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MAT 397— Fall 2020
Applied Problems:
Lagrange Multipliers

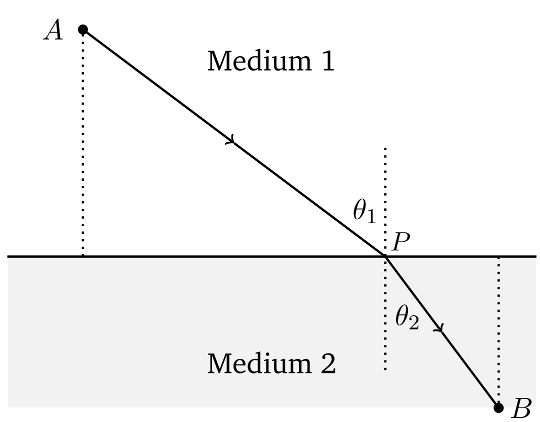
"I regarded as quite useless the reading of large treatises of pure analysis: too large a number of methods pass at once before the eyes. It is in the works of application that one must study them; one judges their utility there and appraises the manner of making use of them."
–Joseph-Louis Lagrange

Snell's Law

In the broadest sense, Physics is the study of matter. One might mistake this for Chemistry except that Physics is primarily interested in the forces between matter, and how this dictates the behavior of matter in space and time. Physics studies objects not only in the very small (Quantum Mechanics) and the very large (General Relativity) but also on 'ordinary' scales. The success of Physics is testified in the great progress in engineering in modern society.

Useful Physics can be done without any actual computations or numbers! For example, it is possible, using only a sheet of paper, some tape, and some careful thinking, to show that the drag force for objects moving at high speed follows $F_{\text{drag}} \sim \rho A v^2$, where ρ is the density of the medium, A is the area the drag force acts upon, and v is the velocity of the body. This shows that for objects moving at high speed the drag force is independent of the viscosity of the medium! Furthermore, this explains why, all other things equal, cars traveling at lower speeds get better gas milage than those traveling at higher speeds—they experience less drag!

While Physics can be done without numbers, as Kurt Lewin said, "there is nothing as practical as a good theory." Ultimately, we want to invoke physical observations or other empirical facts, and use the power of Mathematics, especially Calculus, to develop a theory which might explain (or at least predict) the forces on and motion of objects. For example, you may have experienced that when you reach down into clear water to grab something, where you see the object is not quite where the object actually is. This is because light refracts (or 'bends') as it enters water. This is a physical observation, but a theory is needed to explain it. This was ultimately explained by Fermat's Principle: light travels between points along a path that minimizes time (not necessarily distance), i.e. light need not travel in a straight line, but rather in a path which minimizes time. We are used to the adage, 'a straight line is the shortest distance between two points.' But the shortest distance does not mean the shortest *time*, and light travels between points in a medium in shortest time. Consider the scenario posed below, where you are shining a light at point A in some medium into the water at angle θ_1.



The speed of light changes as it enters the second medium.[1] Rather than travel straight through the second medium, light travels according to Fermat's Principle that it travels in the *least time*. We observe this in the refraction of objects in water, i.e. them not being quite where we see them. Suppose light travels at velocity v_1 in medium 1 and velocity v_2 in medium 2. Snell's was able to use Fermat's Principle to show (what is now called Snell's law)

$$\frac{v_1}{v_2} = \frac{\sin\theta_1}{\sin\theta_2}$$

Assuming the mediums have refractive index $n_1 := c/v_1, n_2 := c/v_2$, respectively, we see that the above is also equivalent to n_2/n_1. This implies $n_1 \sin\theta_1 = n_2 \sin\theta_2$, which is another common way Snell's law is stated. Snell's law appears not only in Physics, but is also used in the design of optical components, such as fibre optic cables, contact lenses, cameras, glasses, etc., and even in the production of sweets!

Problem:

(a) Using the fact that objects moving at constant velocity and direction satisfy $d = vt$, i.e. distance traveled is the product of the velocity and the time, find the total time the light takes to travel from A to B.

(b) Use Lagrange multipliers to derive Snell's law. [Hint: The horizontal and vertical separation of A and B must be constant.]

(c) What happens when $n_1 = n_2$?

(d) Assuming that the distances and v_1, v_2 are constants depending on the mediums you are using, and that the only controllable variable is the entry angle of the light, write the minimal total travel time in terms of this entry angle, θ_1. [Hint: Use trigonometric identities to replace the θ_2 term in (a).]

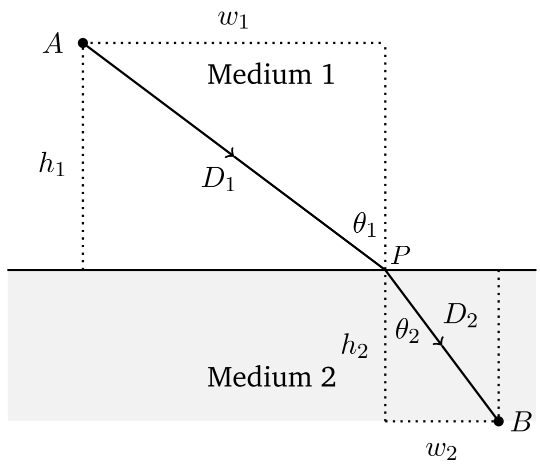
(e) Show your answer to (d) is plausible by showing that it gives the expected result in the case where the mediums are essentially uniform, i.e. $n_1 = n_2$.

Solution.

(a) The distance the light travels is the sum of the lengths of the two arrowed lines, which we will denote by D_1, D_2, respectively. Denote also by T, the total time, and by T_1 and T_2, the time light travels along paths D_1, D_2, respectively. Finally, denote by h_1, h_2 and w_1, w_2 the vertical distance and horizontal, respectively, traveled in the direction given by D_1, D_2, respectively. Then noting $D_i = v_i T_i$ and using some trigonometry, we have

$$\begin{aligned} T &= T_1 + T_2 \\ &= \frac{D_1}{v_1} + \frac{D_2}{v_2} \\ &= \frac{h_1}{v_1\cos\theta_1} + \frac{h_2}{v_2\cos\theta_2} \end{aligned}$$

[1]The speed of light is only constant in a vacuum. Light does travel faster or slower in different mediums, i.e. glass, water, oil, etc., and the ratio of the speed of light to its velocity in the medium, $n := c/v$, is called the *refractive index* of the material.



(b) The horizontal and vertical distances between A and B are constant. In the previous part, we have involved information about the vertical distance, so let us use the fact that the horizontal distance is fixed, i.e. $w_1 + w_2 = C$ for some constant C. Replacing w_1, w_2 in terms of the constants and variables from the previous part, i.e. using the fact that $\tan\theta_i = w_i/h_i$, we have the constraint $h_1 \tan\theta_1 + h_2 \tan\theta_2 = C$. Using the method of Lagrange multipliers, we have the following system of equations:

$$\frac{h_1 \sin\theta_1}{v_1 \cos^2\theta_1} = \lambda \frac{h_1}{\cos^2\theta_1}$$
$$\frac{h_2 \sin\theta_2}{v_2 \cos^2\theta_2} = \lambda \frac{h_2}{\cos^2\theta_2}$$
$$h_1 \tan\theta_1 + h_2 \tan\theta_2 = C$$

Note that you could have used either the power (and chain rule) to find the derivative of $1/\cos\theta_i$ or replace $1/\cos\theta_i$ with $\sec\theta_i$, differentiated, and then simplify the resulting trigonometric expression, to find the equations given above. Simplifying the first two equations, we have

$$\frac{\sin\theta_1}{v_1} = \lambda$$
$$\frac{\sin\theta_2}{v_2} = \lambda$$

But then we have $\lambda = \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$ so that

$$\frac{v_1}{v_2} = \frac{\sin\theta_1}{\sin\theta_2}$$

Note that this whole derivation relies on the assumption that $n_2 > n_1$. Otherwise, there is an angle (called the critical angle) at which all of the light is internally reflected, and no light is transmitted through. Transmission of the light gets weaker as the angle of incidence increases (which implies internal reflection grows stronger).

(c) If $n_1 = n_2$, then $n_2/n_1 = 1$. But then from Snell's law, we know that $\sin\theta_1 = \sin\theta_2$. Because θ_1, θ_2 are quadrant I angles, this implies $\theta_1 = \theta_2$, i.e. the light travels 'straight' through the materials. Furthermore, from the fact that $n_2/n_1 = v_1/v_2$, we know that $v_1 = v_2$. So as far as the light is concerned, it is passing through only one type of material (in terms of how the lights path and velocity behaves).