Name: *Caleb McWhorter — Solutions* **MAT 296: HW 9**
Spring 2018 **Due: 03/30**

Problem 1: Use the Comparison Test or Limit Comparison Test to decide whether the following series converge or diverge. Be sure to justify your answer.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

$$\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}} \geq \sum_{n=2}^{\infty} \frac{1}{n}$$

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ *diverges by the p-test. Therefore,* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ *diverges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{1}{n-\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n-\sqrt{n}} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{\sqrt{n}}} = 1$$

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ *diverges by the p-test. Therefore,* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ *diverges by the Limit Comparison Test.*

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+5}$

$$\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+5} < \sum_{n=1}^{\infty} \frac{n^2}{3n^4} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+5}$ *converges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{n^2-1}{3n^4+5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4-n^2}{3n^4+5} = \frac{1}{3}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+5}$ *converges by the Limit Comparison Test.*

(iii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ *converges by the p-test. Therefore, the series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ *converges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3+2}}}{\frac{1}{\sqrt{n^3}}} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+2}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+2}} = \sqrt{1} = 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ *converges by the p-test. Therefore, the series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ *converges by the Limit Comparison Test.*

(iv) $\sum_{n=1}^{\infty} \frac{3n^2-2n-1}{n^3+n+1}$

Note that $-2n-1 = -(2n+1)$ *and that* $2n+1 < 2n^2$ *for* $n > 1$. *[To see this, note that let* $f(x) = 2x^2$ *and* $g(x) = 2x+1$. *Now* $f(2) = 8$ *and* $g(2) = 5$ *and* $f'(x) = 4x > 2 = g'(x)$ *for* $x > 1$.*] The series* $\sum_{n=1}^{\infty} \frac{3n^2-2n-1}{n^3+n+1}$ *converges if and only if the series* $\sum_{n=2}^{\infty} \frac{3n^2-2n-1}{n^3+n+1}$ *converges.*

$$\sum_{n=2}^{\infty} \frac{3n^2-2n-1}{n^3+n+1} > \sum_{n=2}^{\infty} \frac{3n^2-2n^2}{n^3+n^3+n^3} = \sum_{n=2}^{\infty} \frac{n^2}{3n^3} = \frac{1}{3}\sum_{n=2}^{\infty} \frac{1}{n}$$

The series $\sum_{n=2}^{\infty} \frac{1}{n}$ *diverges by the p-test. Therefore,* $\sum_{n=2}^{\infty} \frac{3n^2-2n-1}{n^3+n+1}$ *diverges by the Comparison Test so that* $\sum_{n=1}^{\infty} \frac{3n^2-2n-1}{n^3+n+1}$ *diverges.*

OR

$$\lim_{n\to\infty} \frac{\frac{3n^2-2n-1}{n^3+n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^3-2n^2-n}{n^3+n+1} = 3$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by the p-test. Therefore,* $\sum_{n=1}^{\infty} \frac{3n^2-2n-1}{n^3+n+1}$ *diverges by the Limit Comparison Test.*

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{n^4}$

$$\sum_{n=1}^{\infty} \frac{1+\cos n}{n^4} < \sum_{n=1}^{\infty} \frac{1+1}{n^4} = 2\sum_{n=1}^{\infty} \frac{1}{n^4}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ *converges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{n^4}$ *converges by the Comparison Test.*

(vi) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3^n+1}$

$$\sum_{n=1}^{\infty} \frac{5}{3^n+1} < \sum_{n=1}^{\infty} \frac{5}{3^n} = \sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$$

The series $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$ *is geometric with* $r = \frac{1}{3}$ *so that the series converges. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3^n+1}$ *converges by the Comparison Test.*

<u>OR</u>

$$\lim_{n\to\infty} \frac{\frac{5}{3^n+1}}{5\left(\frac{1}{3}\right)^n} = \lim_{n\to\infty} \frac{3^n}{3^n+1} = 1$$

The series $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$ *is geometric with* $r = \frac{1}{3}$ *so that the series converges. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3^n+1}$ *converges by the Limit Comparison Test.*

(vii) $\sum_{n=1}^{\infty} \frac{\ln n}{n+5}$

Since $\ln 1 = 0$, $\sum_{n=1}^{\infty} \frac{\ln n}{n+5} = \sum_{n=2}^{\infty} \frac{\ln n}{n+5}$.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n+5} > \sum_{n=2}^{\infty} \frac{1}{n+5n} = \frac{1}{6} \sum_{n=2}^{\infty} \frac{1}{n}$$

The series $\sum_{n=2}^{\infty} \frac{1}{n}$ *diverges by the p-test. Therefore,* $\sum_{n=1}^{\infty} \frac{\ln n}{n+5}$ *diverges by the Comparison Test.*

(viii) $\sum_{n=1}^{\infty} \frac{n}{(n^2+3)^2}$

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+3)^2} < \sum_{n=1}^{\infty} \frac{n}{(n^2)^2} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ *converges by the p-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n}{(n^2+3)^2}$ *converges by the Comparison Test.*

<u>OR</u>

$$\lim_{n\to\infty} \frac{\frac{n}{(n^2+3)^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^4}{(n^2+3)^2} = \lim_{n\to\infty} \frac{n^4}{n^4+6n^2+9} = 1$$

or of course one could have computed the limit as

$$\lim_{n\to\infty} \frac{\frac{n}{(n^2+3)^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^4}{(n^2+3)^2} = \lim_{n\to\infty} \frac{(n^2)^2}{(n^2+3)^2} = \lim_{n\to\infty} \left(\frac{n^2}{n^2+3}\right)^2 = 1^2 = 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ *converges by the p-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n}{(n^2+3)^2}$ *converges by the Limit Comparison Test.*

(ix) $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} \frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \cdot \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \cdot 1 = 1$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$ *converges by the Limit Comparison Test.*

(x) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n+\sqrt{2n^2+5}}$

$$\sum_{n=1}^{\infty} \frac{6}{n+\sqrt{2n^2+5}} > \sum_{n=1}^{\infty} \frac{6}{n+\sqrt{2n^2+5n^2}} = \sum_{n=1}^{\infty} \frac{6}{n+\sqrt{7n^2}} = \sum_{n=1}^{\infty} \frac{6}{n+\sqrt{7}n} = \frac{6}{1+\sqrt{7}} \sum_{n=1}^{\infty} \frac{1}{n}$$

or if one wanted things to look a bit nicer, one could do...

$$\sum_{n=1}^{\infty} \frac{6}{n+\sqrt{2n^2+5}} > \sum_{n=1}^{\infty} \frac{6}{n+\sqrt{2n^2+7n^2}} = \sum_{n=1}^{\infty} \frac{6}{n+\sqrt{9n^2}} = \sum_{n=1}^{\infty} \frac{6}{n+3n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

In either case, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n+\sqrt{2n^2+5}}$ *diverges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{6}{n+\sqrt{2n^2+5}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{6n}{n+\sqrt{2n^2+5}} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{6}{1+\frac{\sqrt{2n^2+5}}{n}} = \lim_{n\to\infty} \frac{6}{1+\sqrt{\frac{2n^2+5}{n^2}}} = \frac{6}{1+\sqrt{2}}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n+\sqrt{2n^2+5}}$ *diverges by the Limit Comparison Test.*

(xi) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\ln n}{n^4+5}$

$$\sum_{n=1}^{\infty} \frac{n^2+\ln n}{n^4+5} < \sum_{n=1}^{\infty} \frac{n^2+n^2}{n^4} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the p-test. Therefore, the series* $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\ln n}{n^4+5}$ *converges by the Comparison Test.*

(xii) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}}$

$$\sum_{n=1}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}} < \sum_{n=1}^{\infty} \frac{2n}{\sqrt{3n^6-2n^6}} = \sum_{n=1}^{\infty} \frac{2n}{n^3} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}}$ *converges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{2n-5}{\sqrt{3n^6-2}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^3-5n^2}{\sqrt{3n^6-2}} \cdot \frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{2-5/n}{\frac{\sqrt{3n^6-2}}{n^3}} = \lim_{n\to\infty} \frac{2-5/n}{\sqrt{\frac{3n^6-2}{n^6}}} = \frac{2}{\sqrt{3}}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}}$ *converges by the Limit Comparison Test.*

(xiii) $\displaystyle\sum_{n=1}^{\infty} \frac{7+4\sqrt[3]{n^4}}{n^2+2n+1}$

$$\sum_{n=1}^{\infty} \frac{7+4\sqrt[3]{n^4}}{n^2+2n+1} > \sum_{n=1}^{\infty} \frac{\sqrt[3]{n^4}}{n^2+2n^2+n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^4}}{n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ *diverges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{7+4\sqrt[3]{n^4}}{n^2+2n+1}$ *diverges by the Comparison Test.*

<u>OR</u>

$$\lim_{n\to\infty} \frac{\dfrac{7+4\sqrt[3]{n^4}}{n^2+2n+1}}{\dfrac{1}{n^{2/3}}} = \lim_{n\to\infty} \frac{7n^{2/3}+4n^2}{n^2+2n+1} = 4$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ *diverges by the p-test. Therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{7+4\sqrt[3]{n^4}}{n^2+2n+1}$ *diverges by the Limit Comparison Test.*