Name: *Caleb McWhorter— Solutions*

MAT 397— Fall 2020
Applied Problems:
Differentials

"A computer lets you make more mistakes faster than any other invention with the possible exceptions of handguns and Tequila."
–Mitch Ratcliffe

Error Analysis

All scientific measurements are subject to error and uncertainty, and the resulting errors are important. Any 'real life' computation that comes without a measurement of error is useless. For example, if you were installing an elevator in a building and told that it could hold up to 2500 lb, you might feel confident that the elevator will be safe because this is roughly fifteen 167 lb persons. But if an error analysis of this prediction was performed, and the actual predicted value was 2500 lb $\pm$ 1750 lb, then you have great reason to be concerned, as the elevator may fail after just five people enter it. The computation of errors is also critical for the Experimental Sciences. If you have a scientific theory in place which predicts a value of $\phi = 0.00132$ for some constant ϕ, does a scientist's measurement of 0.005 disprove the theory? You might question the theory based on the proposed value for ϕ if the scientist's computed value was 0.005 ± 0.00033, but not if the computed value were 0.005 ± 0.004 because ϕ lies in this interval.

There are many examples of both human/programming error and computational error costing money and lives: a small computational error in the design of the Mars Climate Orbiter resulted in the destruction of the over $100 million dollar satellite. In 1991, a patriot missile air system failed to defend American troops against incoming missiles because of compiled rounding errors. A mistake in bond calculation by Bank of America caused a $9 billion drop in stock.

Of course, there are examples of well done science and engineering preventing human or programming error from wreaking havoc. For example, good design of nuclear denotation instruments (though given the error, probably not the missile or facility itself) may have been the key factor in preventing a Damascus Titan nuclear missile from exploding on US soil when a maintenance worker dropped a wrench that pierced the missile's fuel cells. The explosion in Arkansas would have killed millions, and the fallout would have killed people as far as NYC. These design securities also saved lives on many occasions when planes carrying nuclear weapons 'accidentally' dropped nuclear weapons onto US soil, e.g. near Seymour Johnson Air Force Base, Mather Air Force Base, Columbus Air Force Base, Whidbey Island, etc.

Error analysis is the subfield of Applied Mathematics that deals with these computational issues, and it is an absolute necessity for persons in the sciences. A broad understanding of not only the mathematical aspects of error analysis, but deeper understanding of computer analysis and errors (such as floating numbers) and human error (especially human behavior) should be understood to accurately and safely use computations and design systems. For accurate error analysis, Statistics and Probability are required. This is because while you may have a maximum/minimum errors from measurements, not all these numbers are equally likely. So to each range of values there is an attached probability of occurrence that needs to be taken into account. But useful error analysis can still be performed without the need to appeal to Probability or Statistics.

Problem:

(a) We denote error by δ, e.g. δw is the error in the measurement of w. One of the simplest notion of errors is simply a max/min analysis. Suppose you measure $x =$ 205 cm $\pm$ 7 cm and $y =$ 147 cm $\pm$ 13 cm. Let $S = x + y$ and $D = x - y$. Find the the best estimate of S and D, along with a maximum and minimum value for S and D. Use these to find the errors δS and δD.

(b) How does your computation in the previous part support an early measurement of error in error analysis, that given $w = x \pm y$, we have $\delta w \approx \delta x + \delta y$?

(c) Try to create a formula for the error, δw, if $w = xy$, where x and y are measured with error δx, δy, respectively. Do you notice any problems? Explain.

(d) Error alone is not usually the full story.[1] The magnitude of the error relative to the measurement is far more informative. So instead of absolute error, we often examine the *fractional uncertainty*, $\dfrac{\delta w}{|w_{\text{best}}|}$. If you measure $x =$ 15.0 g $\pm$ 2.4 g, find the fractional uncertainty in x.

(e) Suppose $w = xy$ with $x - \delta x, y - \delta y \geq 0$. Use the previous part, along with the notions of Calculus and fractional uncertainty, to show that

$$\frac{\delta w}{|w_{\text{best}}|} \approx \frac{\delta x}{|x_{\text{best}}|} + \frac{\delta x}{|x_{\text{best}}|}$$

(f) Use the previous part to find the best estimate of the distance traveled, along with its fractional and absolute error, if an object is measured to travel at constant velocity (in a fixed direction) $v =$ 981 km/hr $\pm$ 120 km/hr for a length of time $t =$ 3.2 hrs $\pm$ 0.13 hrs.

(g) Explain how you would use (e) to find the fractional uncertainty for $w = x^n$.

(h) Combine parts (b), (e), and (g) to find the best estimate and error in $w = xy + x^2$, where $x =$ 3 m $\pm$ 0.2 m and $y =$ 1.5 m $\pm$ 1 m.

(i) The previous parts are straightforward, in that they do not assume any quadrature (so they assume 'maximal' error, or an interdependence in errors) and only involve basic operations. However, error with functions behave differently. For example, suppose $w = \sin\theta$. How do you find the error δw given θ and its error, $\delta\theta$? This is not as simple as using the maximum and minimum θ values. Explain why the 'max/min' approach for error breaks down for $w = \sin\theta$. [Hint: Consider θ values near $\pi/2$.]

(j) Luckily, the notion of differentials allows us to compute the uncertainty from (i): if $q = q(x)$, then

$$\delta w \approx \underbrace{\left|\frac{dw}{dx}\right|}_{\text{Chance in } w \text{ per change in } x} \cdot \underbrace{\delta x}_{\text{Amount } x \text{ changes}} = \underbrace{\left|\frac{dw}{dx}\right| \cdot \delta x}_{\text{total change in } w}$$

Use this to find the best estimate and error in $w = 1500e^{2(1-x)}$, where $x =$ 0.95 $\pm$ 0.02.[2]

(k) Suppose you are designing an assembly line which fills oil drums to be sold and shipped. Each drum is constructed on an assembly line in a cylindrical shape with radius $r =$ 11.125 in $\pm$ 0.10 in. and height $h =$ 34.5 in $\pm$ 0.12 in. Find the best estimate for the volume of the drums. Use the method of differentials estimate the error in the volume.

[1] If New York State incorrectly predicts its budget by \$10 million, you might be furious. But this number is out of a typical yearly budget of \$85 billion. That would be predicting the budget a year in advance with more than 2% accuracy—can you predict all your income/expenditures within 2% a year in advance? This is exactly what the budget creation process tries to do, and it is indeed a difficult problem.

[2] This is a disguised interest computation. Can you see it?

(l) If oil is loaded into the drums at a measured rate of G in^3/min, write a formula to represent the time taken to fill the drum. Use this formula to find the estimate of the time taken to fill an oil drum if $G = 8700$ in^3/min $\pm$ 1300 in^3/min, and use differentials to compute its error. How does this answer compare to simply using the given value of G, the answer for V from (k), and the error method from (e)?

Solution.

(a) The best estimates are

$$S_{\text{best}} = 205 + 147 = 352 \text{ cm}$$
$$D_{\text{best}} = 205 - 147 = 58 \text{ cm}$$

The sum will be maximized/minimized when x, y are maximized/minimized, respectively. Therefore,

$$S_{\text{max}} = (205 + 7) + (147 + 13) = 212 + 160 = 372 \text{ cm}$$
$$S_{\text{min}} = (205 - 7) + (147 - 13) = 198 + 134 = 332 \text{ cm}$$

The difference will be maximized when x is maximized and y is minimized and vice versa for when the difference is minimized. Therefore,

$$D_{\text{max}} = (205 + 7) - (147 - 13) = 212 + 134 = 78 \text{ cm}$$
$$D_{\text{min}} = (205 - 7) - (147 + 12) = 198 + 160 = 38 \text{ cm}$$

This gives error $\delta S = (S_{\text{max}} - S_{\text{min}})/2 = 20$ cm and $\delta D = (D_{\text{max}} - D_{\text{min}})/2 = 20$ cm.

(b) We have error $\delta S = 20$ cm and $\delta D = 20$ cm. Using the estimate from error analysis, we should have error $\delta x + \delta y = 7 + 13 = 20$ cm. The terms are exactly the same! This is because if $w = x \pm y$, then

$$\begin{aligned} w &= (x \pm \delta x) \pm (y \pm \delta y) \\ &= x \pm \delta x \pm y \pm \delta y \\ &= (x \pm y) \pm (\delta x + \delta y) \end{aligned}$$

Because $x \pm y$ is our best estimate of w, the error in the measurement of w is $\delta w = \delta a + \delta y$. Of course, this is only an initial crude measurement of error, because it will often be the case that the error will be less than this error term.

(c) The best estimate is clearly $w = x \cdot y$. You might think $w_{\text{max}} = x_{\text{max}} \cdot y_{\text{max}}$, but this does not work! What if $x = -2 \pm 1$ and $y = -4 \pm 5$. This would give maximum $w_{\text{max}} = (-1)(1) = -1$. But this is less than even the best predicted value of $w = (-2)(-4) = 8$! So we clearly will have issues with this naïve method if have number(s) whose interval of possible values contains negative numbers or zero.

(d) The fractional uncertainty in x would be

$$\frac{\delta x}{|x_{\text{best}}|} = \frac{2.4}{15.0} = 0.16$$

Notice this measurement is also dimensionless (an added bonus). This comes from the fact that $x_{\text{max}} = x + \delta x$ and $x_{\text{min}} = x - \delta x$. Then the fractional errors are

$$\frac{x_{\text{max}} - x}{x} = \frac{(x + \delta x) - x}{x} = \frac{\delta x}{x}$$
$$\frac{x_{\text{min}} - x}{x} = \frac{(x - \delta x) - x}{x} = -\frac{\delta x}{x}$$

Because $\delta x \geq 0$, the fractional magnitude you are off by is then $\delta x/|x|$, which is precisely the fractional uncertainty.

(e) We know $w_{\text{best}} = x_{\text{best}} y_{\text{best}}$. Now for w_{max}, using fractional uncertainty, we know this is

$$\begin{aligned}
w_{\text{max}} &= (x_{\text{best}} + \delta x)(y_{\text{best}} + \delta y) \\
&= x_{\text{best}} \left(1 + \frac{\delta x}{x_{\text{best}}}\right) \cdot y_{\text{best}} \left(1 + \frac{\delta y}{y_{\text{best}}}\right) \\
&= x_{\text{best}}\, y_{\text{best}} \left(1 + \frac{\delta x}{x_{\text{best}}}\right)\left(1 + \frac{\delta y}{y_{\text{best}}}\right) \\
&= x_{\text{best}}\, y_{\text{best}} \left(1 + \frac{\delta x}{x_{\text{best}}} + \frac{\delta y}{y_{\text{best}}} + \frac{\delta x}{x_{\text{best}}}\frac{\delta y}{y_{\text{best}}}\right) \\
&\approx x_{\text{best}}\, y_{\text{best}} \left(1 + \frac{\delta x}{x_{\text{best}}} + \frac{\delta y}{y_{\text{best}}}\right) \\
&= w_{\text{best}} \left(1 + \frac{\delta x}{x_{\text{best}}} + \frac{\delta y}{y_{\text{best}}}\right)
\end{aligned}$$

where we have used the fact that if δx, δy are 'small' relative to x, y, respectively, then $\frac{\delta x}{x_{\text{best}}}\frac{\delta y}{y_{\text{best}}}$ is negligible. One can find a similar value if one computes w_{min}. Then examining the last line from the computation above, we can see that the fractional error is

$$\frac{\delta w}{|w_{\text{best}}|} \approx \frac{\delta x}{|x_{\text{best}}|} + \frac{\delta x}{|x_{\text{best}}|}$$

(f) We know that $d = vt$. Then $d_{\text{best}} = (981)(3.2) = 3139.2$ km. The fractional error is

$$\frac{\delta d}{|d_{\text{best}}|} \approx \frac{\delta v}{|v_{\text{best}}|} + \frac{\delta t}{|t_{\text{best}}|} = \frac{120}{981} + \frac{0.13}{3.2} = 0.122324 + 0.040625 = 0.1629$$

This gives absolute error

$$\delta d = \delta d \cdot \frac{d}{d} = d \cdot \frac{\delta d}{d} = 3139.2 \cdot 0.1629 = 511.4 \text{ km}$$

Therefore, we measure $d = 3139.2 \text{ km} \pm 511.4 \text{ km}$.

(g) Consider the case where $w = x^2$. We know $x^2 = x \cdot x$, then

$$\frac{\delta w}{|w_{\text{best}}|} \approx \frac{\delta x}{|x_{\text{best}}|} + \frac{\delta x}{|x_{\text{best}}|} = 2\,\frac{\delta x}{|x_{\text{best}}|}$$

Then for $w = x^3$, we know that $x^3 = x \cdot x^2$. Let $a = x^2$. Then $w = xa$, and using the formula above, we have

$$\frac{\delta w}{|w_{\text{best}}|} \approx \frac{\delta x}{|x_{\text{best}}|} + \frac{\delta a}{|a_{\text{best}}|} = \frac{\delta x}{|x_{\text{best}}|} + 2\,\frac{\delta x}{|x_{\text{best}}|} = 3\,\frac{\delta x}{|x_{\text{best}}|}$$

We can then quickly see that if $w = x^n$, we have

$$\frac{\delta w}{|w_{\text{best}}|} \approx n\,\frac{\delta x}{|x_{\text{best}}|}$$

A repeated analysis replacing x with $1/x$ allows us to extend this to any nonzero integer power of x via

$$\frac{\delta w}{|w_{\text{best}}|} \approx |n|\,\frac{\delta x}{|x_{\text{best}}|}$$

(h) Let $a = xy$, and $b = x^2$. Then we can write $w = a + b$. But then

$$w_{\text{best}} = 3(1.5) + 3^2 = 13.5 \text{ m}^2$$

$$a_{\text{best}} = 3(1.5) = 4.5 \text{ m}^2$$
$$b_{\text{best}} = 3^2 = 9 \text{ m}^2$$

$$\frac{\delta a}{a} = \frac{0.2}{3} + \frac{1}{1.5} = 0.7333$$
$$\delta a = a \cdot \frac{\delta a}{a} = 4.5 \cdot 0.7333 = 3.29985 \text{ m}^2$$

$$\frac{\delta b}{b} = \frac{0.2}{3} + \frac{0.2}{3} = 0.1333$$
$$\delta b = b \cdot \frac{\delta b}{b} = 9 \cdot 0.1333 = 1.1997 \text{ m}^2$$

$$\delta w = \delta a + \delta b = 3.29985 + 1.1997 = 4.50 \text{ m}^2$$
$$\frac{\delta w}{w} = \frac{4.50}{13.5} = 0.33$$

(i) Suppose $\theta = 89° \pm 2°$. One would think that we have $w_{\text{max}} = \sin(89° + 2°) = \sin(91°)$. But clearly, the maximum value is $\sin(90°) = 1$. The issue is that for a general function $w = f(x)$, given $x, \delta x$, the behavior of $f(x)$ on the interval $[x - \delta x, x + \delta x]$ may vary, i.e. the function may vary between increasing and decreasing. We need to understand the function in question, $f(x)$, much better to know what its maximum and minimum values on the interval $[x - \delta x, x + \delta x]$ will be.

(j)

$$w_{\text{best}} = 1500e^{2(1-0.95)} = 1657.76$$
$$w' = 1500e^{2(1-x)} \cdot -2 = -3000e^{2(1-x)}$$
$$\delta w = \left|-3000e^{2(1-0.95)}\right| \cdot 0.02 = 66.3103$$
$$\frac{\delta w}{w} = \frac{66.3103}{1657.76} = 0.04$$

(k) We know that $V = \pi r^2 h$. Then we know $V_{\text{best}} = \pi(11.125)^2 \cdot 34.5 = 13414.3 \text{ in}^3$. We can find the error using the methods given above:

$$\frac{\delta V}{V} = 2\frac{\delta r}{r} + \frac{\delta h}{h} = 2\frac{0.10}{11.125} + \frac{0.12}{34.5} = 0.0214558$$

Therefore, $\delta V = V \cdot \frac{\delta V}{V} = 13414.3 \cdot 0.0214558 = 287.815 \text{ in}^3$. Then $V = 13414.3 \text{ in}^3 \pm$

287.815 in^3. We can see if this matches up with the method from differentials.

$$\begin{aligned} dV &= 2\pi rh\ dr + \pi r^2\ dh \\ &= 2\pi(11.125)(34.5)\cdot 0.10 + \pi(11.125)^2 \cdot 0.12 \\ &= 241.16 + 46.66 \\ &= 287.82 \text{ in}^3 \end{aligned}$$

Observe up to rounding, the answers are equivalent.

(l) We would have time $T = V/G = \dfrac{\pi r^2 h}{G}$. Using the given values and differentials, we have

$$\begin{aligned} T_{\text{best}} &= \frac{\pi(11.125)^2(34.5)}{8700} = 1.54188 \text{ min} \\ dT &= \frac{2\pi rh}{G}\ dr + \frac{\pi r^2}{G}\ dh - \frac{\pi r^2 h}{G^2}\ dG \\ &= \frac{2\pi(11.125)(34.5)}{8700}\cdot 0.10 + \frac{\pi(11.125)^2}{8700}\cdot 0.12 - \frac{\pi(11.125)^2(34.5)}{8700^2}\cdot 1300 \\ &= 0.263478 \text{ min} \end{aligned}$$

Therefore, $T = 1.54188$ min $\pm\, 0.263478$ min. Using the methods above, we have $T = V/G$. Then

$$\begin{aligned} T_{\text{best}} &= \frac{13414.3}{8700} = 1.54187 \text{ min} \\ \frac{\delta T}{T} &= \frac{\delta V}{V} + \frac{\delta G}{G} \\ &= \frac{287.815}{13414.3} + \frac{1300}{8700} \\ &= 0.0214558 + 0.149425 \\ &= 0.170881 \end{aligned}$$

Then we have $\delta T = T \cdot \dfrac{\delta T}{T} = 1.54187 \cdot 0.170881 = 0.263476$ min. Once again, the answers are the same up to rounding. Then observe to fill 1,000 drums, you would require *at least* $1000(T \pm \delta T) = 1541.88$ min $\pm\, 263.476$ min to fill all the drums.