Math 296: Exam 3 **Name:** *Caleb McWhorter — Solutions*
Spring – 2018
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80 Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 9 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	15	
3	10	
4	15	
5	10	
6	10	
7	15	
8	15	
Total:	100	

1. (10 points) Determine whether the following series converge or diverge. Be sure to justify your answer completely.

$$\sum_{n=1}^{\infty} n^3 \sin\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} n^3 \sin\left(\frac{1}{n^2}\right) = \lim_{n\to\infty} n \cdot n^2 \sin\left(\frac{1}{n^2}\right) = \lim_{n\to\infty} n \cdot \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \infty$$

Therefore, $\sum_{n=1}^{\infty} n^3 \sin\left(\frac{1}{n^2}\right)$ *diverges by the Divergence Test.*

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt[3]{n^5}}\right)$$

Note that the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/3}}$ *converges by the* p*-test.*

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{\sqrt[3]{n^5}}\right)}{\frac{1}{\sqrt[3]{n^5}}} = 1$$

Therefore, $\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt[3]{n^5}}\right)$ *converges by the Limit Comparison Test.*

2. (15 points) Determine whether the following series converges or diverges. If the series diverges, explain why. If the series converges, find the sum. Be sure to justify your completely.

$$\sum_{n=1}^{\infty} \frac{3^{n-2}}{2^{2n-1}}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{n-2}}{2^{2n-1}} &= \sum_{n=1}^{\infty} \frac{3^{n-2}}{2^{2n-1}} \\
&= \sum_{n=1}^{\infty} \frac{3^n 3^{-2}}{2^{2n} 2^{-1}} \\
&= \sum_{n=1}^{\infty} \frac{2}{9} \cdot \frac{3^n}{2^{2n}} \\
&= \sum_{n=1}^{\infty} \frac{2}{9} \cdot \frac{3^n}{(2^2)^n} \\
&= \sum_{n=1}^{\infty} \frac{2}{9} \left(\frac{3}{4}\right)^n
\end{aligned}$$

The final series is geometric with $|r| = |3/4| = 3/4 < 1$. *Therefore, the series converges. We have*

$$\sum_{n=1}^{\infty} \frac{3^{n-2}}{2^{2n-1}} = \sum_{n=1}^{\infty} \frac{2}{9} \left(\frac{3}{4}\right)^n = \frac{\frac{2}{9} \cdot \frac{3}{4}}{1 - \frac{3}{4}} = \frac{\frac{1}{6}}{\frac{1}{4}} = \frac{1}{6} \cdot \frac{4}{1} = \frac{2}{3}$$

3. (10 points) Determine whether the following series converge or diverge. Be sure to justify your answer completely.

$$\sum_{n=1}^{\infty} \frac{n^3 + n\sin^2 n}{n^5 + n + 1}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test.*

$$\sum_{n=1}^{\infty} \frac{n^3 + n\sin^2 n}{n^5 + n + 1} \le \sum_{n=1}^{\infty} \frac{n^3 + n \cdot 1}{n^5} \le \sum_{n=1}^{\infty} \frac{n^3 + n^3}{n^5} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Therefore, $\sum_{n=1}^{\infty} \frac{n^3 + n\sin^2 n}{n^5 + n + 1}$ *converges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{n^3 + n\sin^2 n}{n^5 + n + 1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^5 + n^3\sin^2 n}{n^5 + n + 1} \cdot \frac{1/n^5}{1/n^5} = \lim_{n\to\infty} \frac{1 + \frac{\sin^2 n}{n^2}}{1 + \frac{1}{n^4} + \frac{1}{n^5}} = 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test. Therefore, Therefore,* $\sum_{n=1}^{\infty} \frac{n^3 + n\sin^2 n}{n^5 + n + 1}$ *converges by the Limit Comparison Test.*

$$\sum_{n=2}^{\infty} \frac{\sqrt{n^2 + n - 1}}{n^{3/2} - 1}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ *diverges by the* p*-test.*

$$\sum_{n=2}^{\infty} \frac{\sqrt{n^2 + n - 1}}{n^{3/2} - 1} \ge \sum_{n=2}^{\infty} \frac{\sqrt{n^2 + n - n}}{n^{3/2}} = \sum_{n=2}^{\infty} \frac{n}{n^{3/2}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

Therefore, $\sum_{n=2}^{\infty} \frac{\sqrt{n^2 + n - 1}}{n^{3/2} - 1}$ *diverges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^2 + n - 1}}{n^{3/2} - 1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n^3 + n^2 - 1}}{n^{3/2} - 1} \cdot \frac{1/n^{3/2}}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\sqrt{1 + \frac{1}{n} - \frac{1}{n^3}}}{1 - \frac{1}{n^{3/2}}} = 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges by the* p*-test. Therefore,* $\sum_{n=2}^{\infty} \frac{\sqrt{n^2 + n - 1}}{n^{3/2} - 1}$ *diverges by the Limit Comparison Test.*

4. (15 points) Determine whether the following series diverges, converges conditionally, or converges absolutely. Be sure to justify your answer completely.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+4}$$

- $\left\{ \frac{n}{n^2+4} \right\}$ *is a decreasing sequence*
- $\lim_{n\to\infty} \frac{n}{n^2+4} = 0$

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+4}$ *converges by the Alternating Series Test.*

Now consider the series $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$. *The series* $\sum_{n=1}^{\infty} \frac{1}{n}$, *the harmonic series, diverges by the* p*-test.*

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4} \geq \sum_{n=1}^{\infty} \frac{n}{n^2+4n^2} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{n}$$

Therefore, $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ *diverges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ *diverges by the Limit Comparison Test.*

Therefore, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+4}$ *converges conditionally.*

5. (10 points) Determine whether the following series diverges, conditionally converges, or converges absolutely. Justify your answer completely.

$$\sum_{n=1}^{\infty} \left(\frac{2n^2+n-1}{3n^2-n+4} \right)^n$$

$$\lim_{n\to\infty} \left| \left(\frac{2n^2+n-1}{3n^2-n+4} \right)^n \right|^{1/n} = \lim_{n\to\infty} \frac{2n^2+n-1}{3n^2-n+4} = \frac{2}{3} < 1$$

Therefore, the series $\sum_{n=1}^{\infty} \left(\frac{2n^2+n-1}{3n^2-n+4} \right)^n$ *converges absolutely by the Root Test.*

6. (10 points) Determine whether the following series diverges, conditionally converges, or converges absolutely. Justify your answer completely.

$$\sum_{n=0}^{\infty} \frac{n!}{5^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{5^n}{5^{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{5^n}}{5 \cdot \cancel{5^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+1}{5} \right| \\
&= \infty
\end{aligned}$$

Therefore, the series $\sum_{n=0}^{\infty} \frac{n!}{5^n}$ *diverges by the Ratio Test.*

7. (15 points) Determine the center, radius of convergence, and interval of convergence for the following power series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2-1}}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{\sqrt{(n+1)^2-1}}}{\frac{(-1)^n x^n}{\sqrt{n^2-1}}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2-1}}\cdot\frac{\sqrt{n^2-1}}{x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{\sqrt{n^2-1}}{\sqrt{n^2+2n}}\right|$$
$$= \lim_{n\to\infty}\left|x\cdot\sqrt{\frac{n^2-1}{n^2+2n}}\right|$$
$$= |x|$$

Because we want this ratio at most 1, we have $|x| < 1$. *This implies that* $-1 < x < 1$.

$x = -1$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt{n^2-1}} = \sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-1}}$

$$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-1}} \geq \sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2}} = \sum_{n=2}^{\infty}\frac{1}{n}$$

The series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ *diverges by the* p*-test. Therefore,* $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-1}}$ *diverges by the Comparison Test.*

<u>OR</u>

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^2-1}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2-1}}\cdot\frac{1/n}{1/n} = \lim_{n\to\infty}\frac{1}{\sqrt{1-\frac{1}{n}}} = 1$$

The series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ *diverges by the* p*-test. Therefore,* $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-1}}$ *diverges by the Limit Comparison Test.*

$x = 1$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n^2-1}}$. *Observe* $\{\frac{1}{\sqrt{n^2-1}}$ *is a decreasing sequence and* $\lim_{n\to\infty}\dfrac{1}{\sqrt{n^2-1}} = 0$. *Therefore, the series converges by the Alternating Series Test.*

Therefore, the interval of convergence is $(-1, 1]$, *the radius of convergence is* $R = \dfrac{1-(-1)}{2} = 1$, *and the center is* $x = 0$.

8. (15 points) Find the first 4 nonzero terms of the Taylor series $f(x) = \dfrac{1}{x^2}$ centered at $x = 1$.

$$f(x) = \frac{1}{x^2}\Big|_{x=1} = 1$$

$$f'(x) = -\frac{2}{x^3}\Big|_{x=1} = -2$$

$$f''(x) = \frac{6}{x^4}\Big|_{x=1} = 6$$

$$f'''(x) = -\frac{24}{x^4}\Big|_{x=1} = -24$$

Therefore, the Taylor series is...

$$1 + -2 \cdot \frac{(x-1)^1}{1!} + 6 \cdot \frac{(x-1)^2}{2!} + (-24) \cdot \frac{(x-1)^3}{3!}$$

$$1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3$$