Math 296: Exam 2
Spring – 2018
03/07/2018
80 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 9 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Integrate the following:

$$\int \frac{2x^2+7}{(x-2)(x+3)^2}\,dx$$

$$\begin{aligned}\frac{2x^2+7}{(x-2)(x+3)^2} &= \frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}\\ &= \frac{A(x+3)^2+B(x-2)(x+3)+C(x-2)}{(x-2)(x+3)^2}\\ &= \frac{Ax^2+6Ax+9A+Bx^2+Bx-6B+Cx-2C}{(x-2)(x+3)^2}\\ &= \frac{(A+B)x^2+(6A+B+C)x+(9A-6B-2C)}{(x-2)(x+3)^2}\end{aligned}$$

This gives the following system of equations:

$$\begin{aligned}A+B&=2\\ 6A+B+C&=0\\ 9A-6B-2C&=7\end{aligned}$$

which has solution $A=\frac{3}{5}$, $B=\frac{7}{5}$, $C=-5$. *Note we can find* A *and* C *using Heaviside's*

$$\begin{aligned}A&=\frac{2(2^2)+7}{(2+3)^2}=\frac{15}{25}=\frac{3}{5}\\ C&=\frac{2(-3)^2+7}{-3-2}=\frac{25}{-5}=-5\end{aligned}$$

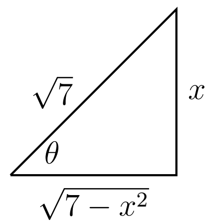
In either case, we obtain...

$$\begin{aligned}\int \frac{2x^2+7}{(x-2)(x+3)^2}\,dx &= \int \frac{3/5}{x-2}+\frac{7/5}{x+3}+\frac{-5}{(x+3)^2}\,dx\\ &= \boxed{\frac{3}{5}\ln|x-2|+\frac{7}{5}\ln|x+3|+\frac{5}{x+3}+K}\end{aligned}$$

2. (10 points) Integrate the following:

$$\int \sqrt{7-x^2}\,dx$$

$$a^2+b^2=c^2$$
$$b^2=\underbrace{c^2-a^2}_{7-x^2}$$



$$\sin\theta=\frac{x}{\sqrt{7}}$$
$$x=\sqrt{7}\sin\theta$$
$$dx=\sqrt{7}\cos\theta\,d\theta$$

$$\cos\theta=\frac{\sqrt{1-x^2}}{\sqrt{7}}$$
$$\sqrt{1-x^2}=\sqrt{7}\cos\theta$$

$$\begin{aligned}
\int \sqrt{7-x^2}\,dx &= \int \sqrt{7}\cos\theta\cdot\sqrt{7}\cos\theta\,d\theta\\
&= 7\int \cos^2\theta\,d\theta\\
&= 7\int \frac{1+\cos 2\theta}{2}\,d\theta\\
&= \frac{7}{2}\int (1+\cos 2\theta)\,d\theta\\
&= \frac{7}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+C\\
&= \frac{7}{2}\left[\theta+\frac{2\sin\theta\cos\theta}{2}\right]+C\\
&= \frac{7}{2}\left[\theta+\sin\theta\cos\theta\right]+C\\
&= \frac{7}{2}\left[\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)+\frac{x}{\sqrt{7}}\cdot\frac{\sqrt{7-x^2}}{\sqrt{7}}\right]+C\\
&= \boxed{\frac{7}{2}\left[\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)+\frac{x\sqrt{7-x^2}}{7}\right]+C}\\
&= \frac{7}{2}\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)+\frac{x\sqrt{7-x^2}}{2}+C
\end{aligned}$$

3. (10 points) Determine whether the following integral converges or diverges. If it converges, find its value.

$$\int_1^\infty \frac{dx}{5x^2 + x}$$

$$\int \frac{dx}{5x^2 + x} = \int \frac{dx}{x(5x+1)}$$

First, we find the partial fraction decomposition of the integrand.

$$\frac{1}{x(5x+1)} = \frac{A}{x} + \frac{B}{5x+1}$$

Heaviside's method yields:

$$A = \frac{1}{5(0)+1} = 1; \qquad B = \frac{1}{-1/5} = -5$$

$$\int \frac{dx}{x(5x+1)} = \int \left(\frac{1}{x} - \frac{5}{5x+1}\right)\, dx = \ln|x| - \ln|5x+1| + C = \ln\left|\frac{x}{5x+1}\right| + C$$

Therefore, we have

$$\begin{aligned}
\int_1^\infty \frac{dx}{5x^2+x} &:= \lim_{b\to\infty} \int_1^b \frac{dx}{5x^2+x} \\
&= \lim_{b\to\infty} \ln\left|\frac{x}{5x+1}\right|\Bigg|_1^b \\
&= \lim_{b\to\infty} \ln\left|\frac{b}{5b+1}\right| - \ln\left|\frac{1}{5+1}\right| \\
&= \ln\left(\frac{1}{5}\right) - \ln\left(\frac{1}{6}\right) \\
&= \ln\left(\frac{1}{5}\right) + \ln 6 \\
&= \boxed{\ln\left(\frac{6}{5}\right)}
\end{aligned}$$

4. (10 points) Integrate the following:

$$\int \frac{9x^2+2x+15}{4x^3+12x^2+9x+27}\,dx$$

$$\frac{9x^2+2x+15}{4x^3+12x^2+9x+27} = \frac{9x^2+2x+15}{4x^2(x+3)+9(x+3)} = \frac{9x^2+2x+15}{(x+3)(4x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{4x^2+9}$$

Then...

$$\begin{aligned}\frac{A}{x+3} + \frac{Bx+C}{4x^2+9} &= \frac{A(4x^2+9)+(Bx+C)(x+3)}{(x+3)(4x^2+9)} \\ &= \frac{4Ax^2+9A+Bx^2+3Bx+Cx+3C}{(x+3)(4x^2+9)} \\ &= \frac{(4A+B)x^2+(3B+C)x+(9A+3C)}{(x+3)(4x^2+9)}\end{aligned}$$

Relating numerators gives the following system of equations:

$$\begin{aligned}4A+B&=9\\ 3B+C&=2\\ 9A+3C&=15\end{aligned}$$

Using Heaviside's, we have...

$$A = \frac{9(-3)^2+2(-3)+15}{4(-3)^2+9} = \frac{81-6+15}{45} = \frac{90}{45} = 2$$

But then $B = 9-4A = 9-4(2) = 1$ *and hence* $C = 2-3B = 2-3(1) = -1$. *Therefore,*

$$\begin{aligned}\int \frac{9x^2+2x+15}{4x^3+12x^2+9x+27}\,dx &= \int \frac{2}{x+3} + \frac{x-1}{4x^2+9}\,dx \\ &= \int \frac{2}{x+3} + \frac{x}{4x^2+9} - \frac{1}{4x^2+9}\,dx \\ &= \int \frac{2}{x+3} + \frac{x}{4x^2+9} - \frac{1/9}{(2x/3)^2+1}\,dx \\ &= \boxed{2\ln|x+3| + \frac{1}{8}\ln|4x^2+9| - \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + K}\end{aligned}$$

5. (10 points) Find the length of the curve $y = \ln(\sec\theta)$ between $0 \leq \theta \leq \frac{\pi}{4}$.

$$y = \ln(\sec\theta)$$

$$y' = \frac{1}{\sec\theta} \cdot \sec\theta\tan\theta = \tan\theta$$

$$\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_0^{\pi/4} \sqrt{1 + \tan^2\theta}\, d\theta \\
&= \int_0^{\pi/4} \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4} \\
&= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| \\
&= \ln|\sqrt{2} + 1| - \ln|1 + 0| \\
&= \boxed{\ln|1 + \sqrt{2}|}
\end{aligned}$$

6. (10 points) Determine whether the following integral converges or diverges. If it converges, find its value.

$$\int_0^1 x^{3/2} \ln x \, dx$$

$\ln x$	$\frac{2}{5}x^{5/2}$
$\frac{1}{x}$	$x^{3/2}$

$$\begin{aligned}\int_0^1 x^{3/2} \ln x \, dx &= \frac{2}{5}x^{5/2}\ln x - \int \frac{2x^{5/2}}{5x} \, dx \\ &= \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{3/2} \, dx \\ &= \frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2} + C\end{aligned}$$

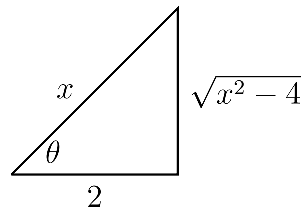
Therefore, we have

$$\begin{aligned}\int_0^1 x^{3/2} \ln x \, dx &:= \lim_{b\to 0^+} \int_b^1 x^{3/2} \ln x \, dx \\ &= \lim_{b\to 0^+} \left[\frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2}\right]_b^1 \\ &= \left(0 - \frac{4}{25}\right) - \lim_{b\to 0^+}\left(\frac{2}{5}b^{5/2}\ln b - \frac{4}{25}b^{5/2}\right) \\ &= -\frac{4}{25} - \lim_{b\to 0^+} \frac{2\ln b}{5\,\frac{1}{b^{5/2}}} + 0 \\ &\overset{L.H.}{=} -\frac{4}{25} - \lim_{b\to 0^+} \frac{2/b}{-25/(2b^{7/2})} \\ &= -\frac{4}{25} - \lim_{b\to 0^+} \frac{4b^{5/2}}{25} \\ &= \boxed{-\frac{4}{25}}\end{aligned}$$

7. (10 points) Integrate the following:

$$\int \frac{dx}{x^2\sqrt{x^2-4}}$$

$$a^2 + b^2 = c^2$$
$$b^2 = \underbrace{c^2 - a^2}_{x^2-4}$$



$$\sec\theta = \frac{x}{2}$$
$$x = 2\sec\theta$$
$$dx = 2\sec\theta\tan\theta\ d\theta$$

$$x^2 = 4\sec^2\theta$$
$$\tan\theta = \frac{\sqrt{x^2-4}}{2}$$
$$\sqrt{x^2-4} = 2\tan\theta$$

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\cdot 2\tan\theta}\ d\theta \\
&= \int \frac{1}{4\sec\theta}\ d\theta \\
&= \frac{1}{4}\int \cos\theta\ d\theta \\
&= \frac{1}{4}\cdot\sin\theta + C \\
&= \frac{1}{4}\cdot\frac{\sqrt{x^2-4}}{x} + C \\
&= \boxed{\frac{\sqrt{x^2-4}}{4x} + C}
\end{aligned}$$

8. (10 points) Suppose $y = f(x)$ is a curve containing the point $(0, 1)$ and has the property that

$$\frac{dy}{dx} = x\, e^{x^2 - \ln(y^2)}$$

Find the curve $f(x)$.

$$\frac{dy}{dx} = x\, e^{x^2 - \ln(y^2)}$$

$$\frac{dy}{dx} = x\, e^{x^2}\, e^{-\ln(y^2)}$$

$$\frac{dy}{dx} = x\, e^{x^2}\, e^{\ln(1/y^2)}$$

$$\frac{dy}{dx} = x\, e^{x^2}\, \frac{1}{y^2}$$

$$y^2\, dy = xe^{x^2}\, dx$$

$$\int y^2\, dy = \int x\, e^{x^2}\, dx$$

$$\frac{y^3}{3} = \frac{e^{x^2}}{2} + C$$

$$y^3 = \frac{3\, e^{x^2}}{2} + C$$

$$y^3 = \frac{3\, e^{x^2} + C}{2}$$

$$y = \sqrt[3]{\frac{3\, e^{x^2} + C}{2}}$$

Since the curve contains the point $(0, 1)$*, if* $x = 0$ *then* $y = 1$.

$$y = \sqrt[3]{\frac{3\, e^{x^2} + C}{2}}$$

$$1 = \sqrt[3]{\frac{3\, e^{0} + C}{2}}$$

This gives $C = -1$*. Therefore,* $\boxed{y = \sqrt[3]{\frac{3\, e^{x^2} - 1}{2}}}$.