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MAT 397— Fall 2020
Applied Problems:
Spherical Coordinates

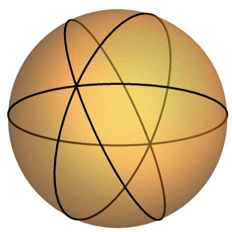
"I know every mile would be worth my while. When I go the distance, I'll be right where I belong."

–Hercules, Hercules

Measuring Global Distances

When measuring distances between places which are 'close' together, e.g. across a field or city, one can neglect the curvature of the Earth, and use the 'straight-line' distance. But for locations which are further apart the Earth's curvature has to be taken into account.

A 'straight line' on a sphere is a circle (see the figures below) because traveling in a straight line on the sphere will result in a path which is a circle. To measure distance on a sphere we use *great circles*, which are circles lying on the surface of a sphere whose center is the center of the sphere. For example, the circles on the sphere in the figure on the left below are great circles, while the circles on the sphere in the figure on the right below are *not* great circles (these are sometimes called *small circles*).



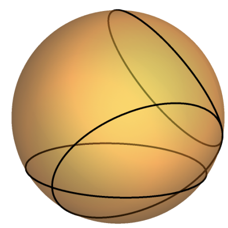


Figure 1. Some great circles on a sphere (left), and some small circles on a sphere (right).

Great and small circles are just the intersection of a sphere with a plane, the only difference being is in the case of great circles the plane passes through the center of the sphere. The distance between two points on a sphere is then the length of the shorter arc of the great circle connecting the two points. This arc is really the arc you would get if you 'flew' from one point 'straight' to the other point along the sphere.

We approximate the Earth as a sphere. We will use a spherical coordinate system $\{(\rho, \theta, \phi)\}$ on this spherical Earth using latitude and longitude (the Geographic coordinate system). To simplify things a tiny bit, we will only consider points in the Northern and Western hemispheres. Take the origin to be the center of the Earth, and the positive z-axis to pass through the North Pole. The positive x-axis will pass through where the prime meridian intersects the equator. Given this coordinate system, $\{(\rho, \theta, \phi)\}$, the latitude, La, and longitude, Lo, are given by $(\text{La}, \text{Lo}) = (90° - \phi, 360° - \theta)$. Assume the radius of the Earth is 3958.8 mi.

Problem:

(a) Find the spherical coordinates for Syracuse, NY (latitude $43.0469°$ N, longitude $76.1444°$ W) and Tucson, AZ (latitude $32.2217°$ N, longitude $110.9264°$ W).

(b) Convert the spherical coordinates for Syracuse, NY and Tucson, AZ to Cartesian coordinates.

(c) Find the angle between the two position vectors from (b).

(d) Find the distance between Syracuse, NY and Tucson, NY. [Hint: $s = \theta\, r$.]

(e) Use Google Maps to discuss whether your answer to (d) is reasonable.

(a) We simply use the definition of latitude and longitude along to find the spherical coordinates for the cities.

Syracuse, NY	Tucson, AZ
$\rho = 3958.8$ mi $\theta = 360° - 76.14 = 283.856°$ $\phi = 90° - 43.05 = 46.9531°$	$\rho = 3958.8$ mi $\theta = 360° - 110.93 = 249.074°$ $\phi = 90° - 32.22 = 57.7783°$
$(\rho, \theta, \phi) = (3958.8 \text{ mi}, 283.856°, 46.9531°)$	$(\rho, \theta, \phi) = (3958.8 \text{ mi}, 249.074°, 57.7783°)$

(b) We can use the conversion formulas from spherical to Cartesian coordinates.

Syracuse, NY	Tucson, AZ
$x = 3958.8 \sin(46.9531°) \cos(283.856°) = 692.82$ mi $y = 3958.8 \sin(46.9531°) \sin(283.856°) = -2808.89$ mi $z = 3958.8 \cos(46.9531°) = 2702.26$ mi	$x = 3958.8 \sin(57.7783°) \cos(249.074°) = -1196.2$ mi $y = 3958.8 \sin(57.7783°) \sin(249.074°) = -3128.2$ mi $z = 3958.8 \cos(57.7783°) = 2110.82$ mi
$(x, y, z) = (692.82 \text{ mi}, -2808.89 \text{ mi}, 2702.26 \text{ mi})$	$(x, y, z) = (-1196.2 \text{ mi}, -3128.2 \text{ mi}, 2110.82 \text{ mi})$

(c) We treat the coordinates as position vectors, i.e. vectors pointing from the center of the Earth to the cities.

$$\text{Syracuse, NY: } \mathbf{S} := \langle 692.82 \text{ mi}, -2808.89 \text{ mi}, 2702.26 \text{ mi} \rangle$$
$$\text{Tucson, AZ: } \mathbf{T} := \langle -1196.2 \text{ mi}, -3128.2 \text{ mi}, 2110.82 \text{ mi} \rangle$$

$$\mathbf{S} \cdot \mathbf{T} = \langle 692.82 \text{ mi}, -2808.89 \text{ mi}, 2702.26 \text{ mi} \rangle \cdot \langle -1196.2 \text{ mi}, -3128.2 \text{ mi}, 2110.82 \text{ mi} \rangle = 13662011.626214 \text{ mi}^2$$

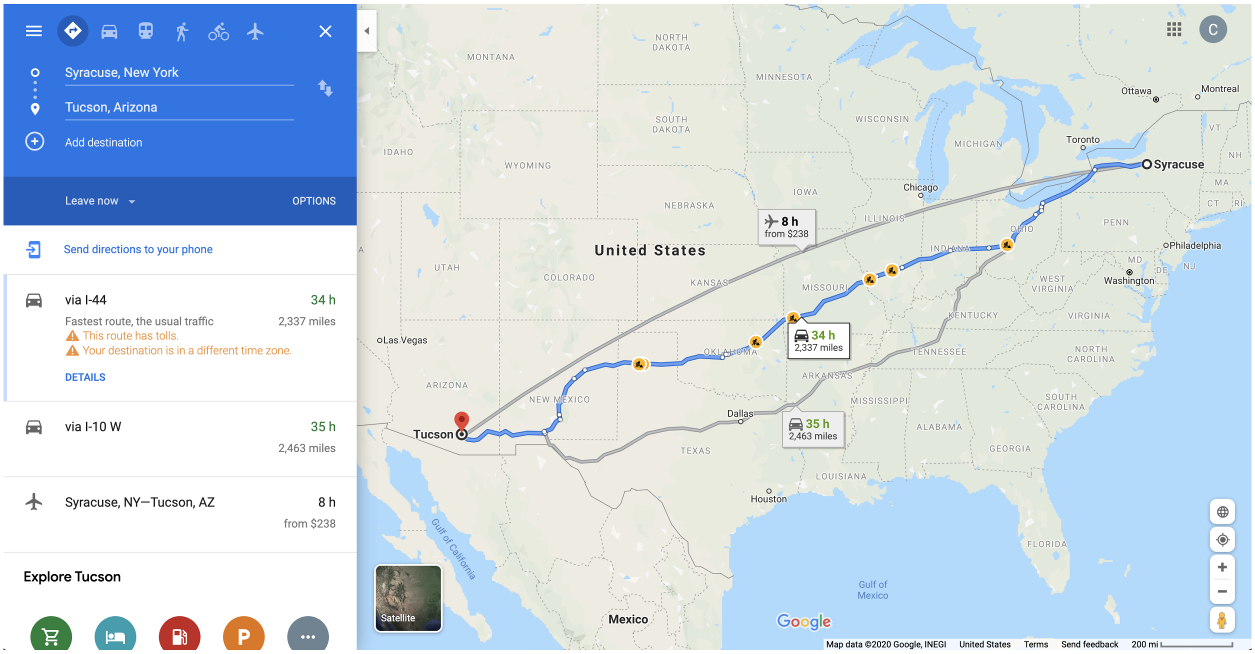
$$\begin{aligned} \mathbf{S} \cdot \mathbf{T} &= |\mathbf{S}|\,|\mathbf{T}| \cos\theta \\ 13662011.626214 \text{ mi}^2 &= (3958.8 \text{ mi})(3958.8 \text{ mi}) \cos\theta \\ \cos\theta &= \frac{13662011.626214 \text{ mi}^2}{15672097.44 \text{ mi}^2} \\ \cos\theta &= 0.871741 \\ \theta &= \arccos(0.871741) \approx 0.512052 \text{ radians} \end{aligned}$$

(d)

$$\text{Distance: } s = \theta\rho = (0.512052)3958.8 \text{ mi} = 2,027.11 \text{ mi}$$

(e) Certainly, the driving distance will have to be greater than the flying distance (which is essentially what we are measuring). Using GoogleMaps, we find that the fastest route is 2,337 miles, which is very close to our answer, and this driving is larger—as expected. So this answer seems reasonable.

Note that if you ask the website http://trippy.com, it tells you that the driving distance is 2,334 miles (essentially the same as Google), but it also tells you that the flight distance is 2,031 miles. We computed the 'flying along the ground' distance from Syracuse, NY to Tucson,



NY, so trippy's answer should be larger than ours—which it is. This makes us more confident in our answer.

In fact, we can go further. We could feel very confident that we computed correctly if we redid our calculation using the same method but instead using trippy's probable 'flying in the air' distance. To do this, we can add the height at which a plane would fly to our Earth radius (because this would be the distance from the flight path to the center of the Earth). Looking up figures, typically a plane flies between 5.9 miles and 7.2 miles in the air. We will use the average of the two figures—6.55 miles. Adding this to our ρ and performing the same computations, mutatis mutandis, we find a distance of 2,030.45 miles—nearly exactly that of trippy's!