Name: *Caleb McWhorter — Solutions* **MAT 296: HW 5**

Spring 2018 **Due: 02/21**

Problem 1: Integrate the following:

$$\int \frac{x}{\sqrt{9-x^2}}\,dx$$

Solution. *Let* $u = 9 - x^2$. *Then* $du = -2x\,dx \iff dx = \dfrac{du}{-2x}$. *Then...*

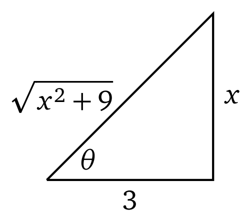
$$\int \frac{x}{\sqrt{9-x^2}}\,dx = \int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{du}{-2\cancel{x}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\cdot 2\sqrt{u} + C = -\sqrt{u} + C = -\sqrt{9-x^2} + C$$

Problem 2: Integrate the following:

$$\int \frac{3}{x^2\sqrt{x^2+9}}\,dx$$

Solution.

$$\underbrace{a^2+b^2}_{x^2+9} = c^2$$



$$\tan\theta = \frac{x}{3}$$
$$x = 3\tan\theta$$
$$dx = 3\sec^2\theta\,d\theta$$

$$x^2 = 9\tan^2\theta$$
$$\sec\theta = \frac{\sqrt{x^2+9}}{3}$$
$$\sqrt{x^2+9} = 3\sec\theta$$

$$\int \frac{3}{9\tan^2\theta\cdot 3\sec\theta}\cdot 3\sec^2\theta\,d\theta = \int \frac{\sec\theta}{3\tan^2\theta}\,d\theta$$
$$= \frac{1}{3}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

Now let $u = \sin\theta$. Then $du = \cos\theta \; d\theta$.

$$\begin{aligned}
\frac{1}{3}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta &= \frac{1}{3}\int \frac{du}{u^2} \\
&= \frac{1}{3}\cdot -\frac{1}{u} + C \\
&= \frac{1}{3}\cdot -\frac{1}{\sin\theta} + C \\
&= \frac{1}{3}\cdot -\csc\theta + C \\
&= \frac{1}{3}\cdot -\frac{\sqrt{x^2+9}}{x} + C \\
&= \frac{\sqrt{x^2+9}}{-3x} + C
\end{aligned}$$

Therefore, we must have

$$\int \frac{3}{x^2\sqrt{x^2+9}}\, dx = \frac{\sqrt{x^2+9}}{-3x} + C$$

Problem 3: Integrate the following:

$$\int \frac{x^2+5x+5}{x+3}\,dx$$

Solution. *Observe that the degree of the numerator is at least the degree of the denominator, so we must long divide first.*

$$\begin{array}{r|l} & x+2 \\ \hline x+3 & x^2+5x+5 \\ & -x^2-3x \\ \hline & 2x+5 \\ & -2x-6 \\ \hline & -1 \end{array}$$

Therefore,

$$\int \frac{x^2+5x+5}{x+3}\,dx = \int \left(x+2+\frac{-1}{x+3}\right)dx = \frac{x^2}{2}+2x-\ln|x+3|+C$$

Problem 4: Integrate the following:

$$\int \frac{x+4}{x^2-4}\,dx$$

Solution.

$$\int \frac{x+4}{x^2-4}\,dx = \int \frac{x+4}{(x-2)(x+2)}\,dx$$

$$\frac{x+4}{(x-2)(x+2)} = \frac{A}{x-2}+\frac{B}{x+2}$$

We use Heaviside's to find A and B:

$$A = \frac{2+4}{2+2} = \frac{6}{4} = \frac{3}{2}$$
$$B = \frac{-2+4}{-2-2} = \frac{2}{-4} = -\frac{1}{2}$$

Therefore,

$$\int \frac{x+4}{(x-2)(x+2)}\,dx = \int \left(\frac{3/2}{x-2}+\frac{-1/2}{x+2}\right)dx = \frac{3}{2}\ln|x-2|-\frac{1}{2}\ln|x+2|+C$$

Problem 5: Integrate the following:

$$\int \frac{x+4}{x(x-2)^2}\,dx$$

Solution.

$$\frac{x+4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Heaviside's will allow us to obtain A, C:

$$A = \frac{0+4}{(0-2)^2} = \frac{4}{4} = 1$$

$$C = \frac{2+4}{2} = \frac{6}{2} = 3$$

To find B, plug in $x = 1$ (nearly any values of x will suffice) to both sides and use the fact we know A, C:

$$\frac{1+4}{1(1-2)^2} = \frac{A}{1} + \frac{B}{1-2} + \frac{C}{(1-2)^2}$$

$$5 = A - B + C$$

$$5 = 1 - B + 3$$

$$5 = -B + 4$$

$$B = -1$$

Therefore,

$$\int \frac{x+4}{x(x-2)^2}\,dx = \int \left(\frac{1}{x} + \frac{-1}{x-2} + \frac{3}{(x-2)^2}\right)dx = \ln|x| - \ln|x-2| - \frac{3}{x-2} + K$$

Problem 6: Integrate the following:

$$\int \frac{2x^2 - 2x + 5}{(x-1)(x^2+4)}\, dx$$

Solution.

$$\frac{2x^2 - 2x + 5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+4}$$

Heaviside's will obtain A:

$$A = \frac{2(1)^2 - 2(1) + 5}{1^2 + 4} = \frac{2 - 2 + 5}{5} = \frac{5}{5} = 1$$

Now plug in $x = 0$ to both sides:

$$\frac{2(0) - 2(0) + 5}{(0-1)(0^2+4)} = \frac{A}{0-1} + \frac{B(0) + C}{0^2 + 4}$$
$$-\frac{5}{4} = -A + \frac{C}{4}$$
$$-5 = -4A + C$$
$$-5 = -4(1) + C$$
$$-1 = C$$

Now plug in $x = -1$ to both sides:

$$\frac{2(-1)^2 - 2(-1) + 5}{(-1-1)((-1)^2+4)} = \frac{A}{-1-1} + \frac{B(-1) + C}{(-1)^2 + 4}$$

$$\frac{2+2+5}{-2(5)} = \frac{A}{-2} + \frac{-B + C}{5}$$

$$\frac{9}{-10} = \frac{A}{-2} + \frac{-B + C}{5}$$

$$9 = 5A - 2(-B + C)$$

$$9 = 5A + 2B - 2C$$

$$9 = 5(1) + 2B - 2(-1)$$

$$9 = 5 + 2B + 2$$

$$9 = 2B + 7$$

$$2 = 2B$$

$$B = 1$$

Therefore,

$$\int \frac{2x^2-2x+5}{(x-1)(x^2+4)}\,dx = \int \left(\frac{1}{x-1}+\frac{x-1}{x^2+4}\right)\,dx$$

$$= \int \left(\frac{1}{x-1}+\frac{x}{x^2+4}-\frac{1}{x^2+4}\right)\,dx$$

$$= \int \left(\frac{1}{x-1}+\frac{x}{x^2+4}-\frac{1}{x^2+4}\cdot\frac{1/4}{1/4}\right)\,dx$$

$$= \int \left(\frac{1}{x-1}+\frac{x}{x^2+4}-\frac{1/4}{x^2/4+1}\right)\,dx$$

$$= \int \left(\frac{1}{x-1}+\frac{x}{x^2+4}-\frac{1/4}{(x/2)^2+1}\right)\,dx$$

$$= \ln|x-1|+\frac{1}{2}\ln|x^2+4|-\frac{1}{4}\tan^{-1}(x/2)\cdot 2+K$$

$$= \ln|x-1|+\frac{1}{2}\ln|x^2+4|-\frac{1}{2}\tan^{-1}(x/2)+K$$