Name: *Caleb McWhorter — Solutions* **MAT 296: HW 8**
Spring 2018 **Due: 03/28**

Problem 1: Use the Integral Test (if possible) to determine whether the following series converge or diverge. Justify your use of the Integral Test and be sure to show all your work.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/2}}$

Observe

- $f(x) := \dfrac{1}{x(\ln x)^{1/2}}$ *is a decreasing function.*
- $f(x) > 0$
- $f(x)$ *is continuous.*

Therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/2}}$ *converges if and only if the following integral converges*

$$\begin{aligned}
\int_2^{\infty} \frac{dx}{x(\ln x)^{1/2}} &:= \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^{1/2}} \\
&= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^{1/2}} \qquad u = \ln x;\ du = \frac{dx}{x} \iff x\,du = dx \\
&= \lim_{b\to\infty} 2\sqrt{u}\,\Big|_{\ln 2}^{\ln b} \\
&= \lim_{b\to\infty} 2\sqrt{\ln b} - 2\sqrt{\ln 2} \\
&= \infty
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/2}}$ *diverges by the Integral Test.*

(ii) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Observe

- $f(x) := \frac{n^2}{e^{n^3}}$ *is a decreasing function.*
- $f(x) > 0$
- $f(x)$ *is continuous.*

Therefore, the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ *converges if and only if the following integral converges*

$$\begin{aligned}
\int_1^{\infty} \frac{x^2}{e^{x^3}}\, dx &:= \lim_{b\to\infty} \int_1^{b} \frac{x^2}{e^{x^3}}\, dx \\
&= \lim_{b\to\infty} \int_1^{b^3} \frac{\cancel{x^2}}{e^u} \frac{du}{3\cancel{x^2}} \qquad u = x^3, du = 3x^2\, dx \iff \frac{du}{3x^2} = dx \\
&= \lim_{b\to\infty} \frac{1}{3} \int_1^{b^3} e^{-u}\, du \\
&= \lim_{b\to\infty} -\frac{e^{-u}}{3}\Big|_1^{b^3} \\
&= \lim_{b\to\infty} -\frac{e^{-b^3}}{3} - \left(-\frac{e^{-1}}{3}\right) \\
&= 0 + \frac{e^{-1}}{3} \\
&= \frac{1}{3e}
\end{aligned}$$

Therefore, the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ *converges by the Integral Test.*

(iii) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln(n^3)}$

The Integral Test does not apply. The function $f(x) := \frac{n}{\ln(n^3)}$ *is increasing. In fact,*

$$\lim_{n\to\infty} \frac{n}{\ln(n^3)} = \lim_{n\to\infty} \frac{n}{3\ln n} \overset{L.H.}{=} \lim_{n\to\infty} \frac{n}{3} = \infty$$

Therefore, $\displaystyle\sum_{n=2} \frac{n}{\ln(n^3)}$ *diverges by the Divergence Test.*

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{1+n^2}$

Observe

- $f(x) := \dfrac{\tan^{-1} x}{1+x^2}$ *is a decreasing function (for* $x > 1$ *but this does not affect convergence).*
- $f(x) > 0$
- $f(x)$ *is continuous.*

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{1+n^2}$ *converges if and only if the following integral converges*

$$\begin{aligned}
\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2}\,dx := \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{1+x^2}\,dx \\
&= \lim_{b\to\infty} \int_{\pi/4}^{\tan^{-1} b} \frac{u}{\cancel{1+x^2}} \cdot \cancel{(1+x^2)}\,du \qquad u = \tan^{-1} x, du = \frac{dx}{1+x^2} \iff (1+x^2)du = dx \\
&= \lim_{b\to\infty} \int_{\pi/4}^{\tan^{-1} b} u\,du \\
&= \lim_{b\to\infty} \frac{u^2}{2}\bigg|_{\pi/4}^{\tan^{-1} b} \\
&= \lim_{n\to\infty} \frac{(\tan^{-1}(b))^2}{2} - \frac{(\pi/4)^2}{2} \\
&= \frac{(\pi/2)^2}{2} - \frac{\pi^2}{32} = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{1+n^2}$ *converges by the Integral Test.*

(v) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$

Observe

- $f(x) := \dfrac{1}{x(\ln x)^5}$ *is a decreasing function.*
- $f(x) > 0$
- $f(x)$ *is continuous.*

Therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ *converges if and only if the following integral converges*

$$\begin{aligned}
\int_2^{\infty} \frac{dx}{x(\ln x)^5} &:= \lim_{b\to\infty} \int_2^{b} \frac{dx}{x(\ln x)^5} \\
&= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^5} \qquad u = \ln x;\ du = \frac{dx}{x} \iff x\,du = dx \\
&= \lim_{b\to\infty} -\frac{1}{4u^4}\bigg|_{\ln 2}^{\ln b} \\
&= \lim_{b\to\infty} -\frac{1}{4(\ln b)^4} - \left(-\frac{1}{4(\ln 2)^4}\right) \\
&= 0 + \frac{1}{4(\ln 2)^4} \\
&= \frac{1}{4(\ln 2)^4}
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ *converges by the Integral Test.*

(vi) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln(n))}$

Observe

- $f(x) := \dfrac{1}{x \ln x \ln(\ln(x))}$ *is a decreasing function.*
- $f(x) > 0$
- $f(x)$ *is continuous.*

Therefore, the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln(n))}$ *converges if and only if the following integral converges*

$$\int_3^{\infty} \frac{dx}{x \ln x \ln(\ln(x))} := \lim_{b\to\infty} \int_3^b \frac{dx}{x \ln x \ln(\ln(x))}.$$

Let $u = \ln(\ln(x))$. *Then* $du = \dfrac{1}{\ln x} \cdot \dfrac{1}{x}\, dx = \dfrac{1}{x \ln x}\, dx$ *so that* $dx = x \ln x \, du$. *But then*

$$\begin{aligned}
\int_3^{\infty} \frac{dx}{x \ln x \ln(\ln(x))} &:= \lim_{b\to\infty} \int_3^b \frac{dx}{x \ln x \ln(\ln(x))} \\
&= \lim_{b\to\infty} \int_{\ln(\ln 3)}^{\ln(\ln b)} \frac{\cancel{x \ln x}}{\cancel{x \ln x}\, u}\, du \\
&= \lim_{b\to\infty} \int_{\ln(\ln 3)}^{\ln(\ln b)} \frac{du}{u} \\
&= \lim_{b\to\infty} \ln u \Big|_{\ln(\ln 3)}^{\ln(\ln b)} \\
&= \lim_{b\to\infty} \ln(\ln(\ln(b))) - \ln(\ln(3)) \\
&= \infty
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln(n))}$ *diverges by the Integral Test.*

(vii) $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Observe

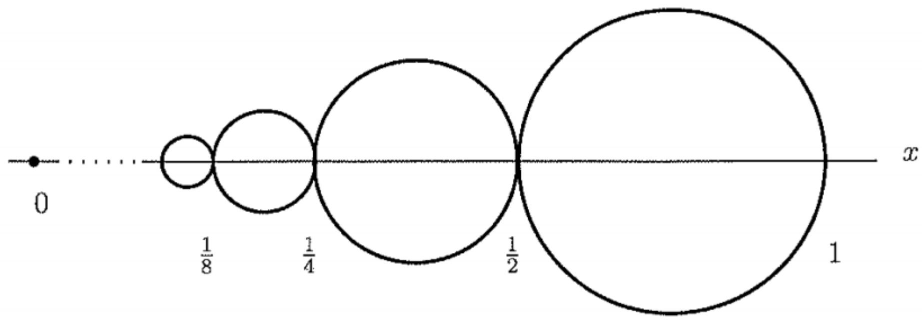
- $f(x) := \frac{x}{1+x^2}$ *is a decreasing function.*
- $f(x) > 0$
- $f(x)$ *is continuous.*

Therefore, the series $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ *converges if and only if the following integral converges*

$$\begin{aligned}
\int_1^{\infty} \frac{x}{1+x^2}\,dx &:= \lim_{b\to\infty} \int_1^{b} \frac{x}{1+x^2}\,dx \\
&= \lim_{b\to\infty} \int_2^{1+b^2} \frac{\cancel{x}}{u} \frac{du}{2\cancel{x}} \qquad u = 1+x^2, du = 2x\,dx \iff \frac{du}{2x} = dx \\
&= \lim_{b\to\infty} \frac{1}{2} \int_2^{1+b^2} \frac{du}{u} \\
&= \lim_{b\to\infty} \left. \frac{\ln u}{2} \right|_2^{1+b^2} \\
&= \lim_{b\to\infty} \frac{\ln(1+b^2)}{2} - \frac{\ln 2}{2} \\
&= \infty
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ *diverges by the Integral Test.*

Problem 2: Find the total area of the infinitely many circles on the interval $[0,1]$, as show in the figure below.



Solution. *Observe the radius of the first circle is* $r_1 = \frac{1-1/2}{2} = \frac{1}{4} = \frac{1}{2^2}$. *The radius of the second circle is* $r_2 = \frac{1/2-1/4}{2} = \frac{1}{8} = \frac{1}{2^3}$. *The radius of the third circle is* $r_3 = \frac{1/4-1/8}{2} = \frac{1}{16} = \frac{1}{2^4}$. *Therefore, the radius of the* n^{th} *circle is* $r_n = \frac{1}{2} \cdot \frac{1}{2^n}$. *Therefore, the sum of the areas of all the circles is*

$$A = \sum_{n=1}^{\infty} \pi r^2 = \sum_{n=1}^{\infty} \pi \left(\frac{1}{2} \cdot \frac{1}{2^n}\right)^2 = \sum_{n=1}^{\infty} \frac{\pi}{4} \cdot \frac{1}{2^{2n}} = \sum_{n=1}^{\infty} \frac{\pi}{4} \cdot \frac{1}{4^n} = \sum_{n=1}^{\infty} \frac{\pi}{4} \cdot \left(\frac{1}{4}\right)^n$$

This series is geometric with $r = 1/4$. *Since* $|r| = 1/4 < 1$, *the series converges (so that the total area is finite). Therefore,*

$$A = \sum_{n=1}^{\infty} \pi r^2 = \sum_{n=1}^{\infty} \frac{\pi}{4} \cdot \left(\frac{1}{4}\right)^n = \frac{\pi/4 \cdot 1/4}{1-\frac{1}{4}} = \frac{\pi/16}{3/4} = \frac{\pi}{12}$$