Math 296: Exam 1
Spring – 2018
02/09/2018
80 Minutes

Name: *Caleb M^c Whorter — Solutions*

Write your name on the appropriate line on the exam cover sheet. This exam contains 11 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	25	
8	15	
Total:	100	

1. (10 points) Integrate the following:

(a) $\displaystyle\int \sin^2\theta\, d\theta$

$$\int \sin^2\theta\, d\theta = \int \left(\frac{1-\cos(2\theta)}{2}\right) d\theta = \frac{1}{2}\int (1-\cos(2\theta))\, d\theta = \frac{1}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right] + C$$
$$= \boxed{\frac{\theta}{2} - \frac{\sin(2\theta)}{4} + C}$$
$$= \frac{2\theta - \sin(2\theta)}{4} + C$$

(b) $\displaystyle\int \frac{x+2}{3x^2+4}\, dx$

Note that $\displaystyle\int \frac{x+2}{3x^2+4}\, dx = \int \left(\frac{x}{3x^2+4} + \frac{2}{3x^2+4}\right) dx$. *We integrate the two pieces separately.*
For the first integral, let $u = 3x^2+4$. *Then* $du = 6x\, dx \iff dx = \frac{du}{6x}$.

$$\int \frac{x}{3x^2+4}\, dx = \int \frac{\cancel{x}}{u}\cdot\frac{du}{6\cancel{x}} = \frac{1}{6}\int \frac{du}{u} = \frac{\ln|u|}{6} + C = \frac{\ln|3x^2+4|}{6} + C = \ln\sqrt[6]{3x^2+4} + C$$

For the second integral,

$$\int \frac{2}{3x^2+4}\, dx = \int \frac{2}{3x^2+4}\cdot\frac{1/4}{1/4}\, dx = \frac{1}{2}\int \frac{dx}{\frac{3}{4}x^2+1} = \frac{1}{2}\int \frac{dx}{\left(\frac{\sqrt{3}x}{2}\right)^2+1}$$

Let $u = \frac{\sqrt{3}\,x}{2}$. *Then* $du = \frac{\sqrt{3}}{2}\, dx \iff dx = \frac{2}{\sqrt{3}}\, du$.

$$\int \frac{2}{3x^2+4}\, dx = \frac{1}{2}\int \frac{dx}{\left(\frac{\sqrt{3}x}{2}\right)^2+1} = \frac{\cancel{2}}{\sqrt{3}}\cdot\frac{1}{\cancel{2}}\int \frac{du}{u^2+1} = \frac{1}{\sqrt{3}}\tan^{-1} u + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}\,x}{2}\right) + C$$

Therefore,

$$\int \frac{x+2}{3x^2+4}\, dx = \boxed{\frac{\ln|3x^2+4|}{6} + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}\,x}{2}\right) + C}$$

2. (10 points) Evaluate the following:

$$\int_0^{\pi/2} \sin^3\theta \ \cos^2\theta \ d\theta$$

We know that $\sin^2\theta + \cos^2\theta = 1$ *so that* $\sin^2\theta = 1 - \cos^2\theta$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^3\theta \ \cos^2\theta \ d\theta &= \int_0^{\pi/2} \sin^2\theta \ \cos^2\theta \cdot \sin\theta \ d\theta \\
&= \int_0^{\pi/2} (1 - \cos^2\theta) \ \cos^2\theta \cdot \sin\theta \ d\theta \\
&= -\int_0^{\pi/2} (1 - \cos^2\theta) \ \cos^2\theta \cdot (-\sin\theta) \ d\theta
\end{aligned}$$

Let $u = \cos\theta$. *Then* $du = -\sin\theta \ d\theta$. *If* $\theta = 0$, *then* $u = \cos 0 = 1$; *if* $\theta = \pi/2$, *then* $u = \cos(\pi/2) = 0$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^3\theta \ \cos^2\theta \ d\theta &= -\int_0^{\pi/2} (1 - \cos^2\theta) \ \cos^2\theta \cdot (-\sin\theta) \ d\theta \\
&= -\int_1^0 (1 - u^2)u^2 \ du \\
&= \int_0^1 (1 - u^2)u^2 \ du \\
&= \int_0^1 (u^2 - u^4) \ du \\
&= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1 \\
&= \left(\frac{1}{3} - \frac{1}{5}\right) - (0 - 0) \\
&= \frac{1}{3} - \frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \boxed{\frac{2}{15}}
\end{aligned}$$

3. (10 points) Integrate the following:

$$\int \cos^3\theta \, \ln(\sin\theta)\, d\theta$$

$$\begin{aligned}\int \cos^3\theta \, \ln(\sin\theta)\, d\theta &= \int \cos^2\theta \, \ln(\sin\theta)\cdot\cos\theta\, d\theta \\ &= \int (1-\sin^2\theta)\, \ln(\sin\theta)\cdot\cos\theta\, d\theta\end{aligned}$$

Let $u = \sin\theta$. *Then* $du = \cos\theta\, d\theta$.

$$\int \cos^3\theta \, \ln(\sin\theta)\, d\theta = \int (1-\sin^2\theta)\, \ln(\sin\theta)\cdot\cos\theta\, d\theta = \int (1-u^2)\ln(u)\, du$$

$\ln u$	$u - \frac{u^3}{3}$
$\frac{1}{u}$	$1-u^2$

$$\begin{aligned}\int (1-\sin^2\theta)\, \ln(\sin\theta)\cdot\cos\theta\, d\theta &= \int (1-u^2)\ln(u)\, du \\ &= \ln u\left(u-\frac{u^3}{3}\right) - \int\left(1-\frac{u^2}{3}\right)du \\ &= \ln u\left(u-\frac{u^3}{3}\right) - \left(u-\frac{u^3}{9}\right) + C \\ &= \boxed{\ln(\sin\theta)\left(\sin\theta-\frac{\sin^3\theta}{3}\right) - \sin\theta + \frac{\sin^3\theta}{9} + C}\end{aligned}$$

4. (10 points) Integrate the following:

$$\int x^3 \sin(2x)\; dx$$

u	sign	dv
x^3	$+$	$\sin(2x)$
$3x^2$	$-$	$-\dfrac{\cos(2x)}{2}$
$6x$	$+$	$-\dfrac{\sin(2x)}{4}$
6	$-$	$\dfrac{\cos(2x)}{8}$
0		$\dfrac{\sin(2x)}{16}$

$$\int x^3 \sin(2x)\; dx = \boxed{-\frac{1}{2}x^3\cos(2x) + \frac{3}{4}x^2\sin(2x) + \frac{6}{8}x\cos(2x) - \frac{6}{16}\sin(2x) + C}$$

$$= -\frac{1}{2}x^3\cos(2x) + \frac{3}{4}x^2\sin(2x) + \frac{3}{4}x\cos(2x) - \frac{3}{8}\sin(2x) + C$$

5. (10 points) Integrate the following:

$$\int \tan^3\theta \sec^6\theta \, d\theta$$

Note that $\sin^2\theta + \cos^2\theta = 1$ *so dividing by* $\cos^2\theta$*, we obtain* $\tan^2\theta + 1 = \sec^2\theta \iff \tan^2\theta = \sec^2\theta - 1$.

$$\int \tan^3\theta \sec^6\theta \, d\theta = \int \tan^3\theta \ \sec^4\theta \cdot \sec^2\theta \, d\theta = \int \tan^3\theta \, (\sec^2\theta)^2 \cdot \sec^2\theta \, d\theta$$
$$= \int \tan^3\theta(\tan^2\theta + 1)^2 \cdot \sec^2\theta \, d\theta$$

Let $u = \tan\theta$. *Then* $du = \sec^2\theta \, d\theta$.

$$\int \tan^3\theta(\tan^2\theta + 1)^2 \cdot \sec^2\theta \, d\theta = \int u^3(u^2+1)^2 \, du = \int u^3(u^4 + 2u^2 + 1) \, du$$
$$= \int (u^7 + 2u^5 + u^3) \, du$$
$$= \frac{u^8}{8} + \frac{2u^6}{6} + \frac{u^4}{4} + C$$
$$= \boxed{\frac{\tan^8\theta}{8} + \frac{\tan^6\theta}{3} + \frac{\tan^4\theta}{4} + C}$$

OR

$$\int \tan^3\theta \sec^6\theta \, d\theta = \int \tan^2\theta \ \sec^5\theta \cdot \sec\theta \ \tan\theta \, d\theta = \int (\sec^2\theta - 1) \sec^5\theta \cdot \sec\theta \ \tan\theta \, d\theta$$

Let $u = \sec\theta$. *Then* $du = \sec\theta \tan\theta \, d\theta$.

$$\int (\sec^2\theta - 1)\sec^5\theta \cdot \sec\theta \ \tan\theta \, d\theta = \int (u^2 - 1)u^5 \, du = \int (u^7 - u^5) \, du$$
$$= \frac{u^8}{8} - \frac{u^6}{6} + C$$
$$= \boxed{\frac{\sec^8\theta}{8} - \frac{\sec^6\theta}{6} + C}$$

6. (10 points) Integrate the following:

$$\int e^x \cos\left(\frac{x}{3}\right)\, dx$$

u		dv
$\cos\left(\frac{x}{3}\right)$	$+$	e^x
$-\frac{1}{3}\sin\left(\frac{x}{3}\right)$	$-$	e^x
$-\frac{1}{9}\cos\left(\frac{x}{3}\right)$	$+$	e^x

$$\int e^x \cos\left(\frac{x}{3}\right)\, dx = e^x\cos\left(\frac{x}{3}\right) + \frac{1}{3}e^x\sin\left(\frac{x}{3}\right) + \int -\frac{1}{9}e^x\cos\left(\frac{x}{3}\right)\, dx$$

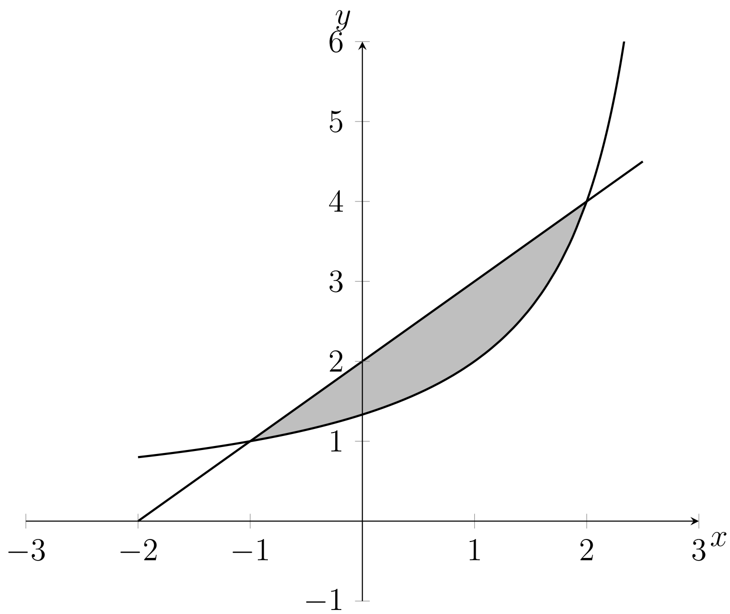
$$\int e^x \cos\left(\frac{x}{3}\right)\, dx = e^x\cos\left(\frac{x}{3}\right) + \frac{1}{3}e^x\sin\left(\frac{x}{3}\right) - \frac{1}{9}\int e^x\cos\left(\frac{x}{3}\right)\, dx$$

$$\frac{10}{9}\int e^x \cos\left(\frac{x}{3}\right)\, dx = e^x\cos\left(\frac{x}{3}\right) + \frac{1}{3}e^x\sin\left(\frac{x}{3}\right)$$

$$\int e^x \cos\left(\frac{x}{3}\right)\, dx = \boxed{\frac{9}{10}\left(e^x\cos\left(\frac{x}{3}\right) + \frac{1}{3}e^x\sin\left(\frac{x}{3}\right)\right) + C}$$

$$\int e^x \cos\left(\frac{x}{3}\right)\, dx = \frac{3e^x}{10}\left(3\cos\left(\frac{x}{3}\right) + \sin\left(\frac{x}{3}\right)\right) + C$$

7. (25 points) Throughout this problem, let $f(x) := x + 2$ and $g(x) := \dfrac{4}{3-x}$. Both $f(x)$ and $g(x)$ are plotted below.



(a) Set up *but do not evaluate* an integral to compute the area bound by $f(x)$ and $g(x)$.

$$\boxed{\int_{-1}^{2} (x+2) - \frac{4}{3-x}\, dx}$$

(b) Set up as completely as possible *both* the integral for the Dish/Washer Method and the integral for the Shell method used to calculate the volume of solid formed by rotating the region bound by $f(x)$ and $g(x)$ about the line $x = -2$. Be sure to label each integral. ***Do not evaluate these integrals.***

Note: $y = x + 2 \iff x = y - 2$ *and* $y = \dfrac{4}{3-x} \iff x = 3 - \dfrac{4}{y} = \dfrac{3y-4}{y}$.

Disks: $$\pi \int_1^4 \left(\frac{3y-4}{y} - (-2)\right)^2 - ((y-2) - (-2))^2 \, dy$$

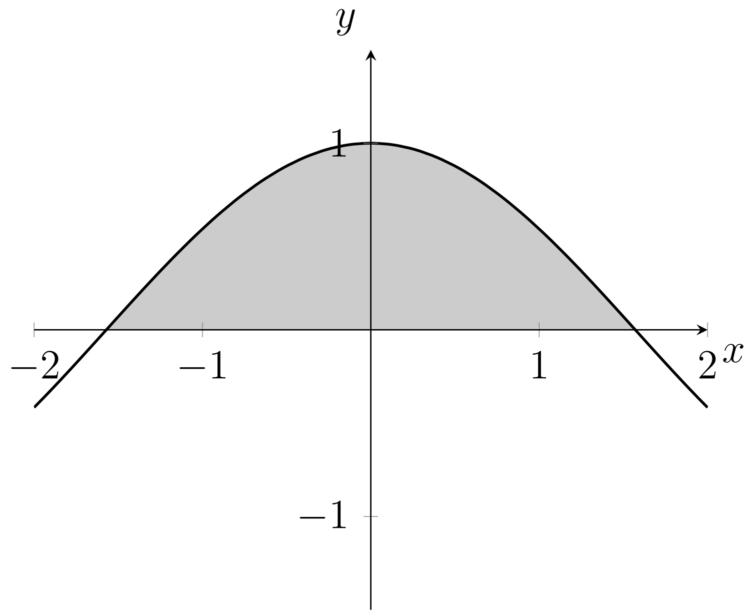
Shells: $$2\pi \int_{-1}^2 (x - (-2)) \left((x+2) - \frac{4}{3-x}\right) \, dx$$

(c) Set up as completely as possible *both* the integral for the Dish/Washer Method and the integral for the Shell method used to calculate the volume of solid formed by rotating the region bound by $f(x)$ and $g(x)$ about the line $y = -1$. Be sure to label each integral. ***Do not evaluate these integrals.***

Disks: $$\pi \int_{-1}^2 (1 + (x+2))^2 - \left(1 + \frac{4}{3-x}\right)^2 \, dx$$

Shells: $$2\pi \int_1^4 (y+1) \left(\frac{3y-4}{y} - (y-2)\right) \, dy$$

8. (15 points) Let R be the region bound by the curve $y = \cos x$ and the x-axis; the graph of R is given below.



If S is a solid whose base is the region R, set up *but do not evaluate* integrals to calculate the volume of S if...

(a) Slices perpendicular to the x-axis are squares.

$$\textit{Area} = \int A(x)\ dx = \int_{-\pi/2}^{\pi/2} s(x)^2\ dx = \boxed{\int_{-\pi/2}^{\pi/2} \cos^2(x)\ dx}$$

(b) Slices perpendicular to the x-axis are semicircles.

$$\textit{Area} = \int A(x)\,dx = \int_{-\pi/2}^{\pi/2} \frac{\pi r(x)^2}{2}\,dx = \frac{\pi}{2}\int_{-\pi/2}^{\pi/2}\left(\frac{\textit{diam}(x)}{2}\right)^2 dx = \boxed{\frac{\pi}{8}\int_{-\pi/2}^{\pi/2}\cos^2(x)\,dx}$$

(c) Slices perpendicular to the x-axis are $30^\circ - 60^\circ - 90^\circ$ triangles with the shortest leg lying in the region R. [Recall in a $30^\circ - 60^\circ - 90^\circ$ triangle, the sides are in ratio $1\colon \sqrt{3}\colon 2$.]

$$\begin{aligned}\textit{Area} &= \int A(x)\,dx = \int_{-\pi/2}^{\pi/2}\frac{1}{2}\,b(x)\,h(x)\,dx \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} b(x)\cdot\sqrt{3}\,b(x)\,dx \\ &= \frac{\sqrt{3}}{2}\int_{-\pi/2}^{\pi/2} b(x)^2\,dx \\ &= \boxed{\frac{\sqrt{3}}{2}\int_{-\pi/2}^{\pi/2}\cos^2(x)\,dx}\end{aligned}$$