**Name:** ________________________________________

**MAT 397— Fall 2020**
**Applied Problems:**
**Vector Functions**

*"A large part of mathematics which becomes useful developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful; and there were no general indications that it ever would be so."*

*–John von Neumann*

# Kinematic Equations

One of the central problems in Physics and Engineering is to accurately predict the path of motion of an object. Broadly speaking, there are two main ways of describing motion: dynamics and kinematics. Dynamics is the more general description, taking into account momentum, forces, energy, etc. Kinematics, on the other hand, is a simpler description of motion, taking into account only the objects position and time. Despite this drawback, kinematics is still accurate in many scenarios and is the starting point for nearly every description of motion and theory of motion (even in dynamics).

We will use kinematics to derive the equations of motion for a projectile. Suppose an object starting at ground level, which we take as position $\mathbf{0}$, is fired at an angle of $\theta$ with the horizontal and initial velocity of $\mathbf{v}_0$. Neglecting air resistance, the only force acting on the object is gravity, which pulls straight downwards with a magnitude of $a$, the acceleration due to gravity (on Earth). For the object, let $\mathbf{a}(t)$ denote the acceleration at time $t$, $\mathbf{v}(t)$ denote the velocity at time $t$, and $\mathbf{r}(t)$ denote the position of the object at time $t$.

**Problem:**

(a) Show that $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0$.

(b) Show that $\mathbf{r}(t) = -\dfrac{1}{2}gt^2\,\mathbf{j} + t\mathbf{v}_0$.

(c) What is $v_0 := \|\mathbf{v}_0\|$? Use $v_0$ to decompose $\mathbf{v}_0$ into its components.

(d) Use (b) and (c) to find $x(t)$ and $y(t)$, the $x$ and $y$ position of the object at time $t$, respectively.

(e) Show that the path the object travels is a parabola. [Hint: Write $t$ as a function of $x$ and use this in $y(t)$.]

(f) Show that the horizontal distance the object travels before impacting the ground is $\dfrac{v_0^2 \sin 2\theta}{g}$.

(g) Show that the total flight time is $\dfrac{2v_0 \sin\theta}{g}$.

(h) Find the angle that maximizes the horizontal distance the object travels.

(i) Find the maximum height the object reaches. [Hint: Use (e).]

(j) Suppose you want to strike a target at coordinate $(x, y)$, i.e. range $x$ and altitude $y$. Assuming $v_0$ is sufficiently large enough to strike the object, show that the required launch angle satisfies

$$\tan\theta = \frac{v_0^2 \pm \sqrt{v_0^4 - g(gx^2 + 2yv_0^2)}}{gx}$$

[Hint: Use the formulas you found in (d) and (e). You will use the identity $\sec^2\theta = 1 + \tan^2\theta$ to form a quadratic equation in $\tan\theta$.] If you want a challenge, use this to find the angle that minimizes the velocity required to strike the target.