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MAT 222
Fall 2019
Homework 4

"Perfection is the enemy of perfectly adequate"
–Jimmy McGill, Better Call Saul

Problem 1: The state of Arizona is trying to determine if there has been an increase in SAT scores for HS students. In 2017, the average SAT score in Arizona was 1116 while in 2018 it was 1149. Researchers then take a SRS of 3,453 students and test the following hypothesis:

$$\begin{cases} H_0 : \mu = 1116 \\ H_a : \mu > 1116 \end{cases}$$

Which of the following would be a correct decision? Which of the following would be a Type I error? Which of the following would be a Type II error?

(a) Concluding that the mean SAT score has increased when it has.

(b) Concluding that the mean SAT score has not increased when it has.

(c) Concluding that the mean SAT score has not increased when it has not.

(d) Concluding that the mean SAT score has increased when it has not.

Solution. *(a) Correct Decision (b) Type II error (c) Correct Decision (d) Type I error*

Problem 2: What are the ways that you can increase the power of a test? What are the ways you can decrease the probability of a Type II error?

Solution. *You can increase the number of samples taken, n, you can decrease the standard deviation, σ, or increase the significance level, α (this corresponds to decreasing z^*). Because the probability of a Type II error is the compliment of power, you would need to do the opposite: decrease n, increase σ, or decrease the significance level α (this corresponds to increasing z^*).*

Problem 3: A chemical manufacturing company uses a machine which must inject 160 μg (micrograms) of a chemical component into one of their mixtures. The company notices that there seems to be an imbalance in their products. To see if this chemical is the source, they take a sample of 33 mixtures and find an average of 166 μg of the chemical in the product. The machines are known to inject the chemical into the mixture with $\sigma = 16$ μg. They test the following hypothesis at the 1% significance level

$$\begin{cases} H_0 : \mu = 160 \\ H_a : \mu > 160 \end{cases}$$

(a) What is the probability of a Type I error?

(b) Assuming H_0 is true, what is the probability that you fail to reject the null hypothesis?

(c) What is the power of the test if $\mu = 165$ μg?

(d) Suppose that $\mu = 170$ μg. Two different researchers perform a test on a SRS of 33 individuals from this population. What is the probability that both the experimenters would reject the given null hypothesis (assuming they use $\alpha = 0.05$)?

Solution.

(a) $P(\text{Type I error}) = \alpha = 0.01$

(b) $P(\text{Do Not Reject} \mid H_0 \text{ True}) = 1 - \alpha = 0.99$

(c) The power is the probability that we reject when the null hypothesis is false. Because $\alpha = 0.01$, the corresponding critical value is $z = 2.33$. But $z = 2.33 = \frac{\overline{x}-160}{16/\sqrt{33}}$ so that $\overline{x} = 166.49$. Therefore, we reject when $\overline{x} > 166.49$. If μ is actually 165 μg (meaning H_0 is false), then

$$z = \frac{166.49 - 165}{16/\sqrt{33}} = \frac{1.49}{2.78524} = 0.53 \rightsquigarrow 0.7019$$

Then we have $1 - 0.7019 = 0.2981$. Therefore, the power is 0.2981.

(d) We know $P(\text{Type II Error}) = 1 - \text{power} = 1 - 0.2981 = 0.7019$.

(e) The probability that they reject is 0.2981. Then the probability that they both reject is $0.2981 \cdot 0.2981 = 0.089$.

Problem 4: A consumer watchdog is testing a car company's claim that their new model car gets 42 mpg. In fact, the car company claims that the distribution of gas milage is $N(42, 1.9)$. They believe this to be an exaggeration. The group tests 20 cars and finds an average of 39 mpg. They will test the car company's claim using a significance level of $\alpha = 0.05$, and they will assume that the given standard deviation is correct.

(a) Write down an appropriate null and alternative hypothesis for the group.

(b) What is the probability of a Type I error?

(c) Assuming H_0 is true, what is the probability that you fail to reject the null hypothesis?

(d) What is the probability of a Type II error if $\mu = 40$ mpg.

(e) What is the power of the test if $\mu = 40$ mpg.

Solution.

(a)

$$\begin{cases} H_0 : \mu = 42 \\ H_a : \mu < 42 \end{cases}$$

(b) $P(\textit{Type I error}) = \alpha = 0.05$

(c) $P(\textit{Do Not Reject} \mid H_0 \textit{ True}) = 1 - \alpha = 0.95$

(d) A Type II error is when we fail to reject a false null hypothesis. Because $\alpha = 0.05$*, the corresponding critical value is* $z = -1.645$*. But* $z = -1.645 = \frac{\overline{x}-42}{1.7/\sqrt{20}}$ *so that* $\overline{x} = 41.37$*. Therefore, we reject when* $\overline{x} < 41.37$*, meaning we fail to reject when* $\overline{x} > 41.37$*. If* μ *is actually 40* μ*g (meaning* H_0 *is false), then*

$$z = \frac{41.37 - 40}{1.9/\sqrt{20}} = \frac{1.37}{0.424853} = 3.22 \rightsquigarrow 0.9994$$

so that we fail to reject with probability $1 - 0.9994 = 0.0006$*. Therefore,* $P(\textit{Type II Error}) = 0.0006$*.*

(e) We know Power $= 1 - P(\textit{Type II Error}) = 1 - 0.0006 = 0.9994$*.*

Problem 5: Suppose you want to test $H_0 : \mu = 124$ against $H_a : \mu \neq 124$ using $\alpha = 0.05$, and to do this you use a sample of size $n = 37$. You know also that $\sigma = 15$.

(a) What is the probability of a Type I error?

(b) Assuming H_0 is true, what is the probability that you fail to reject the null hypothesis?

(c) What is the probability of a Type II error if $\mu = 126$?

(d) What is the power of this test to detect $\mu = 126$?

(e) Give a reasonable explanation in terms of μ for the probabilities you found in (c) and (d).

Solution.

(a) $P(\text{Type I error}) = \alpha = 0.10$

(b) $P(\text{Do Not Reject} \mid H_0 \text{ True}) = 1 - \alpha = 0.90$

(c) A Type II error is when we fail to reject a false null hypothesis. Because $\alpha = 0.10$, the corresponding critical values are $z = -1.96$ and $z = 1.96$. But $z = \frac{\overline{x}-\mu}{\sigma/\sqrt{n}}$ so that we have $\overline{x} = 120.778$ and $\overline{x} = 127.222$. Therefore, we reject when $\overline{x} < 120.778$ or when $\overline{x} > 127.222$. Meaning, we fail to reject when $120.778 < \overline{x} < 127.222$. If μ is actually 130 μg (meaning H_0 is false), then

$$z_{128.833} = \frac{128.833 - 126}{15/\sqrt{37}} = \frac{2.833}{2.46598} = 1.15 \rightsquigarrow 0.8749$$
$$z_{119.167} = \frac{119.167 - 126}{15/\sqrt{37}} = \frac{-6.833}{2.46598} = -2.77 \rightsquigarrow 0.0028$$

so that we fail to reject with probability $0.8749 - 0.0028 = 0.8721$. Therefore, $P(\text{Type II Error}) = 0.8721$.

(d) We know that Power $= 1 - P(\text{Type II Error}) = 1 - 0.8749 = 0.1251$.

(e) The alternative μ is 'close' to the original μ (especially given the size of σ). Therefore, it is hard to detect the difference between them. So if $\mu = 126$, it will very likely be 'mistaken' for $\mu = 124$. Hence, the probability of a Type II error is high. Similarly, because it is difficult to 'tell the difference' between the two, the power is low.