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MAT 397— Fall 2020
Applied Problems:
Green's Theorem

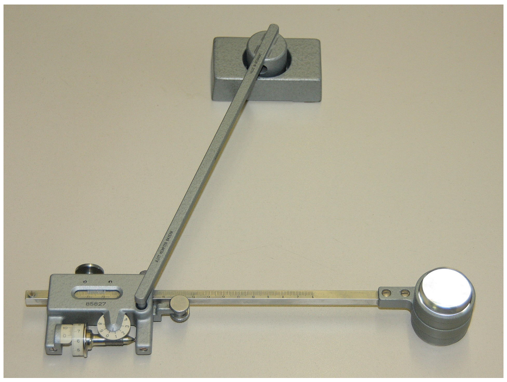
"[Sheldon] But there's some poor woman who's gonna pin her hopes on my sperm. What if she winds up with a toddler who doesn't know if he should use an integral or a differential to solve for the area under a curve? [Leonard] I'm sure she'll still love him. [Sheldon] I wouldn't."
–Sheldon Cooper & Leonard Hofstadter Big Bang Theory

Planimeters

Before there were computers or satellite images, a difficult problem was the calculation of integrals. Of course, if the define integral could be computed directly, and then numerical techniques could be used to compute the final answer, e.g.

$$\int_0^{2\pi} x^3 e^x \sin^2 x \, dx = \frac{8}{625}\left(237e^{2\pi}\left(250\pi^3 - 525\pi^2 + 510\pi - 237\right)\right) \xrightarrow[\text{techniques}]{\text{numerical}} \approx 26976.3$$

However, all other integrals had to be computed numerically. While there were many numerical techniques available, these methods were still computationally expensive to perform. Even pre-computers, engineers, scientists, and mathematicians tried to find a way of 'automatically' computing these integrals. In 1814, J.M. Herman (and independently by Tito Gonnella in 1824) invented the planimeter. However, these early devices were bulky and hard to use. The Swiss mathematician Jacok Amsler-Laffon managed to build the first modern planimeter in 1854. There are many types of planimeters, but among the more common are polar planimeters—the kind developed by Amsler-Laffon. You can see an example of such a device in the image below.[1]



The planimeter consists of a two-bar linkage. At one end there is a pointer which is used to trace a closed curve, while the other arm allows the free movement of the device—though the base of the device remains fixed. The planimeter keeps track of the distance that a wheel fixed in place at the pointer rolls.[2] The planimeter then applies this information, using Vector Calculus, to display on an analog counter the total area enclosed by the curve.

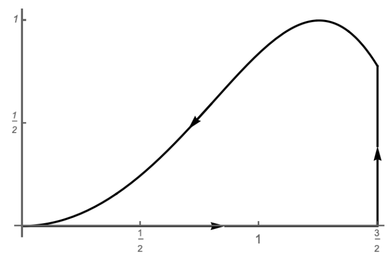
[1] Planimeter image by Schorle - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=7829947

[2] The distance this wheel rolls is not the length of the curve! This is because the wheel is not free to move in every direction. So as you trace along the curve, sometimes the wheel moves, other times, it does not. An old computer mouse could keep track the mouse tracker ball rolled because it was free to move in every direction and the amount of motion was kept track by a laser. Obviously, this device is not so sophisticated. Otherwise, it would be much simpler to measure the area from the length of the curve by using Green's Theorem.

Consider the following definite integral that occurs frequently in optics—the so-called Fresnel integral

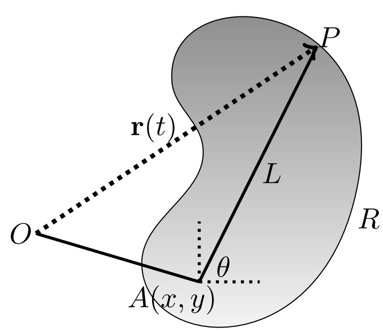
$$S(x) := \int_0^x \sin(t^2)\; dt$$

This integral has no elementary antiderivative. Therefore, numerical techniques are required to evaluate it. One way of doing this would be to sketch the function $\sin(t^2)$ as accurately as possible for $t \in [0, x]$—let's say $x = 3/2$. This can be done very accurately by using rulers and carefully computing many values of $\sin(t^2)$. Then you construct a closed curve consisting of the line segment along the t-axis from 0 to $3/2$, followed by the vertical segment from $(3/2, 0)$ to $(3/2, \sin(9/4))$, followed by the curve $\sin(t^2)$, then the vertical segment from $(0, \sin(0))$ to $(0, 0)$ (here that is the same point so no need for this final segment, but this 'left side' is generally required).[3] This is demonstrated in the figure below.



Because the value of the definite integral is the area under the curve, using Green's Theorem to compute this area, we have found the value of the integral using the planimeter! Of course, the challenge is to design the planimeter to keep track of the area being enclosed using only knowledge of the geometry of the curve being traced.

The planimeter was used not only integral computation but also in land surveying. It is not terribly difficult to compute areas of geographic regions that are rectangular/triangular (or ones which can be triangulated), but the area of general geographic regions is a difficult problem. However, surveying tools existed to create fairly reliable and accurate maps to scale. One could then trace the area of a particular region using a planimeter, hence computing its area. Then one would merely convert from the map area to the physical area.



So suppose that you have a region enclosed by a simple closed curve. You will move the planime-

[3]There is some work to be done if the function crosses the x-axis many times, in which case one need repeat the process above between each root of the function over the interval in question, and then add the results with the appropriate signs.

ter, carefully tracing the curve curve enclosing the region counterclockwise.[4] Suppose you trace out the curve in total time $t_f - t_i$, i.e. $t_i \le t \le t_f$. Let $\mathbf{r}(t)$ denote the straight line segment from the fixed end of the planimeter, denoted O, to your current point P along the curve. Let L denote the fixed length of the tracer arm. The linkage arm will move along some fixed circle with center O. [To see this, think about how the device behaves as you move along the curve.] Let $A(x(t),(y))$ denote the location of A at time t, and $\theta(t)$ denote the angle the tracer arm makes with the horizontal. Note that $A(x,y)$ moves along a circle of fixed radius—the length of the linkage arm. [This together with the fixed tracer arm length forces the region be 'far' from the planimeter but not 'too far.'] There is a small wheel located at P, perpendicular to $\overline{AP}$, that rolls as you move P along the curve.

Problem:

(a) Show $\vec{OP}$ given by

$$\mathbf{r}(t) = \big(x(t) + L\cos\theta(t)\big)\,\mathbf{i} + \big(y(t) + L\sin\theta(t)\big)\,\mathbf{j}$$

(b) Define

$$I_1 := \frac{1}{2}\int_{t_i}^{t_f} L^2\,\frac{d\theta}{dt}\,dt$$
$$I_2 := \frac{1}{2}\int_{t_i}^{t_f} \left(x\,\frac{dy}{dt} - y\,\frac{dx}{dt}\right)\,dt$$

Show that $I_1 = I_2 = 0$. [Hint: The first should be easy to show. For the second, recognize the integrand as a derivative.]

(c) Define

$$I_3 := \frac{1}{2}\int_{t_i}^{t_f} L\left(y\frac{d\theta}{dt}\sin\theta + x\frac{d\theta}{dt}\cos\theta\right)\,dt$$
$$I_4 := \frac{1}{2}\int_{t_i}^{t_f} L\left(-\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta\right)\,dt$$

Show that $I_3 = I_4$. [Hint: Consider the integral $\displaystyle\int_{t_i}^{t_f} \frac{d}{dt}\big(x(t)\sin\theta(t) - y(t)\cos\theta(t)\big)\,dt$.]

(d) Let $\mathbf{N}(\theta) = \langle -\sin\theta, \cos\theta\rangle$, and let $\mathbf{T}$ be the ordinary unit tangent vector to $\mathbf{r}(t)$. Denote by D the distance the wheel rolls. Justify the following:

$$D = \oint_C \mathbf{N}\cdot\mathbf{T}\,ds$$

[Hint: First, expand the integrand to find an integral in terms of $\mathbf{N}$ and $d\mathbf{r}$—do not ever write out $\mathbf{r}$ explicitly throughout this part. Draw a picture to help picture how the wheel moves. Include in this picture $\mathbf{r}'$ and a unit vector in the direction of the wheel. Then write down the distance the wheel moves at time t—you may find projections to be helpful!]

[4] We assume the 'mathematician direction.' However, most planimeters are traced clockwise. But the derivation we perform carries over, mutatis mutandis.

(e) Show the following:

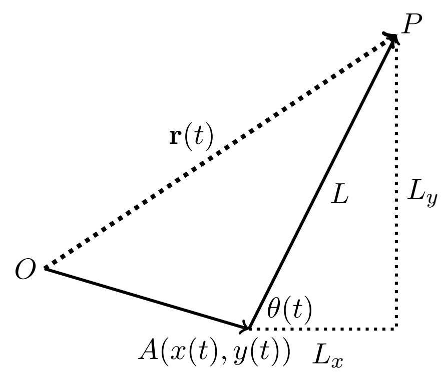
$$\text{area } R = I_1 + I_2 + I_3 + I_4 = DL$$

[Hint: To prove the right equality. Go back to the integral you wrote in the hint from (d). Expand the dot product using the definition of **N** and **r**. Show that the resulting integral is a multiple of I_4. Then use (b), (c), and (d). To prove the left equality, combine the integral I_1, I_2, I_3, I_4 into a single integral. Then recall from Green's Theorem that area $R = \frac{1}{2}\oint_C x\,dy - y\,dx$.]

(f) Suppose you are a land surveyor and use a planimeter to find the area of a plot of land to be sold. You find an area of 15.2 in^2, and the map indicates that 1 in is 5.6 acres. If the land in this region is valued at \$1,800 per square acre, what price should this plot of land be valued at?

Solution.

(a) Consider moving from O alone the linkage arm, then along the tracer arm. Clearly, this will take you to the point given by $\mathbf{r}(t)$. Treating $A(x, y)$ and L as vectors (in the case of L, the vector point from $A(x, y)$ to $\mathbf{r}(t)$, i.e. $\mathbf{r}(t) - A(x, y)$).



Now $L_x = L\cos\theta$ and $L_y = L\sin\theta$. But then, using the fact that $A(t) = (x, y)$, we have

$$\mathbf{r}(t) = A + L = \langle A_x + L_x, A_y + L_y\rangle = \langle x(t) + L\cos\theta(t), y(t) + L\sin\theta(t)\rangle$$

(b) Call the initial point and angle you start at along the curve (x_i, y_i) and θ_i, respectively, and call the final point and angle you start at along the curve (x_f, y_f) and θ_f, respectively. Observe that you will start and end tracing the curve at the same point. Hence, $(x_f, y_f) = (x_i, y_i)$ and $\theta_f = \theta_i$. But then

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{t_i}^{t_f} L^2\,\frac{d\theta}{dt}\,dt \\ &= \frac{L^2}{2}\int_{t_i}^{t_f} \frac{d\theta}{dt}\,dt \\ &= \frac{L^2}{2}\cdot\theta(t)\Big|_{t=t_i}^{t=t_f} \end{aligned}$$

$$
\begin{aligned}
&= \frac{L^2}{2}\big(\theta(t_f) - \theta(t_i)\big) \\
&= \frac{L^2}{2}\big(\theta_f - \theta_i\big) \\
&= 0
\end{aligned}
$$

and also

$$
\begin{aligned}
I_2 &= \frac{1}{2}\int_{t_i}^{t_f} \left(x\,\frac{dy}{dt} - y\,\frac{dx}{dt} \right)\, dt \\
&= \frac{1}{2}\int_{t_i}^{t_f} \frac{d}{dt}\big(x(t)y(t)\big)\, dt \\
&= \frac{1}{2}\cdot x(t)y(t)\bigg|_{t=t_i}^{t=t_f} \\
&= \frac{1}{2}\cdot \big(x(t_f)y(t_f) - x(t_i)y(t_i)\big) \\
&= \frac{1}{2}\cdot \big(x_f y_f - x_i y_i\big) \\
&= 0
\end{aligned}
$$

(c) First, observe that

$$
\begin{aligned}
\int_{t_i}^{t_f} \frac{d}{dt}\big(x(t)\sin\theta(t) - y(t)\cos\theta(t)\big)\, dt &= \big(x(t)\sin\theta(t) - y(t)\cos\theta(t)\big)\bigg|_{t=t_i}^{t=t_f} \\
&= \big(x(t_f)\sin\theta(t_f) - y(t_f)\cos\theta(t_f)\big) - \big(x(t_i)\sin\theta(t_i) - y(t_i)\cos\theta(t_i)\big) \\
&= \big(x_f\sin\theta_f - y_f\cos\theta_f\big) - \big(x_i\sin\theta_i - y_i\cos\theta_i\big) \\
&= \big(x_f\sin\theta_f - x_i\sin\theta_i\big) + \big(y_i\cos\theta_i - y_f\cos\theta_f\big) \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

But we also know that

$$
\begin{aligned}
\int_{t_i}^{t_f} \frac{d}{dt}\big(x(t)\sin\theta(t) - y(t)\cos\theta(t)\big)\, dt &= \int_{t_i}^{t_f} \big(x'(t)\sin\theta(t) + x(t)\theta'(t)\cos\theta(t) - y'(t)\cos\theta(t) \\
&\qquad + y(t)\theta'(t)\sin\theta(t)\big)\, dt \\
&= \int_{t_i}^{t_f} \big(x'(t)\sin\theta(t) - y'(t)\cos\theta(t)\big)\, dt \\
&\qquad + \int_{t_i}^{t_f} \big(x(t)\theta'(t)\cos\theta(t) + y(t)\theta'(t)\sin\theta(t)\big)\, dt
\end{aligned}
$$

But then using the two different computations of the same integral, we have

$$\int_{t_i}^{t_f} \big(x(t)\theta'(t)\cos\theta(t) + y(t)\theta'(t)\sin\theta(t)\big)\;dt = -\int_{t_i}^{t_f} \big(x'(t)\sin\theta(t) - y'(t)\cos\theta(t)\big)\;dt$$
$$\int_{t_i}^{t_f} \big(x(t)\theta'(t)\cos\theta(t) + y(t)\theta'(t)\sin\theta(t)\big)\;dt = \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t)\big)\;dt$$
$$\frac{1}{2}\int_{t_i}^{t_f} \big(x(t)\theta'(t)\cos\theta(t) + y(t)\theta'(t)\sin\theta(t)\big)\;dt = \frac{1}{2}\int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t)\big)\;dt$$
$$\frac{1}{2}\int_{t_i}^{t_f} \left(x\frac{d\theta}{dt}\cos\theta + y\frac{d\theta}{dt}\sin\theta\right)\;dt = \frac{1}{2}\int_{t_i}^{t_f} \left(-\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta\right)\;dt$$

(d) First, observe that

$$\begin{aligned}\oint_C \mathbf{N}\cdot\mathbf{T}\;ds &= \oint \mathbf{N}\cdot\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\|\mathbf{r}'(t)\|\;dt \\ &= \oint \mathbf{N}\cdot\mathbf{r}'(t)\;dt \\ &= \oint \mathbf{N}\cdot d\mathbf{r}\end{aligned}$$

Observe that the wheel does not rotate if $\mathbf{r}(t)$ moves in a direction parallel to $\vec{AP}$. When the direction of $\mathbf{r}(t)$ is perpendicular to $\vec{AP}$, the wheel moves just as much as the point P moved. For every other direction 'in-between' these two cases, the wheel moves some distance 'in-between' zero and the distance P moved.

Suppose we found a unit vector in a direction perpendicular to $\vec{AP}$ at time (t), say $\mathbf{v}(t)$. [For notational ease, we immediately drop the dependence of $\mathbf{v}$ on time.] Given our observations above, the amount the wheel moves is the 'amount of velocity', i.e. the 'amount' of $\mathbf{r}'$', that is in the direction of $\mathbf{v}$, times the amount of time the wheel moved in that direction, i.e. the infinitesimal time dt. The 'amount' of $\mathbf{r}'$ that lies in the direction of $\mathbf{v}$ is $\|\operatorname{proj}_{\mathbf{v}}\mathbf{r}'\|$. But

$$\|\operatorname{proj}_{\mathbf{v}}\mathbf{r}'\| = \left\|\frac{\mathbf{v}\cdot\mathbf{r}'}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right\| = \left\|\frac{\mathbf{v}\cdot\mathbf{r}'}{\|\mathbf{v}\|^2}\mathbf{v}\right\| = \left\|\frac{\mathbf{v}\cdot\mathbf{r}'}{1}\mathbf{v}\right\| = \|(\mathbf{v}\cdot\mathbf{r}')\mathbf{v}\| = |\mathbf{v}\cdot\mathbf{r}'|\,\|\mathbf{v}\| = |\mathbf{v}\cdot\mathbf{r}'|$$

But recall $\mathbf{v}\cdot\mathbf{r}' = |\mathbf{v}|\,|\mathbf{r}'|\cos\theta$. By the construction of the planimeter, we know that $0 \leq \theta \leq \frac{\pi}{2}$. Therefore, $\mathbf{v}\cdot\mathbf{r}' \geq 0$, so that $|\mathbf{v}\cdot\mathbf{r}'| = \mathbf{v}\cdot\mathbf{r}'$. But then the total distance the wheel moves is $\mathbf{v}\cdot\mathbf{r}'\;dt = \mathbf{v}\cdot d\mathbf{r}$. Then the total distance the wheel travels is

$$D = \oint \mathbf{v}\cdot\mathbf{r}'\;dt = \oint \mathbf{v}\cdot d\mathbf{r}$$

We now only need to find $\mathbf{v}$. The displacement vector from $A(x,y)$ to P must be $\langle L\cos\theta, L\sin\theta\rangle$, because to get from $A(x,y)$ to P, you travel in the direction given by θ a distance of L. We know $\|\langle L\cos\theta, L\sin\theta\rangle\| = L$, so that a unit vector pointing in the direction of $\vec{AP}$ is $\langle\cos\theta,\sin\theta\rangle$. But this vector is just a point on a circle, and we know a unit vector perpendicular to such a point is given by $\langle-\sin\theta,\cos\theta\rangle$. [You can also come up with this quickly by trial-and-error.] So we find $\mathbf{v} = \langle-\sin\theta,\cos\theta\rangle$. But notice that $\mathbf{v} = \mathbf{N}$! Therefore,

$$D = \oint \mathbf{v}\cdot d\mathbf{r} = \oint \mathbf{N}\cdot d\mathbf{r} = \oint_C \mathbf{N}\cdot\mathbf{T}\;ds$$

(e) First, observe that by using (b) and (c), we have

$$I_1 + I_2 + I_3 + I_4 = 0 + 0 + I_4 + I_4 = 2I_4$$

Now from (d), we know that

$$D = \oint_C \mathbf{N} \cdot \mathbf{T}\, ds = \oint \mathbf{N} \cdot d\mathbf{r}$$

But we know **N** and **r** as vectors. Computing the rightmost integral, we find

$$\begin{aligned}
D &= \oint \mathbf{N} \cdot d\mathbf{r} \\
&= \int_{t_i}^{t_f} \langle -\sin\theta(t), \cos\theta(t)\rangle \cdot \langle x'(t) - L\theta'(t)\sin\theta(t), y'(t) + L\theta'(t)\cos\theta\rangle\, dt \\
&= \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + L\theta'(t)\sin^2\theta(t) + y'(t)\cos\theta(t) + L\theta'(t)\cos^2\theta(t)\big)\, dt \\
&= \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t) + L\theta'(t)\sin^2\theta(t) + L\theta'(t)\cos^2\theta(t)\big)\, dt \\
&= \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t) + L\theta'(t)\big)\, dt \\
&= \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t)\big)\, dt + \int_{t_i}^{t_f} L\theta'(t)\, dt \\
&= \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t)\big)\, dt + L\int_{t_i}^{t_f} \theta'(t)\, dt \\
&= \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t)\big)\, dt + L\big(\theta(t_f) - \theta(t_i)\big) \\
&= \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t)\big)\, dt + L\big(\theta_f - \theta_i\big) \\
&= \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t)\big)\, dt \\
&= \frac{2}{L} \cdot \frac{L}{2} \int_{t_i}^{t_f} \big(-x'(t)\sin\theta(t) + y'(t)\cos\theta(t)\big)\, dt \\
&= \frac{2}{L} \cdot \frac{1}{2} \int_{t_i}^{t_f} L\left(-\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta \right)\, dt \\
&= \frac{2}{L}\, I_4
\end{aligned}$$

This shows that $I_4 = \dfrac{DL}{2}$. But then using our first computation

$$I_1 + I_2 + I_3 + I_4 = 2I_4 = 2 \cdot \frac{DL}{2} = DL$$

The only thing we have to show is that area $R = I_1 + I_2 + I_3 + I_4$. Green's Theorem gives

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\, dA$$

Choosing $M = -y/2$ and $N = x/2$, we find

$$\frac{1}{2}\oint_C x\ dy - y\ dx = \iint_R \left(\frac{1}{2} - \frac{-1}{2}\right)\ dA = \iint 1\ dA = \text{area } R$$

For our curve $\mathbf{r}$ and region R, we have

$$\begin{aligned}
\text{area } R &= \frac{1}{2}\oint_C x\ dy - y\ dx \\
&= \frac{1}{2}\int_{t_i}^{t_f} \big[(x + L\cos\theta)(y' + L\theta'\cos\theta) - (y + L\sin\theta)(x' - L\theta'\sin\theta)\big]\ dt \\
&= \frac{1}{2}\int_{t_i}^{t_f} \big(xy' + Lx\theta'\cos\theta + Ly'\cos\theta + L^2\theta'\cos^2\theta - (yx' - Ly\theta'\sin\theta + Lx'\sin\theta - L^2\theta'\sin^2\theta)\big)\ dt \\
&= \frac{1}{2}\int_{t_i}^{t_f} \big(xy' + Lx\theta'\cos\theta + Ly'\cos\theta + L^2\theta'\cos^2\theta - yx' + Ly\theta'\sin\theta - Lx'\sin\theta + L^2\theta'\sin^2\theta\big)\ dt \\
&= \frac{1}{2}\int_{t_i}^{t_f} \big(xy' + Lx\theta'\cos\theta + Ly'\cos\theta - yx' + Ly\theta'\sin\theta - Lx'\sin\theta + L^2\theta'\cos^2\theta + L^2\theta'\sin^2\theta\big)\ dt \\
&= \frac{1}{2}\int_{t_i}^{t_f} \big(xy' + Lx\theta'\cos\theta + Ly'\cos\theta - yx' + Ly\theta'\sin\theta - Lx'\sin\theta + L^2\theta'\big)\ dt \\
&= \frac{1}{2}\int_{t_i}^{t_f} \big(L^2\theta' + xy' - yx' + Ly\theta'\sin\theta + Lx\theta'\cos\theta - Lx'\sin\theta + Ly'\cos\theta\big)\ dt
\end{aligned}$$

Finally, observe that

$$\begin{aligned}
I_1 + I_2 + I_3 + I_4 &= \frac{1}{2}\int_{t_i}^{t_f} L^2\ \frac{d\theta}{dt}\ dt + \frac{1}{2}\int_{t_i}^{t_f}\left(x\,\frac{dy}{dt} - y\,\frac{dx}{dt}\right)\ dt + \frac{1}{2}\int_{t_i}^{t_f} L\left(y\frac{d\theta}{dt}\sin\theta + x\frac{d\theta}{dt}\cos\theta\right)\ dt \\
&\qquad + \frac{1}{2}\int_{t_i}^{t_f} L\left(-\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta\right)\ dt \\
&= \frac{1}{2}\int_{t_i}^{t_f} \big(L^2\theta' + xy' - yx' + Ly\theta'\sin\theta + Lx\theta'\cos\theta - Lx'\sin\theta + Ly'\cos\theta\big)\ dt
\end{aligned}$$

Therefore,

$$\text{area } R = I_1 + I_2 + I_3 + I_4 = DL$$

(f) This is mostly just a simple unit conversion:

$$15.2 \text{ in}^2 = 15.2 \text{ in} \cdot \text{ in} = 15.2 \text{ in} \cdot \text{ in} \cdot \frac{5.6 \text{ acres} \cdot 5.6 \text{ acres}}{1 \text{ in} \cdot 1 \text{ in}} = 476.672 \text{ acres}^2$$

Then the land should be priced for at least $\$1800/\text{acre}^2 \cdot 476.672 \text{ acres}^2 = \$858,009.60$.