Name: *Caleb McWhorter — Solutions* **MAT 296: HW 1**
Spring 2018 **Due: 01/29**

Problem 1: Integrate the following:

(i) $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^2}$

Let $u = 1+\sqrt{x}$*. Then* $du = \dfrac{1}{2\sqrt{x}} \iff dx = 2\sqrt{x}\,du$*. Then*

$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^2} = 2\int \frac{\cancel{\sqrt{x}}\,du}{\cancel{\sqrt{x}}\,u^2} = 2\int \frac{du}{u^2} = \frac{-2}{u} + C = \frac{-2}{1+\sqrt{x}} + C$$

(ii) $\displaystyle\int \frac{dx}{4+9x^2}$

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{4+9x^2} \cdot \frac{1/4}{1/4} = \frac{1}{4}\int \frac{dx}{1+\left(\frac{3}{2}x\right)^2}$$

Let $u = \frac{3}{2}x$ *so that* $du = \frac{3}{2}dx \iff dx = \frac{2}{3}du$*. Then*

$$\frac{1}{4}\int \frac{dx}{1+\left(\frac{3}{2}x\right)^2} = \frac{2}{3}\cdot\frac{1}{4}\int \frac{du}{1+u^2} = \frac{1}{6}\arctan(u) + C = \frac{1}{6}\arctan\left(\frac{3x}{2}\right) + C$$

(iii) $\displaystyle\int x\sqrt{x-2}\,dx$

One can use integration-by-parts. However, we will use u-substitution. Let $u = x-2 \iff x = u+2$*, then* $du = dx$*.*

$$\int x\sqrt{x-2}\,dx = \int (u+2)\sqrt{u}\,du = \int \left(u^{3/2}+2\sqrt{u}\right)\,du = \frac{2}{5}u^{5/2}+2\cdot\frac{2}{3}u^{3/2}+C = \frac{2}{5}(x-2)^{5/2}+\frac{4}{3}(x-2)^{3/2}+C$$

(iv) $\displaystyle\int_{2\pi}^{e^{\pi}} \frac{\sin(\ln x)}{x}\,dx$

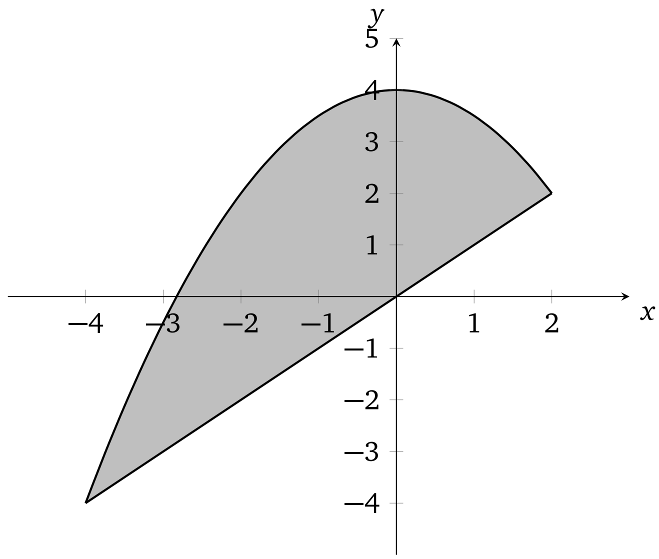
Let $u = \ln x$ *so that* $du = \dfrac{dx}{x} \iff dx = x\,du$*. Now if* $x = e^{\pi}$*, we have* $u = \ln(e^{\pi}) = \pi$*. If* $x = 2\pi$*,* $u = \ln(2\pi)$*. Then we have*

$$\int_{2\pi}^{e^{\pi}} \frac{\sin(\ln x)}{x}\,dx = \int_{\ln(2\pi)}^{\pi} \frac{\sin u}{\cancel{x}}\cdot\cancel{x}\,du = \int_{\ln(2\pi)}^{\pi} \sin u\,du = -\cos u\Big|_{\ln(2\pi)}^{\pi}$$

$$= -(-1)-(-\cos(\ln(2\pi))) = 1+\cos(\ln(2\pi))$$

Problem 2: Find the area of the region bounded by the curves given by $y = -\frac{1}{2}x^2 + 4$ and $y = x$.

Solution. *Setting* $\frac{-1}{2}x^2 + 4 = x$ *yields* $x^2 + 2x - 8 = 0$. *But* $x^2 + 2x - 8 = (x+4)(x-2)$ *so that we have solutions* $x = -4$ *and* $x = 2$. *We plot the region below.*



$$
\begin{aligned}
\int_{-4}^{2} \left(-\frac{1}{2}x^2 + 4\right) - x \, dx &= \int_{-4}^{2} -\frac{1}{2}x^2 - x + 4 \, dx \\
&= \left[-\frac{1}{2} \cdot \frac{x^3}{3} - \frac{x^2}{2} + 4x\right]_{-4}^{2} \\
&= \left[-\frac{1}{6}x^3 - \frac{1}{2}x^2 + 4x\right]_{-4}^{2} \\
&= \left(-\frac{4}{3} - 2 + 8\right) - \left(\frac{32}{3} - 8 - 16\right) \\
&= 18
\end{aligned}
$$

Problem 3: Let $f(x) = \sqrt[3]{x}$ and $g(x) = x/4$. Set up *but do not integrate* an integral expression to find the volume of the region bound by $f(x)$ and $g(x)$ in Quadrant I if this region is rotated about the...

(i) x–axis

(ii) y–axis

Using the method of your choice, find an integral expression for the volume (you do not need to find the value) if the region between $f(x)$ and $g(x)$ is rotated about the line...

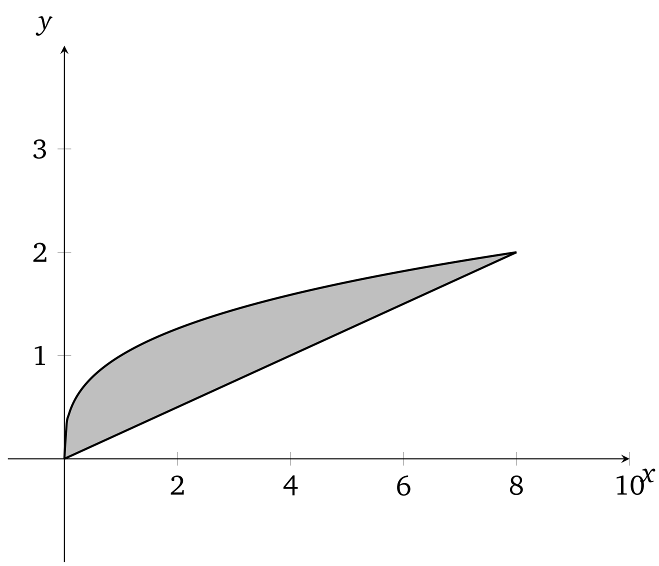
(iii) $x = 10$

(iv) $x = -1$

(v) $y = 5$

(vi) $y = -2$

Solution. *Setting* $\sqrt[3]{x} = \dfrac{x}{4}$, *we have* $x = \left(\dfrac{x}{4}\right)^3$ *so that* $x = \dfrac{x^3}{64}$. *But then* $x^3 - 64x = 0$. *But then we have* $0 = x(x^2 - 64) = x(x-8)(x+8)$. *The solutions are then clearly* $x = 0$ *and* $x = 8$. *Note that if* $y = \sqrt[3]{x}$ *then* $x = y^3$ *and if* $y = \dfrac{x}{4}$ *then* $x = 4y$. *We plot the region below.*



(i)

$$\textit{Disks: } \pi \int_0^8 (\sqrt[3]{x})^2 - (x/4)^2 \, dx$$

$$\textit{Shells: } 2\pi \int_0^2 y \cdot (4y - y^3) \, dy$$

(ii)

$$\textit{Disks: } \pi \int_0^2 (4y)^2 - (y^3)^2 \, dy$$

$$\textit{Shells: } 2\pi \int_0^8 x \cdot \left(\sqrt[3]{x} - \frac{x}{4} \right) \, dx$$

(iii)

$$\textit{Disks: } \pi \int_0^2 (10 - y^3)^2 - (10 - 4y)^2 \, dy$$

$$\textit{Shells: } 2\pi \int_0^8 (10 - x) \left(\sqrt[3]{x} - \frac{x}{4} \right) \, dx$$

(iv)

$$\textit{Disks: } \pi \int_0^2 (1 + 4y)^2 - (1 + y^3)^2 \, dy$$

$$\textit{Shells: } 2\pi \int_0^8 (1 + x) \left(\sqrt[3]{x} - \frac{x}{4} \right) \, dx$$

(v)

$$\textit{Disks: } \pi \int_0^8 \left(5 - \frac{x}{4} \right)^2 - \left(5 - \sqrt[3]{x} \right)^2 \, dx$$

$$\textit{Shells: } 2\pi \int_0^2 (5 - y)(4y - y^3) \, dy$$

(vi)

$$\textit{Disks: } \pi \int_0^8 (2 + \sqrt[3]{x})^2 - \left(2 + \frac{x}{4} \right)^2 \, dx$$

$$\textit{Shells: } 2\pi \int_0^2 (2 + y)(4y - y^3) \, dy$$

Problem 4: If an individual invests P dollars in an account with interest rate r (expressed as a decimal), compounded continuously, the amount in the account after a period of time t is given by Pe^{rt}.

If $10,000 is invested in an account with 8% interest, compounded continuously, for a period of 2 years, how much money will be in the account?

Compared to individuals, the situation with large companies is different. Large companies have money flowing in quickly enough that often one can regard the discrete payments to the company as a continuous stream of income. Let $I(t)$ denote the income stream for a company, i.e. $I(t)$ is the amount of money the company makes at time t. If this income is placed in an account with interest rate r, compounded continuously, then the amount of money in the account after time T is given by

$$\int_0^T I(t)\, e^{r(T-t)}\, dt.$$

If a company has income stream of $10,000 per year deposited into an account with 8% interest, find the value of this account after 10 years.

Solution. *The amount of money in the account after 2 years is* $10000 \cdot e^{0.08\cdot 2} \approx \$11,735.11$. *For the second part, we have*

$$\begin{aligned}\int_0^{10} 10000\, e^{0.08(10-t)}\, dt &= 10000\, e^{0.8} \int_0^{10} e^{-0.08t}\, dt \\ &= 10000\, e^{0.8} \cdot \left.\frac{e^{-0.08t}}{-0.08}\right|_0^{10} \\ &= 10000\, e^{0.8} \left(-\frac{e^{-0.8}}{0.08} - \frac{1}{-0.08} \right) \approx \$153,192.62\end{aligned}$$

Problem 5: (Poiseuille's Law) Blood flow in a blood vessel can be modeled as laminar flow (flow in parallel non-interacting layers). Assuming a laminar flow and a vessel of constant radius, Poiseuille's Law says that the velocity of the blood a distance r from the center of the artery is given by

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2),$$

where l is the length of the blood vessel, R is the radius of the blood vessel, η is the viscosity of blood (viscosity measures resistance of a fluid to flow), and P is the pressure difference between the ends of the blood vessel. If one divides the blood vessel into 'many' concentric circles, then the flow rate is given by multiplying the circumference of each circle by the velocity of the blood flowing through that circle. Adding these numbers together approximates the flow rate of blood through the vessel. Using infinitely many circles, one obtains an integral for the flow rate.

$$\text{Flow Rate } = \int_0^R \frac{2\pi P}{4\eta l}(R^2 - r^2)r\, dr$$

Integrate to find an expression for flow rate of the blood through the vessel.

Solution. *Notice everything in the integral is constant with respect to r, except of course r itself. Then we have*

$$\begin{aligned} \textit{Flow Rate } &= \int_0^R \frac{2\pi P}{4\eta l}(R^2 - r^2)r\, dr \\ &= \frac{\pi P}{2\eta l} \int_0^R (R^2 r - r^3)\, dr \\ &= \frac{\pi P}{2\eta l} \cdot \left[\frac{R^2 r^2}{2} - \frac{r^4}{4} \right]_0^R \\ &= \frac{\pi P}{2\eta l} \cdot \left[\left(\frac{R^4}{2} - \frac{R^4}{4} \right) - 0 \right] \\ &= \frac{\pi P}{2\eta l} \cdot \frac{R^4}{4} \\ &= \frac{\pi P R^4}{8\eta l} \end{aligned}$$

Thus, the flow rate is proportional to R^4. Therefore, assuming constant pressure, a small increase in radius creates a large increase in the flow rate, as expected.