EXTENDING RATIONAL LIMIT EXAMPLES

The trick of finding limits that 'looked like' $\lim\limits_{x\to\pm\infty} \frac{\text{polynomial}}{\text{polynomial}}$ was to multiply by $\frac{1/x^{\text{deg den}}}{1/x^{\text{deg den}}}$. We can do the same thing when the limit is 'close' to being a rational function, but we 'distribute' the root to the power of the degree of the denominator. So instead things like $\frac{1/x^2}{1/x^2}$, the term we multiply by may look like $\frac{1/x^{4/2}}{1/x^{4/2}} = \frac{1/x^2}{1/x^2}$ or $\frac{1/x^{3/2}}{1/x^{3/2}}$. We may have to do some extra algebra to pull one of these x terms into a root, being careful of signs.

Example.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x+1}{\sqrt{2x+1}} &= \lim_{x\to\infty} \frac{x+1}{\sqrt{2x+1}} \cdot \frac{1/\sqrt{x}}{1/\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x} + 1/\sqrt{x}}{\frac{\sqrt{2x+1}}{\sqrt{x}}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x} + 1/\sqrt{x}}{\sqrt{\frac{2x+1}{x}}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x} + 1/\sqrt{x}}{2 + 1/x} \\
&= \infty
\end{aligned}$$

Example.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{3x^2+x-5}{\sqrt{2x^4-5}} &= \lim_{x\to\infty} \frac{3x^2+x-5}{\sqrt{2x^4-5}} \cdot \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to-\infty} \frac{3 + \frac{1}{x} - \frac{5}{x^2}}{\frac{\sqrt{2x^4-5}}{x^2}} \\
&= \lim_{x\to-\infty} \frac{3 + \frac{1}{x} - \frac{5}{x^2}}{\sqrt{\frac{2x^4-5}{x^4}}} \\
&= \lim_{x\to-\infty} \frac{3 + \frac{1}{x} - \frac{5}{x^2}}{\sqrt{2 - \frac{5}{x^4}}} \\
&= \frac{3+0-0}{\sqrt{2-0}} \\
&= \frac{3}{\sqrt{2}}
\end{aligned}$$

Example.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+2x+2}{\sqrt{x^6+4x^2+1}} &= \lim_{x\to\infty} \frac{x^2+2x+2}{\sqrt{x^6+4x^2+1}} \cdot \frac{1/x^3}{1/x^3} \\
&= \lim_{x\to\infty} \frac{\dfrac{1}{x}+\dfrac{2}{x^2}+\dfrac{2}{x^3}}{\dfrac{\sqrt{x^6+4x^2+1}}{x^3}} \\
&= \lim_{x\to\infty} \frac{1/x+2/x^2+2/x^3}{\sqrt{\dfrac{x^6+4x^2+1}{x^6}}} \\
&= \lim_{x\to\infty} \frac{1/x+2/x^2+2/x^3}{\sqrt{1+4/x^4+1/x^6}} \\
&= \frac{0+0+0}{\sqrt{1+0+0}} \\
&= 0
\end{aligned}$$

Example.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+1}} &= \lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+1}} \cdot \frac{1/x}{1/x} \\
&= \lim_{x\to-\infty} \frac{1+1/x}{\dfrac{\sqrt{x^2+1}}{x}} \\
&= \lim_{x\to-\infty} \frac{1+1/x}{-\sqrt{\dfrac{x^2+1}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{1+1/x}{-\sqrt{1+1/x^2}} \\
&= \frac{1+0}{-\sqrt{1+0}} \\
&= -1
\end{aligned}$$

Example.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x+1}{\sqrt{x^3+6}} &= \lim_{x\to\infty} \frac{x+1}{\sqrt{x^3+6}} \cdot \frac{1/x^{3/2}}{1/x^{3/2}} \\
&= \lim_{x\to\infty} \frac{\dfrac{x}{x^{3/2}} + \dfrac{1}{x^{3/2}}}{\dfrac{\sqrt{x^3+6}}{x^{3/2}}} \\
&= \lim_{x\to\infty} \frac{1/x^{1/2} + 1/x^{3/2}}{\sqrt{\dfrac{x^3+6}{x^3}}} \\
&= \lim_{x\to\infty} \frac{1/\sqrt{x} + 1/x^{3/2}}{\sqrt{1+6/x^3}} \\
&= \frac{0+0}{\sqrt{1+0}} \\
&= 0
\end{aligned}$$

Example.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x+4}{\sqrt[3]{x^6+6}} &= \lim_{x\to\infty} \frac{x+4}{\sqrt[3]{x^6+6}} \cdot \frac{1/x^{6/3}}{1/x^{6/3}} \\
&= \lim_{x\to\infty} \frac{\dfrac{x}{x^3} + \dfrac{4}{x^3}}{\dfrac{\sqrt[3]{x^6+6}}{x^{6/3}}} \\
&= \lim_{x\to\infty} \frac{1/x^2 + 4/x^3}{\sqrt[3]{\dfrac{x^6+6}{x^6}}} \\
&= \lim_{x\to\infty} \frac{1/x^2 + 4/x^3}{\sqrt[3]{1+6/x^6}} \\
&= \frac{0+0}{\sqrt[3]{1+0}} \\
&= 0
\end{aligned}$$

Example.

$$
\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{x^4+3}}{2x^2+1} &= \lim_{x\to-\infty}\frac{\sqrt{x^4+3}}{2x^2+1}\cdot\frac{1/x^2}{1/x^2}\\
&= \lim_{x\to-\infty}\frac{\dfrac{\sqrt{x^4+3}}{x^2}}{\dfrac{2x^2+1}{x^2}}\\
&= \lim_{x\to-\infty}\frac{\sqrt{\dfrac{x^4+3}{x^4}}}{2+1/x^2}\\
&= \lim_{x\to-\infty}\frac{\sqrt{1+3/x^4}}{2+1/x^2}\\
&= \frac{\sqrt{1+0}}{2+0}\\
&= \frac{1}{2}
\end{aligned}
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