Quiz 1: Integrate the following: $\displaystyle\int \frac{dx}{1+x^2}$

Solution: $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$

x

Quiz 2: Integrate the following: $\displaystyle\int \frac{x^2+\sqrt{x}-\sqrt[3]{x}}{\sqrt{x}}\,dx$.

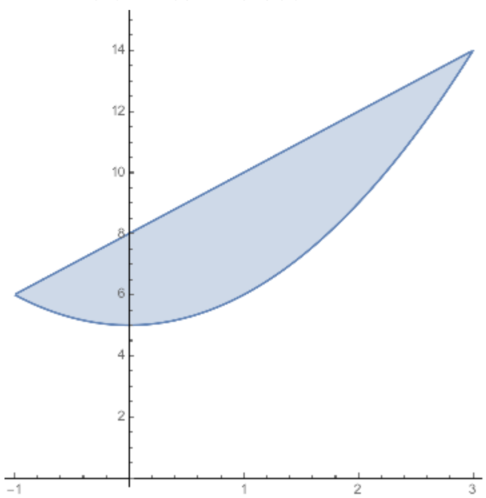
Solution:

$$\int \frac{x^2+\sqrt{x}-\sqrt[3]{x}}{\sqrt{x}}\,dx = \int \left(\frac{x^2}{\sqrt{x}}+1-\frac{\sqrt[3]{x}}{\sqrt{x}}\right)\,dx = \int \left(x^{3/2}+1-x^{-1/6}\right)\,dx = \frac{2}{5}x^{5/2}+x-\frac{6}{5}x^{5/6}+C$$

x

Quiz 3: Set up *but do not evaluate* an integral expression to find the area bound by the curves $y=x^2+5$ and $y=2x+8$.

Solution: Setting $x^2+5=2x+8$, we obtain $x^2-2x-3=0$. But $x^2-2x-3=(x-3)(x+1)$ so that the curves intersect at $x=-1$ and $x=3$. We plot the region below:



$$\int_{-1}^{3} (2x+8)-(x^2+5)\,dx$$

Quiz 4: Evaluate the following integral: $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$.

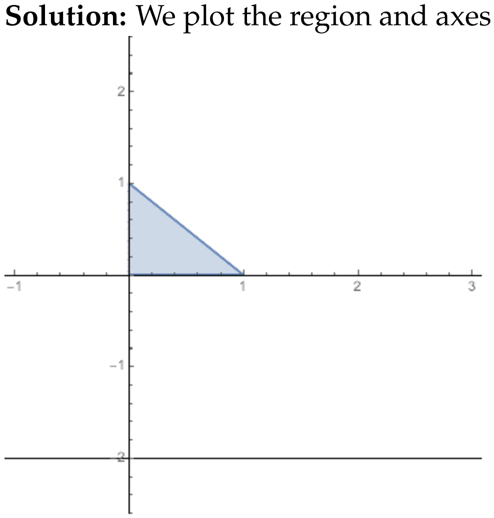
Solution: Let $u = e^x$ so that $du = e^x\,dx \iff dx = \dfrac{du}{e^x}$. Then we have

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{\cancel{e^x}}{1+u^2}\cdot\frac{du}{\cancel{e^x}} = \int \frac{du}{1+u^2} = \tan^{-1}(u) + C = \tan^{-1}(e^x) + C$$

x

Quiz 5: Set-up *but do not evaluate* an integral expression using *both the Disks/Washers and Shells Method* to find the volume resulting from revolving the region bounded by the x-axis, y-axis, and the line $y = 1 - x$ about the line $y = -2$.

Solution: We plot the region and axes of rotation below. Note that $y = 1 - x \iff x = 1 - y$.

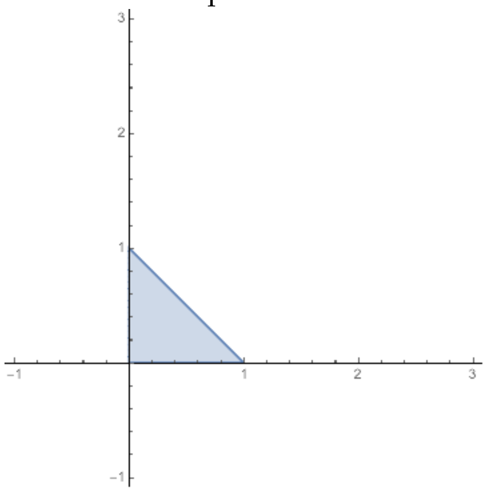


$$\text{Disks: } \pi\int_0^1 \big(2+(1-x)\big)^2 - \big(2+0\big)^2\,dx$$

$$\text{Shells: } 2\pi\int_0^1 (y+2)(1-y)\,dy$$

Quiz 6: Set-up *but do not evaluate* an integral expression for finding the volume of the solid whose base in the xy-plane is given by the lines $y = 0$, $x = 0$, and $y = 1 - x$ and cross-sections perpendicular to the x-axis are squares.

Solution: The region is plotted below. Recall that the area of a square is $A = s^2$. But the length of this side will depend on x. Then we have…



$$V = \int A(x)\, dx = \int_0^1 s(x)^2\, dx = \int_0^1 (1-x)^2\, dx$$

———— x ————

Quiz 7: Evaluate the following expression. [Hint: let $u = \sqrt{x}$.]

$$\int e^{\sqrt{x}}\, dx$$

Solution: Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}} \iff dx = 2\sqrt{x}\, du = 2u\, du$. Then we have…

$$\int e^{\sqrt{x}}\, dx = \int 2u\, e^u\, du$$

Quiz 8: Evaluate the following: $\int_0^{\pi} \sin^3\theta \, \cos^4\theta \; d\theta$

Solution: Let $u = \cos\theta$. Then $du = -\sin\theta \; d\theta$. If $\theta = 0$, then $u = 1$; if $\theta = \pi$, $u = -1$. Then we have...

$$\begin{aligned}
\int_0^{\pi} \sin^3\theta \, \cos^4\theta \; d\theta &= -\int_0^{\pi} \sin^2\theta \, \cos^4\theta \cdot (-\sin\theta) \; d\theta \\
&= -\int_0^{\pi} (1-\cos^2\theta) \, \cos^4\theta \cdot (-\sin\theta) \; d\theta \\
&= -\int_1^{-1} (1-u^2)u^4 \; du \\
&= \int_{-1}^{1} (u^4 - u^6) \; du \\
&= \left[\frac{u^5}{5} - \frac{u^7}{7} \right]_{-1}^{1} \\
&= \left(\frac{1}{5} - \frac{1}{7} \right) - \left(\frac{-1}{5} - \frac{-1}{7} \right) = 2\left(\frac{1}{5} - \frac{1}{7} \right) = \frac{4}{35}
\end{aligned}$$

Quiz 9: Evaluate the following: $\int \frac{x+2}{4x^2+1} \; dx$

Solution:

$$\int \frac{x+2}{4x^2+1} \; dx = \int \frac{x}{4x^2+1} \; dx + \int \frac{2}{4x^2+1} \; dx = \int \frac{x}{4x^2+1} \; dx + \int \frac{2}{(2x)^2+1} \; dx$$

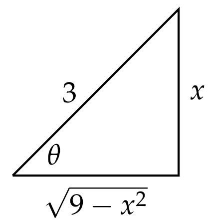
For the first integral, let $u = 4x^2 + 1$. Then $du = 8x \; dx \iff dx = \frac{du}{8x}$. For the second integral, let $v = 2x$. Then $dv = 2 \; dx \iff dx = \frac{dv}{2}$.

$$\begin{aligned}
\int \frac{x+2}{4x^2+1} \; dx &= \int \frac{x}{4x^2+1} \; dx + \int \frac{2}{(2x)^2+1} \; dx \\
&= \int \frac{\cancel{x}}{u} \frac{du}{8\cancel{x}} + \int \frac{\cancel{2}}{v^2+1} \frac{dv}{\cancel{2}} \\
&= \frac{1}{8} \int \frac{du}{u} + \int \frac{dv}{v^2+1} \\
&= \frac{\ln|u|}{8} + \tan^{-1}(v) + C \\
&= \frac{\ln|4x^2+1|}{8} + \tan^{-1}(2x) + C
\end{aligned}$$

Quiz 10: Evaluate the following: $\int \frac{dx}{\sqrt{9-x^2}}$

Solution:

$$a^2 + b^2 = c^2$$
$$b^2 = \underbrace{c^2 - a^2}_{9-x^2}$$



$$\sin\theta = \frac{x}{3}$$
$$x = 3\sin\theta$$
$$dx = 3\cos\theta\, d\theta$$

$$\cos\theta = \frac{\sqrt{1-x^2}}{3}$$
$$\sqrt{1-x^2} = 3\cos\theta$$

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta}{3\cos\theta}\, d\theta = \int 1\, d\theta = \theta + C = \sin^{-1}\left(\frac{x}{3}\right) + C$$

x

Quiz 11: For each of the following problems, circle the number of the answer indicating the correct partial fraction decomposition:

1.

$$\frac{x-7}{x^2(x+3)}$$

(i) $\frac{A}{x} + \frac{Bx+C}{x^2} + \frac{D}{x+3}$

(ii) $\frac{A}{x^2} + \frac{B}{x+3}$

(iii) $\frac{Ax+B}{x^2} + \frac{C}{x+3}$

(iv) $\frac{A}{x} + \frac{B}{x} + \frac{C}{x+3}$

(v) None of the above

The correct decomposition would be:

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

2.

$$\frac{x^2+x+17}{x^2(3x+5)^2}$$

(i) $\dfrac{A}{x^2}+\dfrac{B}{(3x+5)^2}$

(ii) $\boxed{\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{3x+5}+\dfrac{D}{(3x+5)^2}}$

(iii) $\dfrac{A}{x}+\dfrac{Bx+C}{x^2}+\dfrac{D}{3x+5}+\dfrac{Ex+F}{(3x+5)^2}$

(iv) $\dfrac{A}{x}+\dfrac{Bx+C}{x^2}+\dfrac{Dx+E}{3x+5}+\dfrac{Fx+G}{(3x+5)^2}$

(v) None of the above

3.

$$\frac{2x+13}{x^2(2x^2+5)^2}$$

(i) $\dfrac{A}{x^2}+\dfrac{B}{2x^2+5}+\dfrac{C}{(2x^2+5)^2}$

(ii) $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{2x^2+5}+\dfrac{D}{(2x^2+5)^2}$

(iii) $\boxed{\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{2x^2+5}+\dfrac{Ex+F}{(2x^2+5)^2}}$

(iv) $\dfrac{A}{x}+\dfrac{Bx+C}{x^2}+\dfrac{Dx+E}{2x^2+5}+\dfrac{Fx+G}{(2x^2+5)^2}$

(v) None of the above

Quiz 12: Evaluate the following: $\displaystyle\int \frac{3x-5}{x^2-x-12}\,dx$

Solution:

$$\int \frac{3x-5}{x^2-x-12}\,dx = \int \frac{3x-5}{(x+3)(x-4)}\,dx \qquad\qquad \frac{3x-5}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4}$$

$$A = \frac{3(-3)-5}{-3-4} = \frac{-14}{-7} = 2$$

$$B = \frac{3(4)-5}{4+3} = \frac{7}{7} = 1$$

$$\int \frac{3x-5}{x^2-x-12}\,dx = \int \left(\frac{2}{x+3} + \frac{1}{x-4}\right)\,dx = 2\ln|x+3| + \ln|x-4| + C$$

x

Quiz 13: Evaluate the following: $\displaystyle\int \frac{-3x^2+12x-5}{(x+6)(x^2+1)}\,dx$

Solution:

$$\frac{-3x^2+12x-5}{(x+6)(x^2+1)} = \frac{A}{x+6} + \frac{Bx+C}{x^2+1}$$

Using Heaviside's, we have

$$A = \frac{-3(-6)^2+12(-6)-5}{(-6)^2+1} = \frac{-108-72-5}{37} = \frac{-185}{37} = -5$$

If $x = 0$, we have...

$$\frac{-5}{6(1)} = \frac{A}{6} + \frac{C}{1}$$

which gives $-\frac{5}{6} = -\frac{5}{6} + C$ so that $C = 0$. If $x = 1$, we have...

$$\frac{2}{7} = \frac{A}{7} + \frac{B+C}{2}$$

which gives $\frac{2}{7} = -\frac{5}{7} + \frac{B}{2}$. Then $B = 2$. Therefore,

$$\int \frac{-3x^2+12x-5}{(x+6)(x^2+1)}\,dx = \int \frac{-5}{x+6} + \frac{2x}{x^2+1}\,dx$$
$$= -\ln|x+6| + \ln|x^2+1| + K$$

Quiz 14: Evaluate the following: $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Solution:

$$\begin{aligned}
\int_0^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} + \lim_{b\to 1^+}\int_b^3 \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big|_0^b + \lim_{b\to 1^+} 3(x-1)^{1/3}\Big|_b^3 \\
&= \left(\lim_{b\to 1^-} 3(b-1)^{1/3} - (-3)\right) + \left(3(2^{1/3}) - \lim_{b\to 1^+} 3(b-1)^{1/3}\right) \\
&= 0 + 3 + 3(2^{1/3}) - 0 \\
&= 3 + 3(2^{1/3}) \\
&= 3(1+\sqrt[3]{2})
\end{aligned}$$

x

Quiz 15: An astroid is a curve with equation $x^{2/3}+y^{2/3} = a^{2/3}$. Find the length of the astroid $x^{2/3}+y^{2/3}=1$ in Quadrant I.

Solution: If $x^{2/3}+y^{2/3}=1$, then (since we are in Quadrant I) $y=(1-x^{2/3})^{3/2}$. Then

$$\begin{aligned}
y' &= \frac{3}{2}(1-x^{2/3})^{1/2}\cdot -\frac{2}{3}x^{-1/3} = -\frac{(1-x^{2/3})^{1/2}}{x^{1/3}} \\
(y')^2 &= \frac{1-x^{2/3}}{x^{2/3}} = x^{-2/3}-1
\end{aligned}$$

Then the arclength of the curve in Quadrant I is...

$$\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+x^{-2/3}-1}\,dx = \int_0^1 x^{-1/3}\,dx = \frac{3}{2}\,x^{2/3}\Big|_0^1 = \frac{3}{2}$$

Quiz 16: If $y = f(x)$ is a curve containing the point $(0, 1)$ and

$$(3x^2 + 3x^2y^2)dx = dy,$$

then find $f(x)$.

Solution:

$$\begin{aligned}
(3x^2 + 3x^2y^2)dx &= dy \\
3x^2(1 + y^2)dx &= dy \\
3x^2\,dx &= \frac{dy}{1 + y^2} \\
\int 3x^2\,dx &= \int \frac{dy}{1 + y^2} \\
x^3 + C &= \arctan y \\
y &= \tan(x^3 + C)
\end{aligned}$$

Now since the curve contains the point $(0, 1)$, if $x = 0$ then $y = 1$.

$$\begin{aligned}
y &= \tan(x^3 + C) \\
1 &= \tan(0 + C) \\
1 &= \tan(C) \\
C &= \tan^{-1}(1) \\
C &= \frac{\pi}{4}
\end{aligned}$$

Therefore, $y = \tan(x^3 + \frac{\pi}{4})$.

Quiz 17: Find the limit of the following sequences:

(i) $\lim_{n\to\infty} \frac{3n^2 - n + 4}{5n^2 + 6n - 2}$

(ii) $\lim_{n\to\infty} n\sin(1/n)$

(iii) $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^{3n}$

Solution:

(i)

$$\lim_{n\to\infty} \frac{3n^2 - n + 4}{5n^2 + 6n - 2} = \frac{3}{5}$$

(ii)

$$\lim_{n\to\infty} n\sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$$

(iii)

$$\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^{3n} = \lim_{n\to\infty} \left(1 + \frac{1}{n/2}\right)^{n/2\cdot(2\cdot 3)} = \left[\lim_{n\to\infty} \left(1 + \frac{1}{n/2}\right)^{n/2}\right]^6 = e^6$$

x

Quiz 18: Determine if the following series converges or diverges. Justify your answer.

$$\sum_{n=0}^{\infty} \cos\left(\frac{3n-1}{4n+1}\right)$$

Solution:

$$\lim_{n\to\infty} \cos\left(\frac{3n-1}{4n+1}\right) = \cos\left(\lim_{n\to\infty} \frac{3n-1}{4n+1}\right) = \cos\left(\frac{3}{4}\right) \neq 0$$

Therefore, $\sum_{n=0}^{\infty} \cos\left(\frac{3n-1}{4n+1}\right)$ diverges by the Divergence Test.

Quiz 19: Here are four assertions about a sequence $\{\alpha_n\} = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$.

> **A:** $\lim_{n\to\infty} \alpha_n = 0$
>
> **B:** the sequence $\{\alpha_n\}$ either converges to a nonzero number or diverges.
>
> **C:** the series $\sum_{n=1}^{\infty} \alpha_n$ converges
>
> **D:** the series $\sum_{n=1}^{\infty} \alpha_n$ diverges.

Insert each of the letters A, B, C, and D exactly once in the blanks of the following two sentences to make them true statements.

(a) If ______ , then ______.

(b) If ______ , then ______.

Solution:

(a) If **C**, then **A**.

(b) If **B**, then **D**.

x

Quiz 20: Determine whether the following series converges or diverges. Be sure to justify your answer completely.

$$\sum_{n=3}^{\infty} \frac{n+1}{n^2-n-4}$$

Solution:

$$\sum_{n=3}^{\infty} \frac{n+1}{n^2-n-4} > \sum_{n=3}^{\infty} \frac{n}{n^2} = \sum_{n=3}^{\infty} \frac{1}{n}$$

The series $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges by the p-test. Therefore, $\sum_{n=3}^{\infty} \frac{n+1}{n^2-n-4}$ diverges by the Comparison Test.

Quiz 21: Determine whether the following series converges or diverges. Be sure to justify your answer completely.

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt[3]{n^2}}\right)$$

Solution: Observe

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{\sqrt[3]{n^2}}\right)}{\frac{1}{\sqrt[3]{n^2}}} = 1 < \infty.$$

Now the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-test. Therefore, $\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt[3]{n^2}}\right)$ converges by the Limit Comparison Test.

x

Quiz 22: Determine whether the given series is absolutely convergent, conditionally convergent, or divergent. Justify your answer completely.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n-1}}$$

Solution: The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n-1}}$ is alternating. The sequence $\left\{\frac{1}{\sqrt{n-1}}\right\}$ is decreasing and $\lim_{n\to\infty} \frac{1}{\sqrt{n-1}} = 0$. Therefore, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n-1}}$ converges by the Alternating Series Test. Now observe

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges by the p-test. Alternatively,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} \geq \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

and the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-test so that the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$ diverges by the Comparison Test. In any case, this means the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n-1}}$ converges conditionally.

Quiz 23: Determine whether the series converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{n=0}^{\infty} \frac{n^3 6^n}{n!}$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(n+1)^3 6^{n+1}}{(n+1)!} \cdot \frac{n!}{n^3 6^n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \cdot \frac{6^{n+1}}{6^n} \cdot \frac{n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^3 \cdot \frac{6^n \cdot 6}{6^n} \cdot \frac{n!}{(n+1)n!} \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^3 \cdot 6 \cdot \frac{1}{n+1} \right| \\
&= 1 \cdot 6 \cdot 0 = 0 < 1
\end{aligned}$$

Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^3 6^n}{n!}$ converges by the Ratio Test.

x

Quiz 24: Determine whether the series converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} \left(\frac{11n^2+n-1}{n-10n^2}\right)^n$$

Solution:

$$\lim_{n\to\infty} \left| \left(\frac{11n^2+n-1}{n-10n^2}\right)^n \right|^{1/n} = \lim_{n\to\infty} \left| \frac{11n^2+n-1}{n-10n^2} \right| = \frac{11}{10} > 1$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{11n^2+n-1}{n-10n^2}\right)^n$ diverges by the Root Test.

Quiz 25: Find the center, radius of convergence, and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n}$$

Solution:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\dfrac{x^{n+1}}{n+1}}{(-1)^n\dfrac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right| = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|$$

Since we need this ratio at most 1, $|x| < 1$. This implies $|x| < 1$ if and only if $-1 < x < 1$.

$x = 1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. Notice $\{\frac{1}{n}\}$ is a decreasing sequence and $\lim_{n\to\infty}\frac{1}{n} = 0$. Therefore, this series converges by the Alternating Series Test.

$x = -1$: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n}{n}\sum_{n=1}^{\infty}\frac{1}{n}$. This series diverges by the p-test.

Therefore, the interval of convergence is $(-1, 1]$, the radius of convergence is $R = \frac{1-(-1)}{2} = 1$, and the center of the power series is $x = 0$.