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MAT 397— Fall 2020
Applied Problems:
Cross Product

"Give me a lever long enough and a fulcrum on which to place it, and I shall move the world."

–Archimedes

Torque

Torque is a 'twisting' or 'turning' force. Hence, torque is the rotational equivalent of a force. A torque can change the angular velocity of an object, i.e. speed up or slow down the rate of rotation, or cause twisting, bending, or other deformation in an object. Rotating objects have some axis of rotation, upon which torque acts. Therefore, torque must be defined about some rotational axis. Furthermore, because torque is a force applied at some point on an object, it must be defined at a point in addition to being defined relative to an axis of rotation. The magnitude of the torque depends on the size of the force applied and its perpendicular distance from the point of application to the axis of rotation. We experience this in real life—it is easier to push a door open nearer to the handle than it is closer to the hinges.

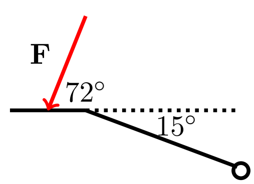
Furthermore, torque has a sign. Forces that rotate the object in the same direction produce torque with the same sign. For example, if you define counterclockwise rotation as positive and have an object rotating counterclockwise, then a torque which accelerates the object's rotation counterclockwise further would have a positive sign. We define torque, $\boldsymbol{\tau}$, as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

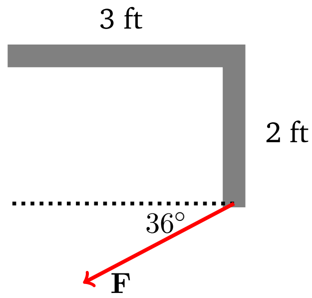
where $\mathbf{r}$ is the vector pointing from the center of rotation to the point of force application and $\mathbf{F}$ is the force vector. This means the amount of torque is given by $|\boldsymbol{\tau}| = |\mathbf{r}||\mathbf{F}| \sin\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$. You find torques in celestial dynamics, particle physics, analytical mechanics, component constructions in engineering, and much more.

Problem:

(a) A rotating arm is being pushed by a 60 N force, as given in the diagram below. If the arm is 16 cm long, find the magnitude of the torque about the anchor of the arm.

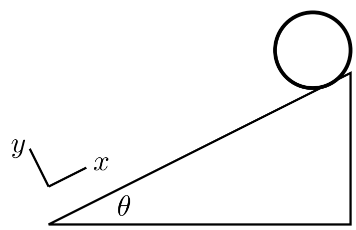


(b) A lever is being acted on by a 40 lb force, as shown in the diagram below. [The diagram is 2D—everything in the diagram lies in a single plane.] Find the magnitude of the torque at the upper left-most point on the lever.



(c) Torque has applications beyond simple rotational forces. We can use torque to find acceleration of objects. Once the acceleration of an object is known, its velocity and position at various moments in time can easily be found. This has immediate application in road construction.

As an example, suppose a spherical object with mass m, radius r, and moment of inertia I rolls without slipping[1] (there is friction) down an inclined plane at an angle of θ with respect to the horizontal. Choose coordinates as shown below



Gravity pulls the object straight downwards with magnitude $|\mathbf{F}| = ma$. Find the x-component of the force vector $\mathbf{F}$ due to gravity.

(d) Relating forces, we must then have

$$-mg\sin\theta + f = -ma$$

Considering rotation about the center of mass, we have $\tau = I\alpha$. But then using $\tau = Rf\sin 90^\circ = Rf$ and $a = \alpha R$. Using these equations, eliminate the friction and solve for the acceleration, a.

(e) For a solid sphere, $I = \dfrac{2}{5}MR^2$, where M is the mass and R is the radius. Assume that $g = 9.8$ m/s^2, $\theta = 30^\circ$, and that the ramp is 10 m long. Find the length of time for the sphere to go from rest at the top of the ramp to the bottom of the ramp.

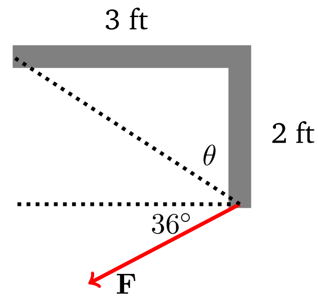
[1] This means that the object only moves when it rolls, and never just 'slides along' the ramp. Because the motion down the ramp is only caused by the rotation of the object, we know that $x = R\theta$. [Think of the case when $\theta = 2\pi$.]

Solution.

(a)

$$|\boldsymbol{\tau}| = | \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta = (0.16\text{ m})(60\text{ N})\sin(87^\circ) = 9.587\text{ N}\cdot\text{m}$$

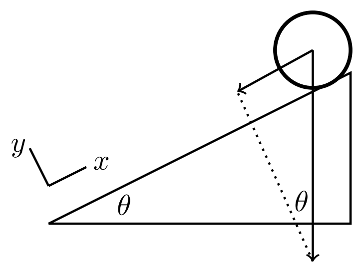
(b) We have $|\mathbf{r}| = \sqrt{2^2+3^2} = \sqrt{13}$ ft. We need to find the angle between $\mathbf{F}$ and $\mathbf{r}$.



We have $\theta = \arctan(3/2) \approx 0.982794$ rad $\approx 56.3^\circ$. Then the angle between $\mathbf{F}$ and $\mathbf{r}$ is $36^\circ + (90^\circ - 56.3^\circ) = 69.7^\circ$. Then we have

$$|\boldsymbol{\tau}| = |\mathbf{r}\times\mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta = (\sqrt{13}\text{ ft})(40\text{ lb})\sin(69.7^\circ) = 135.264\text{ ft}\cdot\text{lb}$$

(c) Draw the gravitational vector.



Applying trigonometry, we find that that the x-component of the gravity vector is $-mg\sin\theta$.

(d) We have the following system of equations:

$$\begin{aligned} -Mg\sin\theta + f &= -Ma \\ fR &= I\alpha \\ R\alpha &= a \end{aligned}$$

Multiplying the first equation by R, using the second equation, multiplying by another R, and using the third equation, we find

$$\begin{aligned} -mg\sin\theta + f &= -ma \\ -mrg\sin\theta + rf &= -rma \\ -mrg\sin\theta + I\alpha &= -rma \\ -mr^2g\sin\theta + rI\alpha &= -r^2ma \\ -mr^2g\sin\theta + Ia &= -r^2ma \\ -mr^2g\sin\theta &= -Ia - r^2ma \\ -mr^2g\sin\theta &= -a(I + r^2m) \\ a &= \frac{mr^2g\sin\theta}{I + r^2m} \end{aligned}$$

(e) For the sphere,

$$a = \frac{mr^2 g \sin\theta}{I + r^2 m} = \frac{mr^2 g \sin\theta}{\frac{2}{5}mr^2 + r^2 m} = \frac{5g\sin\theta}{7}$$

Then using $\theta = 30°$ and $g = 9.8$ m/s^2, we find that $a = 3.5$ m/s^2. Then using the fact that the initial velocity is 0, i.e. $v_0 = 0$, and that the sphere starts at the top of the ramp, i.e. $r_0 = 0$, we have

$$v(t) = \int a(t)\,dt = \int -3.5\,dt = -3.5t + v_0 = -3.5t$$

$$r(t) = \int -3.5t\,dt = -1.75t^2 + r_0 = -1.75t^2$$

Then to find the time down the ramp, we need solve $-1.75t^2 = -10$. Then $t = \sqrt{10/1.75} = 2.39$ seconds.