Name: *Caleb McWhorter — Solutions* **MAT 296: HW 6**
Spring 2018 **Due: 02/26**

Problem 1: Determine whether the following improper integral converges or diverges. If it converges, give the value of the integral:

$$\int_0^\infty \frac{x}{(x^2+1)^2}\,dx$$

Solution. *Let* $u = x^2+1$. *Then* $du = 2x\,dx \iff dx = \dfrac{du}{2x}$. *Then we have*

$$\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} + C = -\frac{1}{2(x^2+1)} + C$$

Therefore, we have

$$\begin{aligned}
\int_0^\infty \frac{x}{(x^2+1)^2}\,dx &:= \lim_{b\to\infty} \int_0^b \frac{x}{(x^2+1)^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{2(x^2+1)}\bigg|_0^b \\
&= \lim_{b\to\infty} -\frac{1}{2(b^2+1)} - \frac{-1}{2(0^2+1)} \\
&= \frac{1}{2}
\end{aligned}$$

Problem 2: Determine whether the following improper integral converges or diverges. If it converges, give the value of the integral:

$$\int_2^6 \frac{dx}{6(x-2)^{3/2}}\,dx$$

Solution.

$$\begin{aligned}
\int_2^6 \frac{dx}{6(x-2)^{3/2}}\,dx &= \lim_{b\to 2^+} \int_b^6 \frac{dx}{6(x-2)^{3/2}}\,dx \\
&= \lim_{b\to 2^+} \left[\frac{1}{6(x-2)^{1/2}} \cdot -2\right]_b^6 \\
&= \lim_{b\to 2^+} \frac{-1}{3(x-2)^{1/2}}\bigg|_b^6 \\
&= \frac{-1}{3(6-2)^{1/2}} - \lim_{b\to 2^+} \frac{-1}{3(b-2)^{1/2}} \\
&= \infty
\end{aligned}$$

Problem 3: Determine whether the following improper integral converges or diverges. If it converges, give the value of the integral:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+9}$$

Solution.

$$\int \frac{dx}{x^2+9} = \int \frac{dx}{x^2+9} \cdot \frac{1/9}{1/9} = \frac{1}{9}\int \frac{dx}{(x/3)^2+1} = \frac{1}{9}\tan^{-1}\left(\frac{x}{3}\right)\cdot 3 + C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{x^2+9} &= \lim_{b\to-\infty}\int_b^0 \frac{dx}{x^2+9} + \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+9} \\
&= \lim_{b\to-\infty} \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_b^0 + \lim_{b\to\infty} \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_0^b \\
&= \left[\frac{1}{3}\tan^{-1}(0) - \lim_{b\to-\infty}\frac{1}{3}\tan^{-1}(b/3)\right] + \left[\lim_{b\to\infty}\tan^{-1}\left(\frac{b}{3}\right) - \tan^{-1}(0)\right] \\
&= \left[0 - \frac{1}{3}\cdot\frac{-\pi}{2}\right] + \left[\frac{1}{3}\cdot\frac{\pi}{2} - 0\right] \\
&= \frac{\pi}{3}
\end{aligned}$$

Problem 4: Determine whether the following integral converges or diverges. If it converges, determine the value of the integral:

$$\int_{-\infty}^{0} \sin\theta \, d\theta$$

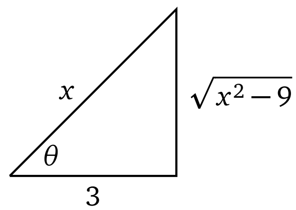
Solution.

$$\int_{-\infty}^{0} \sin\theta\, d\theta := \lim_{b\to-\infty}\int_b^0 \sin\theta\, d\theta = \lim_{b\to-\infty} -\cos\theta\Big|_b^0 = -\cos 0 - \lim_{b\to-\infty} -\cos b = D.N.E.$$

Problem 5: Determine whether the following integral converges or diverges. If it converges, find its value.

$$\int_3^9 \frac{dx}{\sqrt{x^2-9}}$$

$$\begin{aligned} a^2+b^2 &= c^2 \\ b^2 &= \underbrace{c^2-a^2}_{x^2-9} \end{aligned}$$



$$\begin{aligned} \sec\theta &= \frac{x}{3} \\ x &= 3\sec\theta \\ dx &= 3\sec\theta\tan\theta\, d\theta \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{\sqrt{x^2-9}}{3} \\ \sqrt{x^2-9} &= 3\tan\theta \end{aligned}$$

$$\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right| + C$$

$$\begin{aligned} \int_3^9 \frac{dx}{\sqrt{x^2-9}} &= \lim_{b\to 3^+} \int_b^9 \frac{dx}{\sqrt{x^2-9}} \\ &= \lim_{b\to 3^+} \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|\Bigg|_b^9 \\ &= \ln\left|\frac{9}{3}+\frac{\sqrt{9^2-9}}{3}\right| - \lim_{b\to 3^+}\ln\left|\frac{b}{3}+\frac{\sqrt{b^2-9}}{3}\right| \\ &= \ln\left|3+\frac{\sqrt{72}}{3}\right| - \ln|1+0| \\ &= \ln\left|3+\frac{\sqrt{9\cdot 4\cdot 2}}{3}\right| \\ &= \ln|3+2\sqrt{2}| \end{aligned}$$

Problem 6: Determine whether the following integral converges or diverges. If it converges, find its value.

$$\int_{-2}^2 \frac{dx}{x}$$

$$\begin{aligned} \int_{-2}^2 \frac{dx}{x} &:= \lim_{b\to 0^-}\int_{-2}^b \frac{dx}{x} + \lim_{b\to 0^+}\int_b^2 \frac{dx}{x} \\ &= \lim_{b\to 0^-} \ln|x|\Big|_{-2}^b + \lim_{b\to 0^+}\ln|x|\Big|_b^2 \\ &= \left(\lim_{b\to 0^-}\ln|b| - \ln|2|\right) + \left(\ln|2| - \lim_{b\to 0^+}\ln|b|\right) \\ &= D.N.E. \end{aligned}$$

Problem 7: Determine the values of p for which the integral

$$\int_e^\infty \frac{dx}{x(\ln x)^p}$$

converges. Find the value of the integral for these values of p.

Solution. *Let $u = \ln x$. Then $du = \frac{dx}{x} \iff dx = x\,du$. Now if $x = e$, then $u = \ln e = 1$. Since $x \to \infty$, we have* $\lim_{x\to\infty} u = \lim_{x\to\infty} \ln x = \infty$.

$$\int_e^\infty \frac{dx}{x(\ln x)^p} = \int_1^\infty \frac{du}{u^p}$$

By the p-test for integrals, we know this converges if and only if $p > 1$. Now if $p > 1$, we have

$$\begin{aligned}
\int_1^\infty \frac{du}{u^p} &:= \lim_{b\to\infty} \int_1^b \frac{du}{u^p} \\
&= \lim_{b\to\infty} \frac{u^{-p+1}}{-p+1}\bigg|_1^b \\
&= \lim_{b\to\infty} \frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p} \\
&= 0 - \frac{1}{1-p} \\
&= \frac{1}{p-1}
\end{aligned}$$

Problem 8: Determine whether the following integral converges or diverges. [Hint: Try the Comparison Test for Integrals.]

$$\int_1^\infty \frac{\sin^2(e^x) + 3}{\sqrt[3]{8x^9 + 1729}}\,dx$$

Solution. First, observe

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^3} &:= \lim_{b\to\infty} \int_1^\infty \frac{dx}{x^3} \\
&= \lim_{b\to\infty} -\frac{1}{2x^2}\bigg|_1^b \\
&= \lim_{b\to\infty} -\frac{1}{2b^2} - \frac{-1}{2} \\
&= 0 + \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, this integral converges. [One could also observe this via the p-test for Integrals.] Now

$$0 \leq \int_1^\infty \frac{\sin^2(e^x)+3}{\sqrt[3]{8x^9+1729}}\,dx \leq \int_1^\infty \frac{1+3}{\sqrt[3]{8x^9}}\,dx = \frac{4}{\sqrt[3]{8}}\int_1^\infty \frac{dx}{\sqrt[3]{x^9}} = 2\int_1^\infty \frac{dx}{x^3}$$

By the Comparison Test for Integrals, the given integral converges. Furthermore, since $\int_1^\infty \frac{dx}{x^3} = \frac{1}{2}$, we know

$$0 \leq \int_1^\infty \frac{\sin^2(e^x)+3}{\sqrt[3]{8x^9+1729}}\,dx \leq 2 \cdot \frac{1}{2} = 1$$

In fact, $\displaystyle\int_1^\infty \frac{\sin^2(e^x)+3}{\sqrt[3]{8x^9+1729}}\,dx \approx 0.439257$.