**Name:** *Caleb McWhorter— Solutions*
**MAT 397— Fall 2020**
**Applied Problems:**
**Vector Functions**

*"A large part of mathematics which becomes useful developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful; and there were no general indications that it ever would be so."*

*–John von Neumann*

## Kinematic Equations

One of the central problems in Physics and Engineering is to accurately predict the path of motion of an object. Broadly speaking, there are two main ways of describing motion: dynamics and kinematics. Dynamics is the more general description, taking into account momentum, forces, energy, etc. Kinematics, on the other hand, is a simpler description of motion, taking into account only the objects position and time. Despite this drawback, kinematics is still accurate in many scenarios and is the starting point for nearly every description of motion and theory of motion (even in dynamics).

We will use kinematics to derive the equations of motion for a projectile. Suppose an object starting at ground level, which we take as position $\mathbf{0}$, is fired at an angle of $\theta$ with the horizontal and initial velocity of $\mathbf{v}_0$. Neglecting air resistance, the only force acting on the object is gravity, which pulls straight downwards with a magnitude of $a$, the acceleration due to gravity (on Earth). For the object, let $\mathbf{a}(t)$ denote the acceleration at time $t$, $\mathbf{v}(t)$ denote the velocity at time $t$, and $\mathbf{r}(t)$ denote the position of the object at time $t$.

**Problem:**

(a) Show that $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0$.

(b) Show that $\mathbf{r}(t) = -\dfrac{1}{2}gt^2\,\mathbf{j} + t\mathbf{v}_0$.

(c) What is $v_0 := \|\mathbf{v}_0\|$? Use $v_0$ to decompose $\mathbf{v}_0$ into its components.

(d) Use (b) and (c) to find $x(t)$ and $y(t)$, the $x$ and $y$ position of the object at time $t$, respectively.

(e) Show that the path the object travels is a parabola. [Hint: Write $t$ as a function of $x$ and use this in $y(t)$.]

(f) Show that the horizontal distance the object travels before impacting the ground is $\dfrac{v_0^2 \sin 2\theta}{g}$.

(g) Show that the total flight time is $\dfrac{2v_0 \sin\theta}{g}$.

(h) Find the angle that maximizes the horizontal distance the object travels.

(i) Find the maximum height the object reaches. [Hint: Use (e).]

(j) Suppose you want to strike a target at coordinate $(x, y)$, i.e. range $x$ and altitude $y$. Assuming $v_0$ is sufficiently large enough to strike the object, show that the required launch angle satisfies

$$\tan\theta = \frac{v_0^2 \pm \sqrt{v_0^4 - g(gx^2 + 2yv_0^2)}}{gx}$$

[Hint: Use the formulas you found in (d) and (e). You will use the identity $\sec^2\theta = 1 + \tan^2\theta$ to form a quadratic equation in $\tan\theta$.] If you want a challenge, use this to find the angle that minimizes the velocity required to strike the target.

**Solution.**

(a) We know that $\mathbf{v}(0) = \mathbf{r}'(0) = \mathbf{v}_0$, and $\mathbf{a}(t) = -g\,\mathbf{j}$. Then

$$\mathbf{v}(t) = \int \mathbf{a}(t)\;dt = \int -g\,\mathbf{j}\;dt = -gt\,\mathbf{j} + \mathbf{C}$$

But as $\mathbf{v}(0) = \mathbf{v}_0$, we know $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0$.

(b) We know that $\mathbf{r}(0) = \mathbf{0}$. Then

$$\mathbf{r}(t) = \int \mathbf{v}(t)\;dt = \int (-gt\,\mathbf{j} + \mathbf{v}_0)\;dt = -\frac{1}{2}gt^2 + t\mathbf{v}_0 + \mathbf{C}$$

But as $\mathbf{r}(0) = \mathbf{0}$, we know $\mathbf{r}(t) = -\frac{1}{2}gt^2\,\mathbf{j} + t\mathbf{v}_0$.

(c) The scalar $v_0 := \|\mathbf{v}_0\|$ is the initial speed of the object. Breaking $\mathbf{v}_0$ into components, we have

$$\mathbf{v}_0 = v_x\,\mathbf{i} + v_y\,\mathbf{j} = \|\mathbf{v}_0\|\cos\theta\,\mathbf{i} + \|\mathbf{v}_0\|\sin\theta\,\mathbf{j} = v_0\cos\theta\,\mathbf{i} + v_0\sin\theta\,\mathbf{j}$$

(d) We have

$$\begin{aligned}\mathbf{r}(t) &= -\frac{1}{2}gt^2\,\mathbf{j} + t\mathbf{v}_0\\ &= -\frac{1}{2}gt^2\,\mathbf{j} + t(v_0\cos\theta\,\mathbf{i} + v_0\sin\theta\,\mathbf{j})\\ &= \big((v_0\cos\theta)t\big)\,\mathbf{i} + \left(-\frac{1}{2}gt^2 + (v_0\sin\theta)t\right)\mathbf{j}\end{aligned}$$

This immediately gives

$$\begin{cases} x(t) = (v_0\cos\theta)\,t \\ y(t) = -\frac{1}{2}gt^2 + (v_0\sin\theta)\,t \end{cases}$$

(e) We have $t(x) = \dfrac{x}{v_0\cos\theta}$. Then

$$\begin{aligned}y(t) &= -\frac{1}{2}gt^2 + (v_0\sin\theta)t\\ &= -\frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 + (v_0\sin\theta)\left(\frac{x}{v_0\cos\theta}\right)\\ &= -\frac{g}{2(v_0\cos\theta)^2}\,x^2 + (\tan\theta)\,x\end{aligned}$$

which is immediately seen to be a parabola of the form $y = Ax^2 + Bx$, where $A = -\dfrac{g}{2(v_0\cos\theta)^2}$ and $B = \tan\theta$.

(f) The object will hit the ground when $y = 0$. Then

$$\begin{aligned} 0 &= y(t) \\ &= -\frac{g}{2(v_0 \cos\theta)^2} x^2 + \tan\theta\, x \\ &= x\left[-\frac{g}{2(v_0 \cos\theta)^2} x + \tan\theta\right] \end{aligned}$$

Then either the left or right term is zero. The solution $x = 0$ gives $(x, y) = (0, 0)$, which we knew because $\mathbf{r}(0) = \mathbf{0}$. Then setting the right term to zero, we have

$$\begin{aligned} 0 &= -\frac{g}{2(v_0 \cos\theta)^2} x + \tan\theta \\ \frac{g}{2(v_0 \cos\theta)^2} x &= \frac{\sin\theta}{\cos\theta} \\ x &= \frac{2v_0^2 \cos^2\theta \sin\theta}{g \cos\theta} \\ x &= \frac{v_0^2 \cdot 2\cos\theta \sin\theta}{g} \\ x_{\text{max}} &= \frac{v_0^2 \sin 2\theta}{g} \end{aligned}$$

(g) We know that

$$t_{\text{impact}} = \frac{x}{v_0 \cos\theta} = \frac{1}{v_0 \cos\theta} \cdot \frac{v_0^2 \sin 2\theta}{g} = \frac{1}{v_0 \cos\theta} \cdot \frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{2v_0 \sin\theta}{g}$$

(h) We want to maximize $\frac{v_0^2 \sin 2\theta}{g}$. This occurs when $\sin$ is 1, i.e. when $2\theta = \frac{\pi}{2}$. But then an angle of $\theta = \frac{\pi}{4}$, or $45°$, maximizes the (horizontal) distance the object will travel.

(i) The turning point of a parabola occurs at $x = -b/2a$, where we have written the parabola as $y = ax^2 + bx + c$. We have

$$y(t) = -\frac{g}{2(v_0 \cos\theta)^2} x^2 + \tan\theta\, x$$

Using this

$$\begin{aligned} -\frac{b}{2a} &= -\frac{\tan\theta}{2} \cdot -\frac{2v_0^2 \cos^2\theta}{g} \\ &= \frac{\sin\theta}{\cos\theta} \cdot \frac{v_0^2 \cos^2\theta}{g} \\ &= \frac{v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{2g} \end{aligned}$$

One could also use the fact that if the parabola has two real roots, then the turning point has $x$

coordinate the average of the two roots—in this case $0$ and $\dfrac{v_0^2 \sin 2\theta}{g}$. Then

$$\begin{aligned}
y_{\text{max}} &= -\frac{g}{2(v_0 \cos\theta)^2} x^2 + \tan\theta\, x \\
&= -\frac{g}{2(v_0 \cos\theta)^2} \left( \frac{v_0^2 \sin\theta \cos\theta}{g} \right)^2 + \frac{\sin\theta}{\cos\theta} \cdot \frac{v_0^2 \sin\theta \cos\theta}{g} \\
&= -\frac{g}{2v_0^2 \cos^2\theta} \cdot \frac{v_0^4 \sin^2\theta \cos^2\theta}{g^2} + \frac{v_0^2 \sin^2\theta}{g} \\
&= -\frac{v_0^2 \sin^2\theta}{2g} + \frac{2v_0^2 \sin^2\theta}{2g} \\
&= \frac{v_0^2 \sin^2\theta}{2g}
\end{aligned}$$

(j) We want to strike a target at a fixed (hence, known) point $(x, y)$. Recall that from (d), we have

$$\begin{cases} x(t) = (v_0 \cos\theta)\, t \\ y(t) = -\dfrac{1}{2} g t^2 + (v_0 \sin\theta)\, t \end{cases}$$

Then we can consider this a system of two equations in the two unknowns $t$ and $\theta$. From (e), we know that $t = x/(v_0 \cos\theta)$, and using this in $y(t)$, we found $y(t) = -\dfrac{g}{2(v_0 \cos\theta)^2}\, x^2 + (\tan\theta)\, x$. Then

$$\begin{aligned}
y(t) &= -\frac{g}{2(v_0 \cos\theta)^2}\, x^2 + (\tan\theta)\, x \\
&= -\frac{g}{2v_0^2} \cdot \sec^2\theta \cdot x^2 + (\tan\theta)\, x \\
&= -\frac{gx^2}{2v_0^2} \cdot (1 + \tan^2\theta) + (\tan\theta)\, x \\
&= -\frac{gx^2}{2v_0^2} \cdot (1 + \tan^2\theta) + (\tan\theta)\, x \\
&= -\frac{gx^2}{2v_0^2}\, \tan^2\theta + x\tan\theta - \frac{gx^2}{2v_0^2}
\end{aligned}$$

But then we have

$$\begin{aligned}
y &= -\frac{gx^2}{2v_0^2}\, \tan^2\theta + x\tan\theta - \frac{gx^2}{2v_0^2} \\
0 &= -\frac{gx^2}{2v_0^2}\, \tan^2\theta + x\tan\theta - \frac{gx^2}{2v_0^2} - y \\
0 &= -\frac{gx^2}{2v_0^2}\, \tan^2\theta + x\tan\theta - \frac{gx^2 + 2yv_0^2}{2v_0^2}
\end{aligned}$$

This is a quadratic equation in $\tan\theta$. Using the quadratic formula, we have

$$\begin{aligned}
\tan\theta &= \frac{-x \pm \sqrt{x^2 - 4 \cdot \dfrac{-gx^2}{2v_0^2} \cdot \dfrac{-gx^2 - 2yv_0^2}{2v_0^2}}}{2 \cdot \dfrac{-gx^2}{2v_0^2}} \\
&= v_0^2 \cdot \frac{-x \pm \sqrt{x^2 - 4 \cdot \dfrac{-gx^2}{2v_0^2} \cdot -\dfrac{gx^2 + 2yv_0^2}{2v_0^2}}}{-gx^2} \\
&= v_0^2 \cdot \frac{-x \pm \sqrt{x^2 - \dfrac{gx^2(gx^2 + 2yv_0^2)}{v_0^4}}}{-gx^2} \\
&= v_0^2 \cdot \frac{-x \pm \sqrt{\dfrac{x^2}{v_0^4}\left(v_0^4 - g(gx^2 + 2yv_0^2)\right)}}{-gx^2} \\
&= v_0^2 \cdot \frac{-x \pm \dfrac{x}{v_0^2}\sqrt{v_0^4 - g(gx^2 + 2yv_0^2)}}{-gx^2} \\
&= \frac{v_0^2 \mp x\sqrt{v_0^4 - g(gx^2 + 2yv_0^2)}}{gx}
\end{aligned}$$

Note that we can even find the launch angle allowing for the least possible launch velocity, $v_0$. This will be the angle 'on the border' between no solutions, i.e. complex solutions, and two repeated solutions. This occurs when the discriminant of the above quadratic equation is zero, i.e. the $v_0$ for which $v_0^4 - g(gx^2 + 2yv_0^2) = 0$. But then we have

$$0 = v_0^4 - g(gx^2 + 2yv_0^2) = (v_0^2)^2 - (2yg)v_0^2 - (gx)^2$$

This is a quadratic equation in $v_0^2$. Applying the quadratic equation, we find

$$\begin{aligned}
v_0^2 &= \frac{2yg \pm \sqrt{4y^2g^2 + 4g^2x^2}}{2} \\
&= \frac{2yg \pm \sqrt{4g^2(y^2 + x^2)}}{2} \\
&= \frac{2yg \pm 2g\sqrt{y^2 + x^2}}{2} \\
&= g(y \pm \sqrt{x^2 + y^2})
\end{aligned}$$

Because $v_0^2 \geq 0$, we reject the negative solution for $v_0^2$. Now using the discriminant is zero, we have

$$\begin{aligned}
\tan\theta &= \frac{v_0^2}{gx} \\
&= \frac{g(y + \sqrt{x^2 + y^2})}{gx} \\
&= \frac{y + \sqrt{x^2 + y^2}}{x}
\end{aligned}$$

Assume we are striking a point $(x, y)$. Then denote by $\psi$ the angle so that $\tan\psi = y/x$. But then

$$\begin{aligned}
\tan\theta &= \frac{y + \sqrt{x^2 + y^2}}{x} \\
&= \frac{y}{x} + \sqrt{\frac{x^2 + y^2}{x^2}} \\
&= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \\
&= \tan\psi + \sqrt{1 + \tan^2\psi} \\
&= \tan\psi + \sec\psi \\
&= \frac{\sin\psi}{\cos\psi} + \frac{1}{\cos\psi} \\
&= \frac{1 + \sin\psi}{\cos\psi}
\end{aligned}$$

First, we use the identity $\cot\theta = \tan(\pi/2 - \theta)$.

$$\begin{aligned}
\tan\theta &= \frac{1 + \sin\psi}{\cos\psi} \\
\frac{1}{\tan\theta} &= \frac{\cos\psi}{1 + \sin\psi} \\
\cot\theta &= \frac{\cos\psi}{1 + \sin\psi} \\
\tan(\pi/2 - \theta) &= \frac{\cos\psi}{1 + \sin\psi}
\end{aligned}$$

Now we make use of the identity $\cos^2\theta + \sin^2\theta = 1$.

$$\begin{aligned}
\tan(\pi/2 - \theta) &= \frac{\cos\psi}{1 + \sin\psi} \\
\tan^2(\pi/2 - \theta) &= \frac{\cos^2\psi}{(1 + \sin\psi)^2} \\
\tan^2(\pi/2 - \theta) &= \frac{1 - \sin^2\psi}{(1 + \sin\psi)^2} \\
\tan^2(\pi/2 - \theta) &= \frac{(1 - \sin\psi)(1 + \sin\psi)}{(1 + \sin\psi)^2} \\
\tan^2(\pi/2 - \theta) &= \frac{1 - \sin\psi}{1 + \sin\psi} \\
\tan^2(\pi/2 - \theta) &= -\frac{\sin\psi - 1}{1 + \sin\psi} \\
\tan^2(\pi/2 - \theta) &= -\frac{\sin\psi + (1 - 1) - 1}{1 + \sin\psi} \\
\tan^2(\pi/2 - \theta) &= -\frac{1 + \sin\psi - 2}{1 + \sin\psi} \\
\tan^2(\pi/2 - \theta) &= -1 + \frac{2}{1 + \sin\psi} \\
1 + \tan^2(\pi/2 - \theta) &= \frac{2}{1 + \sin\psi}
\end{aligned}$$

Finally, we use the identities $\sec^2\theta = 1+\tan^2\theta$, $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$ (so that $\cos(2\theta) = 2\cos^2\theta - 1$), and $\sin\theta = \cos(\pi/2-\theta)$.

$$\begin{aligned}
1+\tan^2(\pi/2-\theta) &= \frac{2}{1+\sin\psi} \\
\sec^2(\pi/2-\theta) &= \frac{2}{1+\sin\psi} \\
\frac{1}{\cos^2(\pi/2-\theta)} &= \frac{2}{1+\sin\psi} \\
\cos^2(\pi/2-\theta) &= \frac{1+\sin\psi}{2} \\
2\cos^2(\pi/2-\theta)-1 &= \sin\psi \\
\cos\big(2(\pi/2-\theta)\big) &= \sin\psi \\
\cos\big(2(\pi/2-\theta)\big) &= \cos(\pi/2-\psi) \\
\cos(\pi-2\theta) &= \cos(\pi/2-\psi)
\end{aligned}$$

Using the fact that these are Quadrant I angles, we then have $\pi - 2\theta = \pi/2-\psi$. Solving for $\theta$, we find

$$\begin{aligned}
\pi-2\theta &= \frac{\pi}{2}-\psi \\
2\theta &= \frac{\pi}{2}+\psi \\
\theta &= \frac{\pi}{4}+\frac{\psi}{2}
\end{aligned}$$

But we can rewrite this as

$$\theta = \frac{\pi}{4}+\frac{\psi}{2} = \frac{\pi}{2}-\frac{\pi}{4}+\frac{\psi}{2} = \frac{\pi}{2}-\frac{1}{2}\left(\frac{\pi}{2}-\psi\right)$$

That is, the required angle is the angle halfway between the angle to the intended target and the vertical.