Name: ____________________________________

MAT 397— Fall 2020
Applied Problems:
Vector Fields

"Look at that sea, girls—all silver and shadow and vision of things not seen. We couldn't enjoy its loveliness any more if we had millions of dollars and ropes of diamonds."

–Lucy Maud Montgomery, Anne of Green Gables

Ocean Currents

Ocean currents are caused by wind, lunar gravitation, and differences in water density, temperature, and salinity. The science of ocean, river, channel, tidal, etc currents ultimately rests upon partial differential equations (PDEs). For instance, to discuss tidal currents (ignoring friction with the Earth, assuming constant depth, and placing the x-axis in the direction of the tide), one examines the partial differential equations

$$\frac{\partial v_x}{\partial t} = -g\frac{\partial \eta}{\partial x}, \quad \frac{\partial \eta}{\partial t} = -h\frac{\partial v_x}{\partial x}$$

where v_x is the horizontal velocity, g is the acceleration of gravity, and η is the vertical displacement of the surfaces. As another example, Lord Kelvin gave integral equations which are applicable to tidal waves in an infinitely long canal with constant depth and width:

$$v_x = \sqrt{\frac{g}{h}}\,\eta_0\, e^{-(\gamma/c)y}\cos\left(\sigma t - \frac{\sigma}{c}x\right), \quad v_y = 0$$

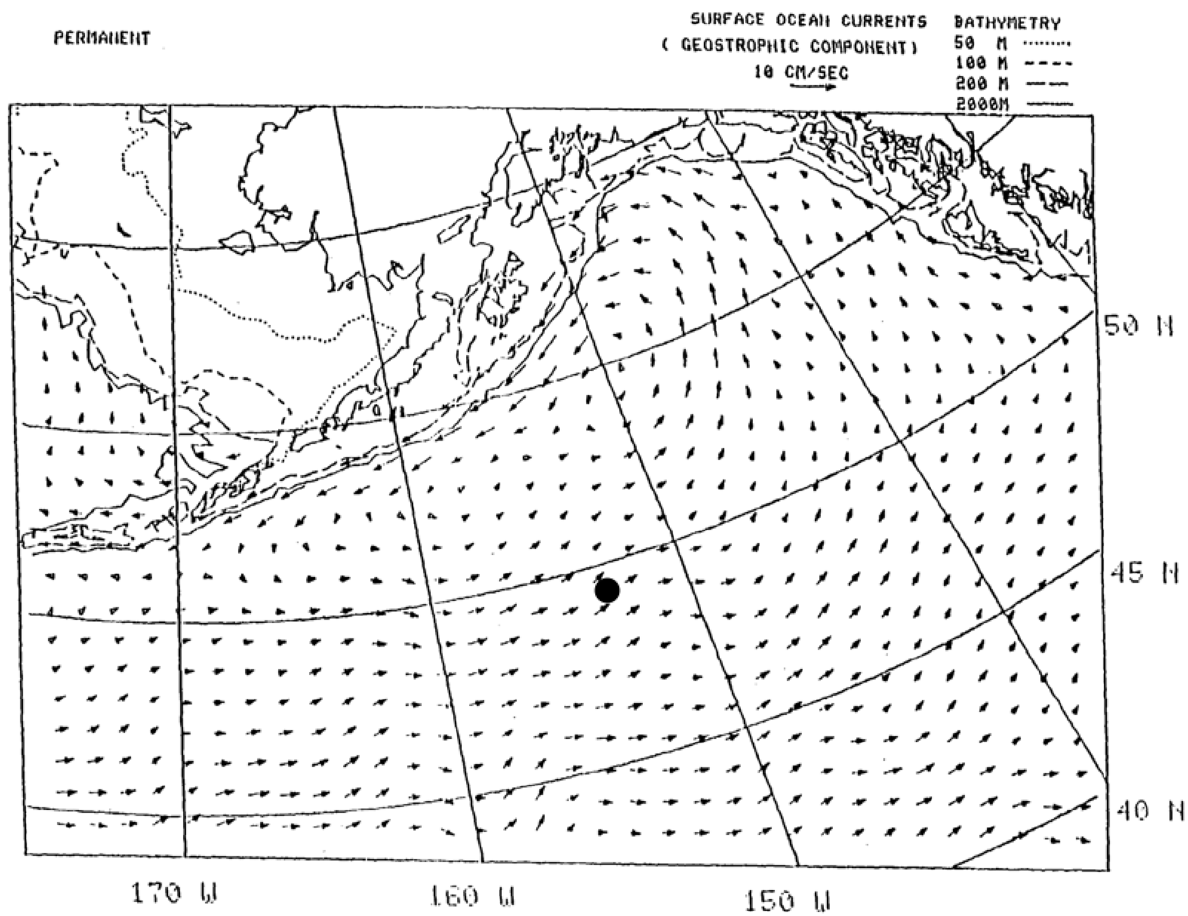
Ultimately, many of these PDE theories require a continuity condition—that net flow in some direction must be the net flow out in another direction. This condition is called the continuity equation, e.g.

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}$$

The difficulty with partial differential equations is that they are incredibly complex, and they can almost never be solved exactly. Instead, numerical solutions to partial differential equations are found. The equations v_x, v_y, v_z can be taken as the components of a vector field $\mathbf{F} = \langle v_x, v_y, v_z\rangle$. Even without equations for v_x, v_y, v_z, if numerical data for each is known, one can still approximate solutions. Given a starting point P_0, we find $\langle v_x, v_y, v_z\rangle$ at P_0, then apply this 'force' to an 'object' at P_0 for time dt. This gives a new point P_1. We repeat the process for P_1, and we obtain a point P_2. Continuing in this manner, we obtain a set of points $\{P_i\}_{i=0}^{n}$. Smoothly connecting these points, we obtain a path which we can take as an approximate solution to the PDE.

These paths are called stream plots—if $\langle v_x, v_y, v_z\rangle$ is the current force in a stream, then the points $\{P_i\}$ form the path of an object floating in the water. This general approach to partial differential equations is an essential tool in the study of heat, fluids, electrical systems, particle physics, and much more. Consider current flows near the Bering Sea. Taking ocean measurements, one can find the direction of ocean currents at various points in the water. Using this data, we can find the motion of objects subject to current forces alone. Suppose there is a ship that capsized in the ocean near the Bering Sea. The passengers have gone overboard into the water and a search plan needs to be developed.

Data and Figure from *Ocean Surface Current Simulations in the North Pacific Ocean and Bering Sea (Oscurs—Numerical Model)* by W. James Ingraham Jr, Robert K Miyahara and made available by the National Technical Information Service, U.S. Department of Commerce.



Problem:

(a) Suppose the ship capsized at the point indicated in the ocean current map above. Assuming the ocean currents remain constant over the next few hours, find a probable path for the passengers in the water.

(b) Is the path from (a) the only possible path? Explain.

(c) Mark a search area for rescuers on the map given the data.