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MAT 397— Fall 2020

Applied Problems:
Surfaces

“[Roy] Are you sure we need this nozzle thing? [Quentin] Are you kidding? The nozzle is the most important part—it directs the flow of the hot gases! [Roy] Hey, cool it, Quentin! Man, talkin’ ’bout your hot gases. . .”

—Roy Lee & Quentin Wilson, Rocket Boys

Ideal Gas Law

Many states of matter are difficult to develop accessible and manageable mathematical frameworks with which to predict their properties. In particular, plasmas, fluids, and gases can be extremely difficult to create working theories with which to predict their behavior. For gases, there are three ‘classical’ types of gases: Maxwell-Boltzmann ideal gases, Bose gases (ideal quantum gases composed of bosons), and Fermi gases (ideal quantum gases composed of fermions). We will consider Maxwell-Boltzmann ideal gases.

Because gases are ultimately probabilistic and stochastic in nature, we imagine a so called *ideal gas*. An ideal gas is a theoretical gas composed of many moving particles (considered as point particles) that do not interact and have perfectly elastic collisions (a collision in which there is no net loss in kinetic energy, so that momentum and kinetic energy are conserved). You can imagine such gases as consisting of many very tiny spheres which can collide without deformation, but otherwise do not interact with each other. Though no actual ideal gases exist, they serve as a good approximation to how many gases actually behave under ‘normal’ conditions. Ideal gases are governed by the ideal gas law, which is a combination of Boyle’s law, Charles’ law, Avogadro’s law, and Gay-Lussac’s law.

The ideal gas law is $PV = nRT$, where P is the pressure of the gas (in pascals), V is the volume of the gas (in cubic meters), n is the amount of substance (in moles), R is the ideal gas constant ($R \approx 8.314462 \text{ J}\cdot\text{K}^{-1}\text{mol}^{-1}$), and T is the temperature (in Kelvins). This law has been generalized to non-ideal gases, e.g. gases which has molecular interaction and whose particles have volume. This generalization is the van der Waals equation:

$$\left(P + a \frac{n^2}{V^2}\right) (V - nb) = nRT$$

where V_m is the molar volume of the gas, a is a constant depending on the gas, and b is the volume occupied by one mole of the gas molecules. [Notice when V is ‘large’ compared to n and nb , this reduces to the ideal gas law.]

Consider the scenario where we are considering some fixed number of moles of an ideal gas. We can then rewrite the ideal gas law as $PV = kT$, where k is some constant. Considering this as a surface, the level curves to this surface become particularly informative.

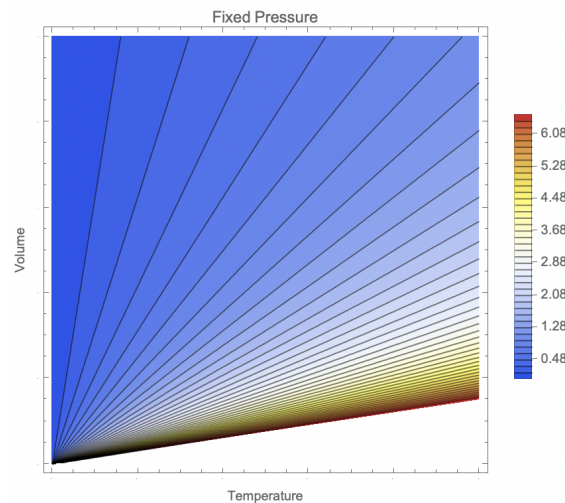
Problem:

- Consider the pressure of an ideal gas, P , as a function of its temperature and volume. Sketch some level curves for this function. Describe what this tells you about the nature of ideal gases with fixed pressure.
- Consider the temperature of an ideal gas, T , as a function of its pressure and volume. Sketch some level curves for this function. Describe what this tells you about the nature of ideal gases with fixed temperature.

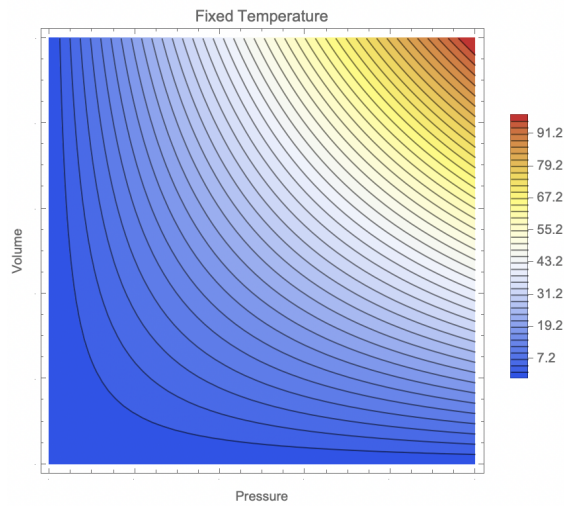
- (c) Consider the volume of an ideal gas, V , as a function of its pressure and temperature. Sketch some level curves for this function. Describe what this tells you about the nature of ideal gases with fixed volume.
- (d) Either from the equation $PV = kT$ or from what you learned from (a), (b), and (c), what type of surface does the ideal gas law describe. Use a computer software system to plot an example of such a surface. [Hint: If you do this algebraically, start with $z = Cxy$ and make a substitution $x = a - b$, $y = a + b$, and $z' = z/C$, see what surface you get in x, y, z' .]

Solution.

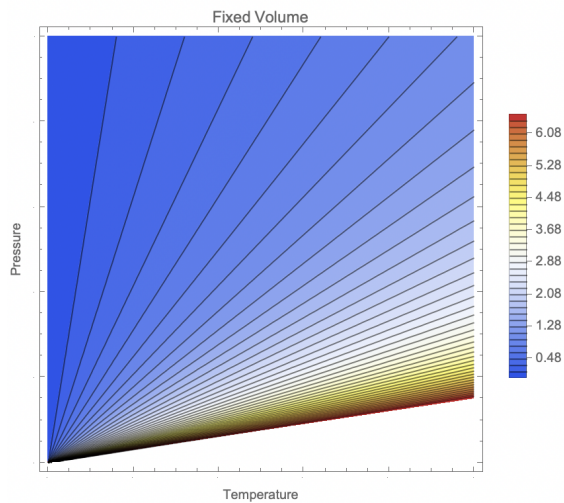
- (a) We have $P := P(V, T) = k \frac{T}{V}$. So for a fixed pressure, we have either $V = \frac{k}{P_0} T$ or $T = \frac{k}{P_0} V$, which are lines. Therefore for a fixed pressure, if temperature increases then the volume must increase, i.e. the gas expands, and if temperature decreases the volume must decrease, i.e. the gas contracts.



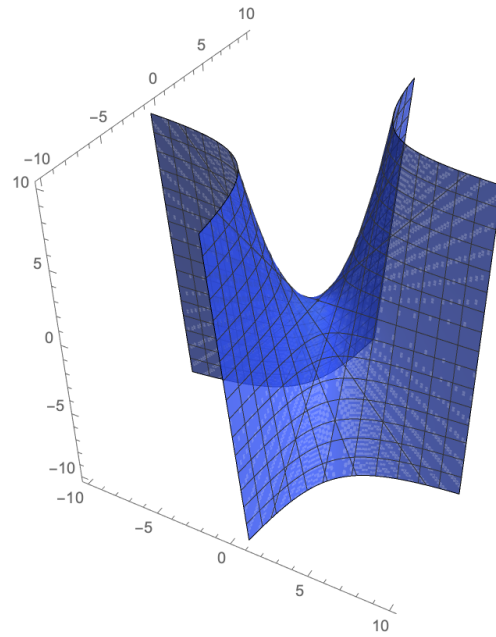
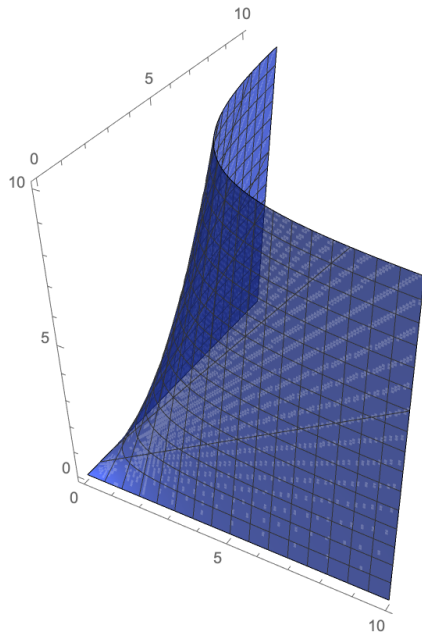
- (b) We have $T := T(P, V) = \frac{1}{k} PV$. So for a fixed temperature, we have either $V = kT_0 \frac{1}{P}$ or $P = kT_0 \frac{1}{V}$, which are hyperbolas. Therefore for a fixed temperature, if the volumes increases the pressure decreases, and if the volume decreases the pressure increases.



- (c) We have $V := V(P, T) = k \frac{T}{P}$. Observe that this is the same situation as in (a), mutatis mutandis. For a fixed volume, we have either $T = \frac{V_0}{k} P$ or $P = \frac{k}{V_0} T$. Therefore, if the temperature increases then the pressure increases, and if the temperature decreases the pressure decreases.



- (d) Because the only valid values for V, T, P are nonnegative, we get the surface on the left in the figures below. This is a section of a hyperbolic paraboloid, seen on the right in the figures below.



Suppose we have a surface given by $z = Cxy$. Setting $x = a - b$ and $y = a + b$, for some a, b , then we have

$$z = Cxy$$

$$z = C(a - b)(a + b)$$

$$z = C(a^2 - b^2)$$

$$\frac{z}{C} = a^2 - b^2$$

$$z' = a^2 - b^2$$

where $z' = z/C$. Observe that this is reversible: given a, b, z' , there are unique x, y, z making the equation $z = Cxy$ hold, and given x, y, z' , there are unique a, b, z making $z' = a^2 - b^2$ hold (namely $a = \frac{x+y}{2}$, $b = \frac{y-x}{2}$, $z' = z/C$). Therefore, the equation $z = Cxy$ is another way of expressing a hyperbolic paraboloid. Writing $T = \frac{1}{k}PV$, we see that taking $z = T$, $C = \frac{1}{k}$, $x = P$, and $y = V$ shows that $PV = kT$ represents a hyperbolic paraboloid.