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MAT 397— Fall 2020
Applied Problems:
Arclength & Curvature

"DNA is like a computer program but far, far more advanced than any software ever created."

–Bill Gates, The Road Ahead

DNA

Deoxyribonucleic acid (DNA) is a molecule consisting of two polynucleotide chains that coil around each other, forming a double helix. This helix contains the genetic information necessary for most forms of life (along with Ribonucleic acid, or RNA). The strands consist of simpler monomeric units called nucleotides—each composed of four nitrogen containing nucleobases: cytosine, guanine, adenine, and thymine. The helix structure allows DNA to split and be rematched with new half-helix strains of DNA to form another new full chain. This allows cells to multiply while still containing the same essential DNA. The replication of DNA in animals is so accurate that, on average, there is only one mistake for every 10 billion nucleotides replicated.

To give an idea of the scale of this accuracy, if you were to write every word in every major world language (over 80 languages) down 1,000 times, to achieve the same accuracy, you could misspell at most 2 of those words. The study of DNA is essential for DNA testing, medical treatments, Bioinformatics, DNA editing, etc. The radius of a human DNA helix is approximately 10 angstroms, i.e. 1 nanometer (10^{-9} meters)—though some measure DNA to be slightly larger. To complete one complete turn in the helix, DNA rises about 34 angstroms. Each strand of DNA contains about 2.9×10^8 turns in a single helix.

Problem:

(a) Find a parametrization for a single helix within a DNA molecule.

(b) Use your answer from (a) to find the length of a strand of DNA.

(c) If you could pull this DNA strand taunt, how long would it be? Give a representation of scale that an 'average person' could understand. [There are over 2.76 quadrillion strands of DNA in a human.]

(d) Compute the curvature of a molecule of DNA.

(e) Reconcile (c) and (d).

DNA image taken from https://www.livescience.com/37247-dna.html

Solution.

(a) We will work in meters so that in the end our answer will be immediately interpretable in (c). The radius of the helix is 10^{-9} meters. Then the circular part of the helix can be parametrized by $\langle 10^{-9}\cos t, 10^{-9}\sin t\rangle$. Each 2π interval in t represents a full turn of the helix, in which it must rise 3.4 nm. Therefore, we can parametrize the helix by

$$\mathbf{r}(t) = \left\langle 10^{-9}\cos t, 10^{-9}\sin t, \frac{3.4\cdot 10^{-9}}{2\pi}t \right\rangle$$

(b) There are 2.9$\times 10^8$ turns and each turn requires a 2π change in t. Therefore, the range for t should be $[0, 2.9\times 10^8\cdot 2\pi] = [0, 5.8\pi\cdot 10^8]$. Then we have

$$\begin{aligned}
L &= \int |\mathbf{r}'(t)|\, dt \\
&= \int_0^{5.8\pi\cdot 10^8} \left|\left\langle -10^{-9}\sin t, 10^{-9}\cos t, \frac{3.4\cdot 10^{-9}}{2\pi} \right\rangle\right| dt \\
&= \int_0^{5.8\pi\cdot 10^8} \sqrt{(-10^{-9}\sin t)^2 + (10^{-9}\cos t)^2 + (3.4\cdot 10^{-9}/(2\pi))^2}\, dt \\
&= \int_0^{5.8\pi\cdot 10^8} \sqrt{10^{-18} + 2.56/\pi^2\cdot 10^{-18}}\, dt \\
&= \int_0^{5.8\pi\cdot 10^8} \sqrt{10^{-18}(1 + 2.56/\pi^2)}\, dt \\
&= \frac{\sqrt{1+2.56/\pi^2}}{10^9}\int_0^{5.8\pi\cdot 10^8} dt \\
&= \frac{\sqrt{1+2.56/\pi^2}}{10^9}\cdot 5.8\pi\cdot 10^8 \\
&= \frac{5.8\pi\sqrt{1+2.56/\pi^2}}{10} \\
&= 0.58\sqrt{\pi^2+2.56} \approx 2.04483 \text{ meters}
\end{aligned}$$

(c) A length 2.04483 meters is about 6 ft 8.5 in. Meaning a single strand of human DNA, stretched taunt, would be taller than well over 99.999% of humans in the world. Then if you stretched every DNA strand in a human straight and laid them end to end, they would stretch 3.5069 trillion miles, or equivalently 0.59654 light years.

(d) Recall that $\kappa = \dfrac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}$. Now

$$\begin{aligned}
\mathbf{r}(t) &= \left\langle 10^{-9}\cos t, 10^{-9}\sin t, \frac{3.4\cdot 10^{-9}}{2\pi}t \right\rangle \\
\mathbf{r}'(t) = \mathbf{v} &= \left\langle -10^{-9}\sin t, 10^{-9}\cos t, \frac{3.4\cdot 10^{-9}}{2\pi} \right\rangle \\
\mathbf{r}''(t) = \mathbf{a} &= \left\langle -10^{-9}\cos t, -10^{-9}\sin t, 0 \right\rangle
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= \left\langle -10^{-9}\sin t, 10^{-9}\cos t, \frac{3.4 \cdot 10^{-9}}{2\pi} \right\rangle \times \langle -10^{-9}\cos t, -10^{-9}\sin t, 0 \rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10^{-9}\sin t & 10^{-9}\cos t & \dfrac{3.4 \cdot 10^{-9}}{2\pi} \\ -10^{-9}\cos t & -10^{-9}\sin t & 0 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 10^{-9}\cos t & \dfrac{3.4 \cdot 10^{-9}}{2\pi} \\ -10^{-9}\sin t & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -10^{-9}\sin t & \dfrac{3.4 \cdot 10^{-9}}{2\pi} \\ -10^{-9}\cos t & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -10^{-9}\sin t & 10^{-9}\cos t \\ -10^{-9}\cos t & -10^{-9}\sin t \end{vmatrix} \\
&= \left\langle \frac{3.4 \cdot 10^{-18}\sin t}{2\pi}, \frac{3.4 \cdot 10^{-18}\cos t}{2\pi}, 10^{-18} \right\rangle \\
&= \frac{1}{10^{18}} \left\langle \frac{3.4\sin t}{2\pi}, \frac{3.4\cos t}{2\pi}, 1 \right\rangle \\
&= \frac{1}{2\pi \cdot 10^{18}} \langle 3.4\sin t, 3.4\cos t, 2\pi \rangle
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathbf{v} \times \mathbf{a}\| &= \left\| \frac{1}{2\pi \cdot 10^{18}} \langle 3.4\sin t, 3.4\cos t, 2\pi \rangle \right\| \\
&= \frac{1}{2\pi \cdot 10^{18}} \|\langle 3.4\sin t, 3.4\cos t, 2\pi \rangle\| \\
&= \frac{1}{2\pi \cdot 10^{18}} \sqrt{11.56\sin^2 t + 11.56\cos^2 + 4\pi^2} \\
&= \frac{\sqrt{11.56 + 4\pi^2}}{2\pi \cdot 10^{18}}
\end{aligned}$$

Now

$$\begin{aligned}
\|\mathbf{v}\| &= \left\| \left\langle -10^{-9}\sin t, 10^{-9}\cos t, \frac{3.4 \cdot 10^{-9}}{2\pi} \right\rangle \right\| \\
&= \left\| 10^{-9} \left\langle -\sin t, \cos t, \frac{3.4}{2\pi} \right\rangle \right\| \\
&= \frac{1}{10^9} \left\| \left\langle -\sin t, \cos t, \frac{3.4}{2\pi} \right\rangle \right\| \\
&= \frac{1}{10^9} \sqrt{\sin^2 t + \cos^2 t + \frac{11.56}{4\pi^2}} \\
&= \frac{1}{10^9} \sqrt{1 + \frac{11.56}{4\pi^2}} \\
&= \frac{1}{10^9} \sqrt{\frac{4\pi^2 + 11.56}{4\pi^2}} \\
&= \frac{1}{2\pi \cdot 10^9} \sqrt{11.56 + 4\pi^2}
\end{aligned}$$

Then we have

$$\|\mathbf{v}\|^3 = \left(\frac{1}{2\pi \cdot 10^9} \sqrt{11.56 + 4\pi^2} \right)^3 = \left(\frac{1}{2\pi \cdot 10^9} \right)^3 (11.56 + 4\pi^2)^{3/2}$$

Therefore,

$$\begin{aligned}
\kappa &= \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} \\
&= \frac{\dfrac{\sqrt{11.56 + 4\pi^2}}{2\pi \cdot 10^{18}}}{\left(\dfrac{1}{2\pi \cdot 10^9}\right)^3 (11.56 + 4\pi^2)^{3/2}} \\
&= \frac{(2\pi \cdot 10^9)^3}{2\pi \cdot 10^{18}} \cdot \frac{(11.56 + 4\pi^2)^{1/2}}{(11.56 + 4\pi^2)^{3/2}} \\
&= \frac{(2\pi)^3 \cdot (10^9)^3}{2\pi \cdot (10^9)^2} \cdot \frac{1}{11.56 + 4\pi^2} \\
&= \frac{(2\pi)^2 \cdot 10^9}{1 \cdot 1} \cdot \frac{1}{11.56 + 4\pi^2} \\
&= \frac{4\pi^2 \cdot 10^9}{11.56 + 4\pi^2} \\
&\approx 7.73504 \cdot 10^8 \text{ m}^{-1}
\end{aligned}$$

(e) A large curvature makes sense because the DNA must wind very tight to be very small. After all, 2.04483 meters is a long strand of DNA to try to fit in a small space! Coiling very tightly would be an efficient way of making this possible.