$$\tan\theta = \frac{h}{200}$$
$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{h}{200}$$
$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{200}\frac{dh}{dt}$$
$$\frac{dh}{dt} = 200\sec^2\theta\,\frac{d\theta}{dt}$$
$$\frac{dh}{dt} = 200(\sec(\pi/3))^2 \cdot \frac{1}{5}$$
$$\frac{dh}{dt} = 200 \cdot (2)^2 \cdot \frac{1}{5}$$
$$\frac{dh}{dt} = 200 \cdot 4 \cdot \frac{1}{5}$$
$$\frac{dh}{dt} = 160 \text{ ft/min}$$

Example. A conical tank (with the tip at the floor) is 4 ft across at the top and 3 ft tall. If the tank is being filled with water at a rate of $\frac{8}{27}$ ft^3/min, what is the rate of change of the depth of the water when the tank is filled to a depth of 1 ft?

$$V = \frac{1}{3}\pi r^2 h$$





$$\text{Want: } \frac{dh}{dt}$$

$$\frac{r}{h} = \frac{2}{3} \Rightarrow r = \frac{2}{3}h \quad \text{and if } h = \frac{1}{4} \Rightarrow r = \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{6}$$

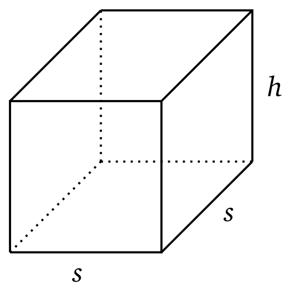
$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h$$
$$V = \frac{4\pi}{27}h^3$$
$$\frac{d}{dt}V = \frac{d}{dt}\frac{4\pi}{27}h^3$$
$$\frac{dV}{dt} = \frac{4\pi}{9}h^2\frac{dh}{dt}$$
$$\frac{8}{27} = \frac{4\pi}{9}\cdot 1^2 \cdot \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{8}{27}\cdot\frac{9}{4\pi}$$
$$\frac{dh}{dt} = \frac{2}{3\pi}\ \text{ft}^3/\text{min}$$

Optimization

- This is an application of finding max/mins (on intervals).

1. Draw a picture.
2. Write down what you want to optimize and any constraint equations.
3. Write down your known values.
4. Use your constraint equation to get your optimization equation into one variable.
5. Maximize or minimize this new equation using derivative methods.

Example. A rectangular box has a square bottom and an open top. If only 2,700 cm^2 of material is available to construct the box, what dimensions maximize the volume of the box? Be sure to draw a picture and justify completely that these dimensions are optimal.



$$\text{Maximize } V = lwh$$
$$\text{Constraint } 2700 = \text{Surface Area} = 4sh + s^2$$

We want to optimize $V = lwh = s \cdot s \cdot h = s^2 h$. But we know that

$$\text{Surface Area} = 2,700\ \text{cm}^2$$
$$\text{Surface Area} = 4sh + s^2$$

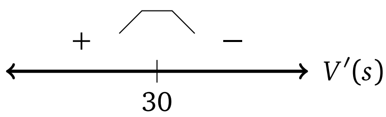
But then we have

$$\begin{aligned} 4sh + s^2 &= 2700 \\ 4sh &= 2700 - s^2 \\ h &= \frac{2700 - s^2}{4s} \end{aligned}$$

But then $V = s^2 h = s^2 \left(\frac{2700 - s^2}{4s} \right) = \frac{2700s - s^3}{4}$. Clearly, $s \in [0, \sqrt{2700}]$, where in each case the box has $V = 0$. We have $V' = \frac{2700 - 3s^2}{4}$. Now setting $V' = 0$, we have

$$\begin{aligned} \frac{2700 - 3s^2}{4} &= 0 \\ 2700 - 3s^2 &= 0 \\ 3s^2 &= 2700 \\ s^2 &= 900 \\ s &= \pm\sqrt{900} = \pm\sqrt{9 \cdot 100} \\ s &= \pm 30 \end{aligned}$$

But then we must have $s = 30$ cm^2. Then $h = \frac{2700 - 900}{120} = \frac{800}{120} = 15$. The dimensions then are $30 \times 30 \times 15$. We confirm this is a maximum,



Mean Value Theorem

- Mean Value Theorem (MVT): If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ so that $f(b) - f(a) = f'(c)(b - a)$.

Example. Verify that $f(x) = x^3 + x - 1$ satisfies the hypotheses of the Mean Value Theorem on $[0, 3]$. Find all numbers c satisfying the conclusions of the Mean Value Theorem on this interval.

Observe $f(x) = x^3 + x - 1$ is continuous on $[0, 3]$. Because $f'(x) = 3x^2 + 1$ is defined on $[0, 3]$, $f(x)$ is differentiable on $(0, 3)$. By the Mean Value Theorem, there exists $c \in (0, 3)$ such that $f(3) - f(0) = f'(c)(3 - 0)$. But then we have

$$\begin{aligned} f(3) - f(0) &= f'(c)(3 - 0) \\ 29 - (-1) &= 3(3c^2 + 1) \\ 30 &= 3(3c^2 + 1) \\ 10 &= 3c^2 + 1 \\ 9 &= 3c^2 \\ c^2 &= 3 \\ c &= \pm\sqrt{3} \end{aligned}$$

Therefore, $c = \sqrt{3}$ satisfies the hypothesis of the Mean Value Theorem on $[0,3]$.

Example. Use the Mean Value Theorem to prove that if $f'(x) = 0$ for all $x \in [a,b]$, then $f(x)$ is constant on $[a,b]$. [Hint: Show $f(x_0) = f(a)$ for all $a \le x_0 \le b$.]

Since $f'(x)$ exists on $[a,b]$, $f(x)$ is differentiable (hence continuous) on $[a,b]$. Therefore, $f(x)$ satisfies the Mean Value Theorem on $[a,x_0]$ for any $x_0 \le b$. Then by the Mean Value Theorem, $f(x_0) - f(a) = f'(c)(x_0 - a)$ for some $c \in [a,x_0]$. But $f'(x) = 0$ for all $x \in [a,b]$. Therefore,

$$\begin{aligned} f(x_0) - f(a) &= f'(c)(x_0 - a) \\ f(x_0) - f(a) &= 0 \\ f(x_0) &= f(a) \end{aligned}$$

This shows $f(x) = f(a)$ for all $x \in [a,b]$. But then $f(x)$ is constant on $[a,b]$.

l'Hôpital's Rule

- l'Hôpital's Rule: If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form, and $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.
- This is used to calculate limits of indeterminate forms, i.e. $\dfrac{0}{0}$, $\pm\dfrac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0. Each one is handled differently.
  - (i) $\dfrac{0}{0}$, $\pm\dfrac{\infty}{\infty}$: Handled 'normally.'
  - (ii) $0 \cdot \infty$: Move either the 0 or ∞ term into the denominator to obtain one of the above forms.
  - (iii) $\infty - \infty$: Combine terms or factor out something to obtain one of the forms above.
  - (iv) $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0: Use logarithms, i.e. set $L = \lim f(x)^{g(x)}$, take logs to obtain $\lim g(x) \ln f(x)$. Then compute this limit, which is now one of the above indeterminate forms above. If this limit is W, then the original limit is e^W.

Example. $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2} \overset{\text{L.H.}}{=} \lim\limits_{x \to 0} \dfrac{e^x - 1}{2x} \overset{\text{L.H.}}{=} \lim\limits_{x \to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

Example.

$$\lim_{x \to \infty} \frac{\ln(1 + e^{6x})}{5x} \overset{\text{L.H.}}{=} \lim_{x \to \infty} \frac{\frac{1}{1 + e^{6x}} \cdot 6e^{6x}}{5} = \lim_{x \to \infty} \frac{6e^{6x}}{5(1 + e^{6x})} \overset{\text{L.H.}}{=} \lim_{x \to \infty} \frac{36e^{6x}}{5(6e^{6x})} = \lim_{x \to \infty} \frac{36}{5(6)} = \frac{6}{5}$$

Example. $\lim\limits_{x \to 0^+} \sqrt[3]{x} \ln x = \lim\limits_{x \to 0^+} \dfrac{\ln x}{x^{-1/3}} \overset{\text{L.H.}}{=} \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{3} x^{-4/3}} = \lim\limits_{x \to 0^+} -\dfrac{3x^{4/3}}{x} = \lim\limits_{x \to 0^+} -3x^{1/3} = 0$

Example. $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{xe^{4x}}\right)=\lim\limits_{x\to 0}\left(\frac{e^{4x}}{xe^{4x}}-\frac{1}{xe^{4x}}\right)=\lim\limits_{x\to 0}\frac{e^{4x}-1}{xe^{4x}}\overset{\text{L.H.}}{=}\lim\limits_{x\to 0}\frac{4e^{4x}}{e^{4x}+4xe^{4x}}=\frac{4}{1+0}=4$

Example. $\lim\limits_{x\to 0}(1+2x)^{3/x}$

Let $L=\lim\limits_{x\to 0}(1+2x)^{3/x}$. Then $\ln L=\lim\limits_{x\to 0}\ln(1+2x)^{3/x}=\lim\limits_{x\to 0}\frac{3}{x}\ln(1+2x)$.

$$\lim_{x\to 0}\frac{3}{x}\ln(1+2x)=\lim_{x\to 0}\frac{3\ln(1+2x)}{x}\overset{\text{L.H.}}{=}\lim_{x\to 0}\frac{3\,\frac{1}{1+2x}\cdot 2}{1}=\lim_{x\to 0}\frac{6}{1+2x}=6.$$

Therefore, $\ln L=6$ so that $L=e^6$.

Riemann Sums

- The definite integral $\int_a^b f(x)\,dx$ finds the (directed) area under the curve $f(x)$ from $x=a$ to $x=b$. Remember, area under the x-axis is negative.
- $\int_a^b f(x)\,dx=\lim\limits_{n\to\infty}\sum f(x_i^*)\Delta x_i$. We can approximate the definite integral using appropriate left/right sums. The values x_i^* depend on whether one is using the left/right values but $\Delta x_i=\frac{b-a}{n}$, where n is the number of rectangles used.

Example. Use right Riemann sums (with 3 subintervals) to approximate $\int_1^{10} 2x\,dx$.

We have $\Delta x_i=\frac{10-1}{3}=3$. Then we have values $\{1,4,7,10\}$. We are using the right end points, so we use $4,7,10$. Then we have

$$\int_1^{10} 2x\,dx\approx 3(f(4)+f(7)+f(1))=3(8+14+20)=3(42)=126.$$

Fundamental Theorem of Calculus

- First Fundamental Theorem of Calculus: If f is continuous on $[a,b]$ and $F:=\int_a^x f(t)\,dt$, then F is continuous on $[a,b]$, differentiable on (a,b), and $F'(x)=f(x)$.
- Second Fundamental Theorem of Calculus: If F is an antiderivative of f in $[a,b]$, i.e. $F'=f$ on $[a,b]$, then $\int_a^b f(x)\,dx=F(b)-F(a)$.
- The First Theorem is mainly for special differentiation problems while the Second Theorem is how we compute integrals in practice.
- For indefinite integrals, ***DO NOT FORGET*** $+$**C** !!!!

- For all intents and purposes, antiderivative = integral.

Example. $\dfrac{d}{dx}\displaystyle\int_2^{2x} \sin(t^2)\,dt = \sin[(2x)^2]\cdot 2$

Example. $\dfrac{d}{dx}\displaystyle\int_{x^{1/2}}^{1} t(2+t^2)^{1/2}\,dt = -\dfrac{d}{dx}\displaystyle\int_1^{x^{1/2}} t(2+t^2)^{1/2}\,dt = -x^{1/2}(2+(x^{1/2})^2)\cdot\dfrac{1}{2\sqrt{x}}$

Example.

$$\begin{aligned}\frac{d}{dx}\int_{2x}^{3x}\sin^2 t\;dt &= \frac{d}{dx}\int_{2x}^{0}\sin^2 t\;dt + \frac{d}{dx}\int_0^{3x}\sin^2 t\;dt\\ &= -\frac{d}{dx}\int_0^{2x}\sin^2 t\;dt + \frac{d}{dx}\int_0^{3x}\sin^2 t\;dt\\ &= -\sin^2(2x)\cdot 2 + \sin^2(3x)\cdot 3\end{aligned}$$

Example. $\displaystyle\int_0^1 (2x+1)\,dx = \left(\frac{2x^2}{2}+x\right)\Bigg|_0^1 = \left(x^2+x\right)\Bigg|_0^1 = (1+1)-(0+0) = 1-0 = 1$

Example. If $f'(x) = 2x+1$ and $f(1) = 3$, find $f(x)$.

$$f(x) = \int f'(x)\,dx = \int (2x+1)\,dx = \frac{2x^2}{2} + x + C = x^2 + x + C.$$

But $1 = f(1) = 1+1+C$ so that $C = -1$. Therefore, $f(x) = x^2 + x - 1$.

u-Substitution

- Used to compute integrals you could not otherwise compute. Specifically, it can compute integrals where there is a function and its derivative in the integrand.
- When compute indefinite integrals using u-sub, do not forget to go back to the original variable at the end.
- When computing definite integrals using u-sub, *do not forget to change the limits of integration*!

Example. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}$ so that $dx = 2\sqrt{x}\,du$. Then

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\cancel{\sqrt{x}}}\,2\cancel{\sqrt{x}}\,du = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$$

Example. $\displaystyle\int_0^{\pi^{1/2}} x^2 \sin x^3 \, dx$

Let $u = x^3$, then $du = 3x^2\, dx$ so that $dx = \dfrac{du}{3x^2}$. If $x = \pi^{1/2}$, then $u = \pi$. If $x = 0$, then $u = 0$. Then

$$\int_0^{\pi^{1/2}} x^2 \sin x^3 \, dx = \int_0^{\pi} \cancel{x^2} \sin u \, \frac{du}{3\cancel{x^2}} = \frac{1}{3}\int_0^{\pi} \sin u \, du = \frac{1}{3} \cdot -\cos u \Big|_0^{\pi} = -\frac{1}{3}(\cos \pi - \cos 0) = -\frac{1}{3}(-1-1) = \frac{2}{3}$$