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MAT 397— Fall 2020
Applied Problems:
Partial Derivatives

"OK, first rule of Wall Street. Nobody, and I don't care if you're Warren Buffet or Jimmy Buffet, nobody knows if a stock's going up, down or f-ing sideways, least of all stockbrokers. But we have to pretend we know."

–Mark Hanna, The Wolf of Wall Street

Financial Mathematics

Mathematical finance (equivalently, Quantitative Finance or Financial Mathematics) is an entire field of Applied Mathematics that deals with modeling and predicting financial markets. Models in Financial Mathematics can be continuous or discrete, and both can have stochastic properties. These models can be arbitrarily created—merely empirically matching real data—or based off of financial theories. Moreover, these models can become very complex rapidly, even in simple scenarios. For example, the Black-Scholes equation is a partial differential equation used in describing pricing an options contract:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where $S(t)$ is the underlying asset at time t, $V(S,t)$ is the price option of the asset at time t, σ is the standard deviation of the stock's return, and r is the annualized risk-free interest rate. Modern Financial Mathematics greatly scales up complexity with advanced time series analysis using metadata such as Twitter sentiment data.

In microeconomics, the goal is to understand the behavior of individual entities in making resource allocation decisions in response to factors such as resources, prices, incentives, production methods, etc. If you consider a simplified model of a company, you might consider the following factors: revenue, costs, profit, and productivity. Revenue is the amount the company takes in from selling their products, e.g. $R(x,y)$ represents the amount of money the company takes in producing x items of one good and y items of another. Costs are the amount the company spends producing and selling their goods, e.g. $C(x,y)$ represents the amount of money spent in producing x items of one good and y items of another. The value $C(0,0,\ldots,0)$ is called the fixed costs, because this is the cost before producing any goods, e.g. the total costs of paying employees, buying/renting space, etc. The profit is then $P := R - C$.

Suppose the revenue and cost function for a particular company, measured in thousands of dollars, producing two goods are $R(x,y) = 5x^2 + 7y^2$ and $C(x,y) = 4x^2 + 5y^2 + 200$, respectively, where x, y are measured in thousands of items.

Problem:

(a) Find the fixed costs, and find the break-even point(s), i.e. the production levels which result in a profit of zero.

(b) The marginal revenue is the change in revenue resulting in producing one additional unit of one of the products. Suppose the company is currently producing 16,000 units of x and 11,000 units of y. Estimate the marginal revenue for the product corresponding to x at this level of production. Use partial derivatives to find this exactly.

(c) The marginal cost is the change in cost resulting in producing one additional unit of one of the products. Suppose the company is currently producing 16,000 units of x and 11,000 units of

y. Estimate the marginal cost for the product corresponding to y at this level of production. Use partial derivatives to find this exactly.

(d) The marginal profit is the change in revenue resulting in producing one additional unit of one of the products. Suppose the company is currently producing 16,000 units of x and 11,000 units of y. Given the limited resources of the company, if the company is only to increase production for one of the products, which should they invest in making more of? Explain.

(e) Marginal utility is the additional 'satisfaction' a consumer gains by consuming additional goods or services. Marginal utility allows producers to determine how much of an item a consumer will be willing to purchase. For example, you are more likely to buy another water bottle if you have three of them than if you have thirty of them. Together, marginal cost and marginal utility are often used together to determine price. Suppose for this company that the utility function is $U(x, y) = 4xy - x^2 - 5y^2$. Find the marginal utilities for x and y given the production levels in the previous parts. Based on this information, which product should the company 'focus on'?

Of course, we can also look at macroeconomic systems: collections of companies, national productivity, etc. For example, the Cobb-Douglas production function examines how the quantity of output behaves with respect to the various levels of production. Generally, the Cobb-Douglas function is

$$C(L, K) := AL^{\beta}K^{\alpha}$$

where C is the aggregate output (value of the goods produced), L is the labor (for instance, total person hours worked), K is the capital output (value of the capital against its price), α, β are output elasticities of capital and labor, respectively, and A is the total factor production (TFP). The factor A is a sort of 'quality' factor, which takes into account skill and education of the work force. Hence, we generally expect A to increase over time. Sometimes, the function is reworked to display constant returns to scale, e.g. doubling L and K results in a doubled value in C. This occurs when $\alpha + \beta = 1$, in which case we rewrite $C(L, K)$ as

$$\widetilde{C}(L, K) := AL^{1-\alpha}K^{\alpha}$$

[If $\alpha + \beta > 1$, then we say there is an increasing returns to scale, and if $\alpha + \beta < 1$, we say that there are decreasing returns to scale.]

(f) Show that greater labor input increases the production of more output, i.e. $\widetilde{C}_L > 0$. The function $\widetilde{C}_L$ is called the marginal product of labor (MPL).

(g) Show that there is 'diminishing returns' on labor, i.e. the more labor used the less the affect on output (all other things equal).

(h) Are parts (e) and (f) the same if 'labor' is replaced by 'capital'? Explain.

(i) What happens to the marginal product of labor when the capital is increased? Explain.

(j) Show that $\widetilde{C}$ gives constant returns to scale, i.e. scaling L and K by a common factor λ results in scaling $\widetilde{C}$ by λ.

(k) In a 'perfectly competitive' economy, the profit maximizing behaviors of individuals tends to force that the factors of production are paid a total return equivalent to their marginal productions. Show that the total earnings per factor are each a percent share of the aggregate output $\widetilde{C}$. [Hint: The 'earnings per unit' are MPL and MPK.] This can be used to examine various GDP factors, such as compensation of employees.

Solution.

(a) The fixed costs are $C(0,0) = 200$. Therefore, the fixed costs are \$200,000. The break-even point(s) will be the point(s) where $P = 0$, i.e. where $R = C$.

$$\begin{aligned} 5x^2 + 7y^2 &= 4x^2 + 5y^2 + 200 \\ x^2 + 2y^2 &= 200 \end{aligned}$$

Of course, this is an ellipse.

(b) We can estimate this using

$$\frac{R(16.001, 11) - R(16, 11)}{16.001 - 16} = \frac{0.160005}{0.001} = \$160.005/\text{thousand unit}$$

To compute this directly, observe $R_x(x, y) = 10x$, so that $R_x(16, 11) = \$160$.

(c) We can estimate this using

$$\frac{C(16, 11.001) - C(16, 11)}{11.001 - 11} = \frac{0.110005}{0.001} = \$110.005/\text{thousand unit}$$

To compute this directly, observe $C_y(x, y) = 10y$, so that $R_x(16, 11) = \$110.00$.

(d) The profit is $P(x, y) := R(x, y) - C(x, y) = x^2 + 2y^2 - 200$. We then find the marginal profits for the products corresponding to x and y are

$$\begin{aligned} P_x(x, y) &= 2x\Big|_{(x,y)=(16,11)} = \$32,000/\text{thousand unit} \\ P_y(x, y) &= 4y\Big|_{(x,y)=(16,11)} = \$44,000/\text{thousand unit} \end{aligned}$$

The company should invest in making more of the product corresponding to y, because they make more profit per unit in y at this production level. Of course, this should be reevaluated as they increase the production level in the product corresponding to y as it happens to track if this trend continues.

(e) We have

$$\begin{aligned} U_x(x, y) &= -2x + 4y\Big|_{(x,y)=(16,11)} = 12 \\ U_y(x, y) &= 4x - 10y\Big|_{(x,y)=(16,11)} = -46 \end{aligned}$$

Positive marginal utility implies increased satisfaction from additional purchases of the good. Negative marginal utility implies decreased satisfaction from additional purchases of the good. Zero marginal utility implies that there is difference in satisfaction from additional purchases of the good. Given there is positive marginal utility for the product x and negative utility for the product y, the company ought to focus on product x as there seems to be a desire for x in the market, while the market seems sated/oversaturated with product y.

(f) This is just a calculation of the corresponding partial,

$$\begin{aligned}
\text{MPL} &= \widetilde{C}_L \\
&= (1-\alpha)AL^{1-\alpha-1}K^{\alpha} \\
&= (1-\alpha)\cdot AL^{1-\alpha}K^{\alpha}\cdot L^{-1} \\
&= (1-\alpha)\,\frac{\widetilde{C}}{L}
\end{aligned}$$

But $0<\alpha<1$ and $C, N>0$. Therefore, the marginal product of labor is positive: great labor input leads to production of more output.

(g) We see from the previous part, all other things equal, increasing L forces $\text{MPL} = (1-\alpha)\,\frac{\widetilde{C}}{L}$ to be less positive; that is, we see 'diminishing returns' on labor. We can also see this from the second partial,

$$\begin{aligned}
\text{MPL}_L &= \widetilde{C}_{LL} \\
&= \frac{\partial}{\partial L}(1-\alpha)AL^{1-\alpha-1}K^{\alpha} \\
&= (1-\alpha)A(1-\alpha-1)L^{1-\alpha-2}K^{\alpha} \\
&= (-\alpha)(1-\alpha)\cdot AL^{1-\alpha}K^{\alpha}\cdot L^{-2} \\
&= (-\alpha)(1-\alpha)\,\frac{\widetilde{C}}{L^2}
\end{aligned}$$

But as $\alpha, 1-\alpha>0$, and $\widetilde{C}/N^2>0$, we see that $\text{MPL}_L = (-\alpha)(1-\alpha)\frac{\widetilde{C}}{L^2}<0$.

(h) Yes! We can perform similar analyses:

$$\begin{aligned}
\text{MPK} &= \widetilde{C}_K \\
&= \alpha AL^{1-\alpha}K^{\alpha-1} \\
&= \alpha\cdot AL^{1-\alpha}K^{\alpha}\cdot K^{-1} \\
&= \alpha\frac{\widetilde{C}}{K}
\end{aligned}$$

which is greater than zero because $\alpha, \widetilde{C}, K>0$, and there are 'diminishing returns' on capital because, all things equal increasing K makes MPK smaller, or equivalently,

$$\begin{aligned}
\text{MPK}_K &= \widetilde{C}_{KK} \\
&= \frac{\partial}{\partial K}\alpha AL^{1-\alpha}K^{\alpha-1} \\
&= (\alpha-1)\alpha AL^{1-\alpha}K^{\alpha-2} \\
&= \alpha(\alpha-1)\cdot AL^{1-\alpha}K^{\alpha}\cdot K^{-2} \\
&= \alpha(\alpha-1)\frac{\widetilde{C}}{K^2}
\end{aligned}$$

and $\text{MPK}_K = \alpha(\alpha-1)\frac{\widetilde{C}}{K^2}<0$ because $\alpha, \widetilde{C}, K>0$ and $0<\alpha<1$ so that $\alpha-1<0$.

(i) An increase in capital raises the marginal product of labor, which can be seen by computing either of the following partials (which are equivalent) and using the fact that $0 < \alpha < 1$ along with $\widetilde{C}, L, K > 0$:

$$\begin{aligned}\text{MPL}_K &= \frac{\partial}{\partial K}(1-\alpha)AL^{1-\alpha-1}K^{\alpha} \\ &= \alpha(1-\alpha)AL^{1-\alpha-1}K^{\alpha-1} \\ &= \alpha(1-\alpha)AL^{1-\alpha}K^{\alpha} \cdot L^{-1}K^{-1} \\ &= \alpha(1-\alpha)\frac{\widetilde{C}}{LK} > 0\end{aligned}$$

$$\begin{aligned}\text{MPK}_L &= \frac{\partial}{\partial L}\alpha AL^{1-\alpha}K^{\alpha-1} \\ &= (1-\alpha)\alpha AL^{1-\alpha-1}K^{\alpha-1} \\ &= \alpha(1-\alpha) \cdot AL^{1-\alpha}K^{\alpha} \cdot L^{-1}K^{-1} \\ &= \alpha(1-\alpha)\frac{\widetilde{C}}{LK} > 0\end{aligned}$$

(j) Suppose we scale L, K by λ to $L' := \lambda L$ and $K' := \lambda K$. We know that $\widetilde{C}(L, K) = AL^{1-\alpha}K^{\alpha}$. Then

$$\begin{aligned}\widetilde{C}(L', K') &= A(L')^{1-\alpha}(K')^{\alpha} \\ &= A(\lambda L)^{1-\alpha}(\lambda K)^{\alpha} \\ &= A\lambda^{1-\alpha}L^{1-\alpha}\lambda^{\alpha}K^{\alpha} \\ &= \lambda AL^{1-\alpha}K^{\alpha} \\ &= \lambda\widetilde{C}(L, K)\end{aligned}$$

(k)

$$\begin{aligned}\text{Labor Earnings } &= L \cdot \text{MPL} = L \cdot (1-\alpha)\,\frac{\widetilde{C}}{L} = (1-\alpha)\widetilde{C} \\ \text{Capital Earnings } &= K \cdot \text{MPK} = K \cdot \alpha\,\frac{\widetilde{C}}{K} = \alpha\widetilde{C}\end{aligned}$$

Then $1-\alpha$ is the labor's share of $\widetilde{C}$, while α is the capital's share of $\widetilde{C}$. Note in total, we have $(1-\alpha)\widetilde{C} + \alpha\widetilde{C} = \widetilde{C}$.

Remark: Note the validity of (f)–(k) relies on the assumptions of the Cobb-Douglas equations holding, as well as it being applicable to the system under consideration!