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MAT 121
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Homework 8

"Alcohol is very very bad... for children. But when you turn 21 it becomes very very good."
–Turanga Leela, Futurama

Problem 1: Explain why, without appealing to algebra or formulas, $\binom{n}{0} = 1$.

Solution. *$\binom{n}{k}$ represents the number of combinations of choosing k objects from n objects. Therefore, $\binom{n}{0}$ is the number of ways of choosing 0 objects from n objects. There is only one way to do this—choose no objects!*

Problem 2: Explain why, without appealing to algebra or formulas, $\binom{n}{1} = n$.

Solution. *$\binom{n}{k}$ represents the number of combinations of choosing k objects from n objects. Therefore, $\binom{n}{1}$ is the number of ways of choosing 1 objects from n objects. But there are n choices—the n objects!*

Problem 3: Explain why, without appealing to algebra or formulas, $\binom{n}{k} = \binom{n}{n-k}$.

Solution. *$\binom{n}{k}$ represents the number of combinations of choosing k objects from n objects. But choosing k objects leaves $n-k$ objects unchosen. Choosing k objects is then the same as choosing $n-k$ objects that one does not want to choose. Therefore, the number of ways of choosing k objects from n objects is the same as the number of ways of choosing $n-k$ objects one does not want from n objects.*

Problem 4: Explain what the Central Limit Theorem says.

Solution. *The Central Limit Theorem states, that for sufficiently large n, the distribution of group averages for groups of size n is normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$, where μ is the mean of the underlying distribution and σ is the standard deviation for the underlying distribution.*

Problem 5: Suppose you have a distribution with mean μ and standard deviation σ. How does the sampling distribution for size $n = 100$ compare to the sampling distribution for size $n = 64$?

Solution. *Because $n \geq 30$, the Central Limit Theorem applies in both situations. The Central Limit Theorem then gives that the sampling distributions are $N(\mu, \sigma/\sqrt{100}) = N(\mu, \sigma/10)$ and $N(\mu, \sigma/\sqrt{64}) = N(\mu, \sigma/8)$, respectively. Therefore, both sampling distributions are normally distributed with mean μ (so that they have the same center) but the sampling distribution for $n = 100$ is narrower than the sampling distribution for $n = 64$.*

Problem 6: Suppose the scores for an exam are normally distributed with mean $\mu = 83$ and standard deviation $\sigma = 4$. Let X represent the score on this exam.

(a) What is $P(X \leq 80)$?

$$z_{80} = \frac{80 - 83}{4} = \frac{-3}{4} = -0.75 \rightsquigarrow 0.2266$$

Therefore, $P(X \leq 80) = 0.2266$.

(b) What percentage of students scored at most 80% on the exam? What percentage of students scored *less than* 80% on the exam?

This is precisely $P(X \leq 80)$*. Therefore,* 22.66% *of students scored at most 80%. The only difference between the first and second question is that a student could score 80%. But the probability of any particular value in a continuous distribution is 0. Therefore, this is again* 22.66%.

(c) What is the minimum score required to be in the top 22% of exam takers?

If you obtained the minimum possible score, then 78% of students scored less than you. But then the z*-value for your score must correspond to this 78%. This is a* z*-score of* $z = 0.77$*. Therefore,* $0.77 = \frac{x - 83}{4}$ *so that* $x = 86.08$.

(d) What is the probability that a student scores at 90% on the exam?

$$z_{90} = \frac{90 - 83}{4} = \frac{7}{4} = 1.75 \rightsquigarrow 0.9599$$

Therefore, the probability is $1 - 0.9599 = 0.0401$.

(e) What is the probability that a group of 15 students score an average of at most 80%?

The original distribution was normal. Therefore, the Central Limit Theorem applies.

$$z_{80} = \frac{80 - 83}{4/\sqrt{15}} = \frac{-3}{1.0328} = -2.90 \rightsquigarrow 0.0019$$

Therefore, the probability is 0.0019.

(f) Could you do (e) if the exam scores were not normally distributed?

No. We needed the Central Limit Theorem to find probabilities in the sampling distribution. Because $n = 15 < 30$*, we could not perform the computation from (e) if the original distribution was not normally distributed.*