

Name: \_\_\_\_\_

MAT 397— Fall 2020

Applied Problems:  
Green's Theorem

*“[Sheldon] But there’s some poor woman who’s gonna pin her hopes on my sperm. What if she winds up with a toddler who doesn’t know if he should use an integral or a differential to solve for the area under a curve? [Leonard] I’m sure she’ll still love him. [Sheldon] I wouldn’t.”*

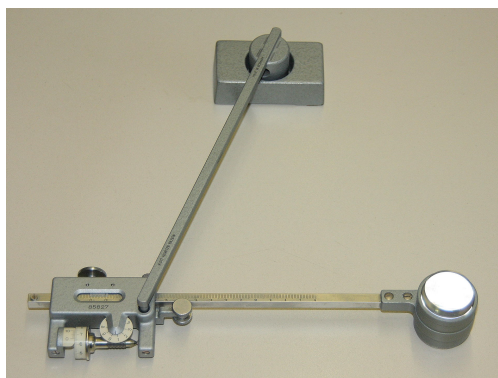
– Sheldon Cooper & Leonard Hofstadter *Big Bang Theory*

## Planimeters

Before there were computers or satellite images, a difficult problem was the calculation of integrals. Of course, if the definite integral could be computed directly, and then numerical techniques could be used to compute the final answer, e.g.

$$\int_0^{2\pi} x^3 e^x \sin^2 x \, dx = \frac{8}{625} (237e^{2\pi} (250\pi^3 - 525\pi^2 + 510\pi - 237)) \overset{\substack{\text{numerical} \\ \text{techniques}}}{\approx} 26976.3$$

However, all other integrals had to be computed numerically. While there were many numerical techniques available, these methods were still computationally expensive to perform. Even pre-computers, engineers, scientists, and mathematicians tried to find a way of ‘automatically’ computing these integrals. In 1814, J.M. Herman (and independently by Tito Gonnella in 1824) invented the planimeter. However, these early devices were bulky and hard to use. The Swiss mathematician Jakob Amsler-Laffon managed to build the first modern planimeter in 1854. There are many types of planimeters, but among the more common are polar planimeters—the kind developed by Amsler-Laffon. You can see an example of such a device in the image below.<sup>1</sup>



The planimeter consists of a two-bar linkage. At one end there is a pointer which is used to trace a closed curve, while the other arm allows the free movement of the device—though the base of the device remains fixed. The planimeter keeps track of the distance that a wheel fixed in place at the pointer rolls.<sup>2</sup> The planimeter then applies this information, using Vector Calculus, to display on an analog counter the total area enclosed by the curve.

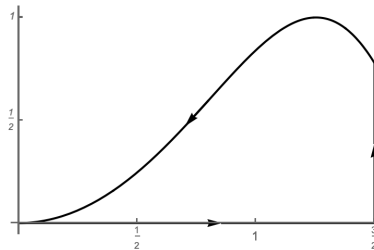
<sup>1</sup>Planimeter image by Schorle - Own work, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=7829947>

<sup>2</sup>The distance this wheel rolls is not the length of the curve! This is because the wheel is not free to move in every direction. So as you trace along the curve, sometimes the wheel moves, other times, it does not. An old computer mouse could keep track the mouse tracker ball rolled because it was free to move in every direction and the amount of motion was kept track by a laser. Obviously, this device is not so sophisticated. Otherwise, it would be much simpler to measure the area from the length of the curve by using Green's Theorem.

Consider the following definite integral that occurs frequently in optics—the so-called Fresnel integral

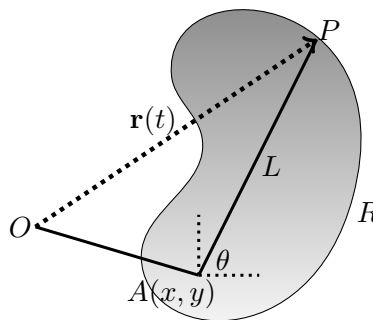
$$S(x) := \int_0^x \sin(t^2) dt$$

This integral has no elementary antiderivative. Therefore, numerical techniques are required to evaluate it. One way of doing this would be to sketch the function  $\sin(t^2)$  as accurately as possible for  $t \in [0, x]$ —let's say  $x = 3/2$ . This can be done very accurately by using rulers and carefully computing many values of  $\sin(t^2)$ . Then you construct a closed curve consisting of the line segment along the  $t$ -axis from 0 to  $3/2$ , followed by the vertical segment from  $(3/2, 0)$  to  $(3/2, \sin(9/4))$ , followed by the curve  $\sin(t^2)$ , then the vertical segment from  $(0, \sin(0))$  to  $(0, 0)$  (here that is the same point so no need for this final segment, but this 'left side' is generally required).<sup>3</sup> This is demonstrated in the figure below.



Because the value of the definite integral is the area under the curve, using Green's Theorem to compute this area, we have found the value of the integral using the planimeter! Of course, the challenge is to design the planimeter to keep track of the area being enclosed using only knowledge of the geometry of the curve being traced.

The planimeter was used not only integral computation but also in land surveying. It is not terribly difficult to compute areas of geographic regions that are rectangular/triangular (or ones which can be triangulated), but the area of general geographic regions is a difficult problem. However, surveying tools existed to create fairly reliable and accurate maps to scale. One could then trace the area of a particular region using a planimeter, hence computing its area. Then one would merely convert from the map area to the physical area.



So suppose that you have a region enclosed by a simple closed curve. You will move the planime-

<sup>3</sup>There is some work to be done if the function crosses the  $x$ -axis many times, in which case one need repeat the process above between each root of the function over the interval in question, and then add the results with the appropriate signs.

ter, carefully tracing the curve enclosing the region counterclockwise.<sup>4</sup> Suppose you trace out the curve in total time  $t_f - t_i$ , i.e.  $t_i \leq t \leq t_f$ . Let  $\mathbf{r}(t)$  denote the straight line segment from the fixed end of the planimeter, denoted  $O$ , to your current point  $P$  along the curve. Let  $L$  denote the fixed length of the tracer arm. The linkage arm will move along some fixed circle with center  $O$ . [To see this, think about how the device behaves as you move along the curve.] Let  $A(x(t), (y))$  denote the location of  $A$  at time  $t$ , and  $\theta(t)$  denote the angle the tracer arm makes with the horizontal. Note that  $A(x, y)$  moves along a circle of fixed radius—the length of the linkage arm. [This together with the fixed tracer arm length forces the region be ‘far’ from the planimeter but not ‘too far.’] There is a small wheel located at  $P$ , perpendicular to  $\overline{AP}$ , that rolls as you move  $P$  along the curve.

**Problem:**

(a) Show  $\vec{OP}$  given by

$$\mathbf{r}(t) = (x(t) + L \cos \theta(t)) \mathbf{i} + (y(t) + L \sin \theta(t)) \mathbf{j}$$

(b) Define

$$I_1 := \frac{1}{2} \int_{t_i}^{t_f} L^2 \frac{d\theta}{dt} dt$$

$$I_2 := \frac{1}{2} \int_{t_i}^{t_f} \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) dt$$

Show that  $I_1 = I_2 = 0$ . [Hint: The first should be easy to show. For the second, recognize the integrand as a derivative.]

(c) Define

$$I_3 := \frac{1}{2} \int_{t_i}^{t_f} L \left( y \frac{d\theta}{dt} \sin \theta + x \frac{d\theta}{dt} \cos \theta \right) dt$$

$$I_4 := \frac{1}{2} \int_{t_i}^{t_f} L \left( -\frac{dx}{dt} \sin \theta + \frac{dy}{dt} \cos \theta \right) dt$$

Show that  $I_3 = I_4$ . [Hint: Consider the integral  $\int_{t_i}^{t_f} \frac{d}{dt} (x(t) \sin \theta(t) - y(t) \cos \theta(t)) dt$ .]

(d) Let  $\mathbf{N}(\theta) = \langle -\sin \theta, \cos \theta \rangle$ , and let  $\mathbf{T}$  be the ordinary unit tangent vector to  $\mathbf{r}(t)$ . Denote by  $D$  the distance the wheel rolls. Justify the following:

$$D = \oint_C \mathbf{N} \cdot \mathbf{T} ds$$

[Hint: First, expand the integrand to find an integral in terms of  $\mathbf{N}$  and  $dx$ —do not ever write out  $\mathbf{r}$  explicitly throughout this part. Draw a picture to help picture how the wheel moves. Include in this picture  $\mathbf{r}'$  and a unit vector in the direction of the wheel. Then write down the distance the wheel moves at time  $t$ —you may find projections to be helpful!]

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<sup>4</sup>We assume the ‘mathematician direction.’ However, most planimeters are traced clockwise. But the derivation we perform carries over, mutatis mutandis.

(e) Show the following:

$$\text{area } R = I_1 + I_2 + I_3 + I_4 = DL$$

[Hint: To prove the right equality. Go back to the integral you wrote in the hint from (d). Expand the dot product using the definition of  $\mathbf{N}$  and  $\mathbf{r}$ . Show that the resulting integral is a multiple of  $I_4$ . Then use (b), (c), and (d). To prove the left equality, combine the integral  $I_1, I_2, I_3, I_4$  into a single integral. Then recall from Green's Theorem that area  $R = \frac{1}{2} \oint_C x dy - y dx$ .]

(f) Suppose you are a land surveyor and use a planimeter to find the area of a plot of land to be sold. You find an area of  $15.2 \text{ in}^2$ , and the map indicates that 1 in is 5.6 acres. If the land in this region is valued at \$1,800 per square acre, what price should this plot of land be valued at?