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MAT 397— Fall 2020
Applied Problems:
Limits & Continuity

> *"Trying to understand the way nature works involves a most terrible test of human reasoning ability. It involves subtle trickery, beautiful tightropes of logic on which one has to walk in order not to make a mistake in predicting what will happen. The quantum mechanical and the relativity ideas are examples of this."*
>
> *–Richard Feynman*

Time Dilation

Newton was arguably one of the greatest minds in human history. He was a mathematician, physicist, astronomer, theologian, and author. Newton independently developed Calculus, laid the groundwork for all of Physics in his *Principia*, established Optics as a proper field in *Opticks*, built the first practical reflecting telescope, developed a theory of gravity, classified cubic curves, served as Master of the Royal Mint, and much more. Despite the theory of gravity empirically working for hundreds of years, it was not the whole story. The Newtonian Theory of Gravity broke down in a few special cases, such as failing to predict the precession of the perihelion of Mercury.

The problems with Newtonian gravity were not fixed until Einstein's Annus Mirabilis papers (the so called "miracle year"). Einstein had a series of four papers published in 1905 which revolutionized Physics: a paper on the theory of the photoelectric effect (for which he won the Nobel Prize), a paper on Brownian motion, a paper on special relativity, and a fourth on mass-energy equivalence. Of these papers, his paper on special relativity is perhaps his most famous.

The ideas of special relativity had been proposed before in works of FitzGerald in 1889 and independently by Lorentz in 1892, and in further work of Larmor, Lorentz, and Poincaré. However, it took Einstein to build the theory as a whole. At the time, Einstein's theory was very controversial and the experiments to confirm the theory nearly failed and/or did not happen. [It is a fascinating story well worth hearing.] Modernly, special relativity stands as one of the most accurately tested and experimentally supported theories in the scientific literature. Einstein would later generalize special relativity to explain the law of gravitation. Einstein and the famed mathematician David Hilbert essentially completed General Relativity at the same time in a famed race to the finish—yet another story well worth hearing.

Relativity (along with Quantum Mechanics) is famous for being rather confusing because it conflicts with the typical human understanding of reality. The human brain was evolved to understand objects which move at 'medium' paces at 'medium' scales—which are not the scales for Relativity (the very large) and Quantum Mechanics (the very small).

Let us look at an example of the conflicting nature of relativity. [As a first fact, even at the time of relativity, it was observed (and now known) that the speed of light is constant.] Imagine a friend is sitting in the middle of a railcar traveling down a track. Suddenly, lightening strikes both ends of the railcar. Light from the bolts at each window race toward your friend, and their eyes flash as they arrive. Your friend sees the flashes of light at the same time. Because they know that the speed of light is constant and that the flashes of light arrived to them at the same time, the lightening must have struck both ends at the same time—namely $t = L/(2c)$ seconds ago, where L is the length of the car. But suppose you were sitting outside the train watching this horror story play out. What you see is the railcar moving forward, and the lightening strike. Because the car is moving forward (and your friend along with it), the light from the right side must arrive at your friend before the light from the left side of the car 'catches up' to your friend. But then your friend says they were killed by both lightening strikes simultaneously, while you state it must be the one on the right that did the deed. You both have measured and observed everything perfectly, but you cannot both be correct! What resolves the apparent contradiction is time dilation/length contraction from special relativity. Though this seems to be very distant an inapplicable to 'real life,' both of these features have to be taken into account when designing GPS or satellite clocks, predicting planetary motion, designing fine circuits, and more!

We will examine a simple case of time dilation in special relativity here. Suppose you have an observer apparently at rest measuring the time of an event as t, and an observer in motion at a constant velocity v relative to the observer at apparent rest measuring the time of an event as t_0. Then according to special relativity,

$$t = \frac{t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \gamma\, t_0$$

where c is the speed of light ($c = 299792458$ m/s, or about 671,000,000 mph if you prefer) and $\gamma := \dfrac{1}{\sqrt{1 - v^2/c^2}}$ is the Lorentz factor.

Problem:

(a) Noting that nothing travels faster than the speed of light, who measures a greater time—the person at apparent rest or the person in motion? Explain.

(b) How does (a) simulate time dilation? How might you summarize this to someone in a sentence?

(c) Treating t as a function of t_0, v, what happens as v approaches the speed of light, i.e. what is $\displaystyle\lim_{(t_0,v)\to(t_0,c)} t(t_0,v)$? What does this say about time to a person traveling at velocities near the speed of light?

(d) Treating t as a function of t_0, v, what happens as t_0 approaches 0, i.e. what is $\displaystyle\lim_{(t_0,v)\to(0,v)} t(t_0,v)$? Explain why this must be so.

(e) Treating t as a function of t_0, v, what is $\displaystyle\lim_{(t_0,v)\to(0,c)} t(t_0,v)$?

(f) Treating t as a function of t_0, v, is $t(t_0,v)$ continuous? Explain.

Solution.

(a) The person at rest measures the longer time. Trying a few values for v (noting that $v < c$), we see that $1/\sqrt{1 - v^2/c^2} > 1$ so that $t > t_0$. We can also prove this directly. First, observe that $v < c$ so that $v^2 < c^2$, which implies $0 < v^2/c^2 < 1$. But then

$$\begin{aligned} 0 < 1 - \frac{v^2}{c^2} &< 1 \\ \sqrt{1 - \frac{v^2}{c^2}} &< 1 \\ \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} &> 1 \end{aligned}$$

so that

$$t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot t_0 > 1 \cdot t_0 = t_0$$

(b) Notice that the time observed by the person at rest is greater than the time observed by the person in motion relative to the person at rest. You could summarize this as 'moving clocks run slower' (relative to those at rest).

(c) We have

$$\lim_{(t_0,v)\to(t_0,c)} t(t_0, v) = \lim_{(t_0,v)\to(t_0,c)} \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \infty$$

This means the time observed by the person at rest is infinite. Equivalently, time slows down to a halt for the person in motion relative to the person at rest.

(d) We have

$$\lim_{(t_0,v)\to(0,v)} t(t_0, v) = \lim_{(t_0,v)\to(t_0,c)} \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = 0$$

This must be the case because if no time has passed for the person in motion, so too must no time pass for the person at rest. For all intents and purposes, the velocity, i.e. motion, is irrelevant if all objects are frozen in time.

(e) Combining (c) and (d), we see that $\lim\limits_{(t_0,v)\to(0,c)} t(t_0, v)$ does not exist.

(f) Yes, $t(t_0, v)$ is continuous. Observe that it is continuous for all t_0 and for $0 \leq v < c$, which is the domain of $t(t_0, v)$. Although $\lim\limits_{(t_0,v)\to(0,c)} t(t_0, v)$ does not exist, $(t_0, v) = (0, c)$ is not in the domain of $t(t_0, v)$.