MAT 397: Exam 1
Fall – 2020
09/21/2020
80 Minutes

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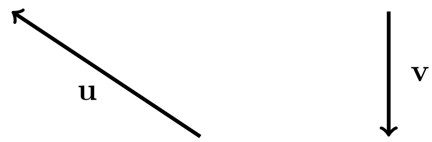
Write your name on the appropriate line on the exam cover sheet. This exam contains 8 pages (including this cover page) and 7 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

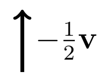
Question	Points	Score
1	14	
2	16	
3	16	
4	16	
5	16	
6	16	
7	16	
Total:	110	

1. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors given below.

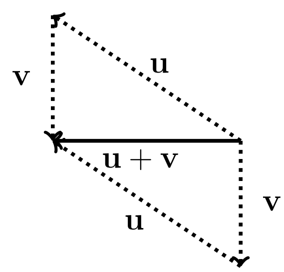


As accurately as possible, sketch the following vectors:

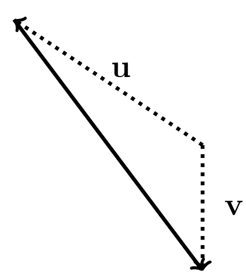
(a) (3 points) $-\frac{1}{2}\mathbf{v}$



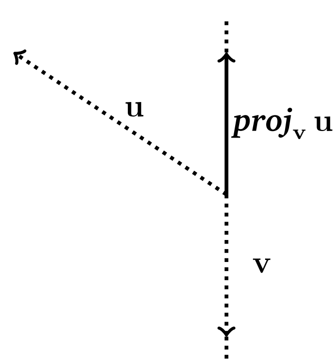
(b) (3 points) $\mathbf{u}+\mathbf{v}$



(c) (4 points) $\mathbf{v}-\mathbf{u}$



(d) (4 points) $\mathbf{proj}_{\mathbf{v}}\,\mathbf{u}$



2. Suppose you are given the following:

$$\begin{aligned} \mathbf{u} &= \langle 2, -1, 3 \rangle & \|\mathbf{v}\| &= 3\sqrt{5} \\ \mathbf{v} &= \langle 4, -2, 5 \rangle & \mathbf{u} \cdot \mathbf{v} &= 25 \\ \mathbf{w} &= \langle 1, -1, 1 \rangle & \mathbf{u} \times \mathbf{w} &= \langle 2, 1, -1 \rangle \end{aligned}$$

(a) (3 points) Find $\|\mathbf{u}\|$.

$$\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

(b) (3 points) Is $\mathbf{u}$ parallel to $\mathbf{v}$? Justify your answer completely.

No. If $\mathbf{u} \parallel \mathbf{v}$*, then there would be* c *so that* $\mathbf{v} = c\mathbf{u}$*. But then* $\langle 4, -2, 5 \rangle = \langle 2c, -c, 3c \rangle$*. Equating the first components, we see that* $c = 2$*. But* $\langle 4, -2, 5 \rangle \neq 2\langle 2, -1, 3 \rangle = \langle 4, -2, 6 \rangle$.

(c) (5 points) Is $\mathbf{u}$ perpendicular to $\mathbf{w}$? Justify your answer.

$$\mathbf{u} \cdot \mathbf{w} = 2(1) + (-1)(-1) + 3(1) = 2 + 1 + 3 = 6 \neq 0$$

Therefore, $\mathbf{u} \not\perp \mathbf{w}$.

(d) (5 points) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \\ 25 &= \sqrt{14} \cdot 3\sqrt{5} \cdot \cos\theta \\ \cos\theta &= \frac{25}{3\sqrt{5}\sqrt{14}} \\ \theta &= \cos^{-1}\left(\frac{25}{3\sqrt{5}\sqrt{14}}\right) = \cos^{-1}\left(\frac{25}{3\sqrt{70}}\right) \approx 5.11109^\circ \end{aligned}$$

3. Suppose you are given the following:

$$\begin{aligned} \mathbf{u} &= \langle 2, -1, 3 \rangle & \|\mathbf{v}\| &= 3\sqrt{5} \\ \mathbf{v} &= \langle 4, -2, 5 \rangle & \mathbf{u} \cdot \mathbf{v} &= 25 \\ \mathbf{w} &= \langle 1, -1, 1 \rangle & \mathbf{u} \times \mathbf{w} &= \langle 2, 1, -1 \rangle \end{aligned}$$

(a) (4 points) Without explicitly computing the angle between $\mathbf{u}$ and $\mathbf{v}$, determine whether the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute, right, or obtuse. Justify your answer.

Note that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$. Now $\|\mathbf{u}\|, \|\mathbf{v}\| > 0$ for nonzero vectors. Therefore, the sign of $\mathbf{u} \cdot \mathbf{v}$ is entirely determined by $\cos\theta$. But $\cos\theta$ is positive when $0 < \theta < \frac{\pi}{2}$, zero when $\theta = \frac{\pi}{2}$, and negative when $\frac{\pi}{2} < \theta < \pi$. Because $\mathbf{u} \cdot \mathbf{v} > 0$, the angle between $\mathbf{u}$ and $\mathbf{v}$ must be acute.

(b) (4 points) Are the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ coplanar? Justify your answer.

We find the volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ by computing the scalar triple product. [One could use the 'shortcut', but we are given partial information already.]

$$\begin{aligned} V &= |\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})| \\ &= |\langle 4, -2, 5 \rangle \cdot \langle 2, 1, -1 \rangle| \\ &= |8 - 2 - 5| \\ &= |1| \\ &= 1 \end{aligned}$$

Because $V \neq 0$, the vectors cannot be coplanar.

(c) (4 points) Compute the projection $\textbf{proj}_{\mathbf{v}}\, \mathbf{u}$.

$$\textbf{\textit{proj}}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\, \mathbf{v} = \frac{25}{(3\sqrt{5})^2} \langle 4, -2, 5 \rangle = \frac{5}{9} \langle 4, -2, 5 \rangle = \left\langle \frac{20}{9}, -\frac{10}{9}, \frac{25}{9} \right\rangle$$

(d) (4 points) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{w}$.

$$\|\mathbf{u} \times \mathbf{w}\| = \|\langle 2, 1, -1 \rangle\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$$

4. Parametrize the following curves:

   (a) (6 points) the directed line segment from $(-2,3,0)$ to $(1,-1,2)$.

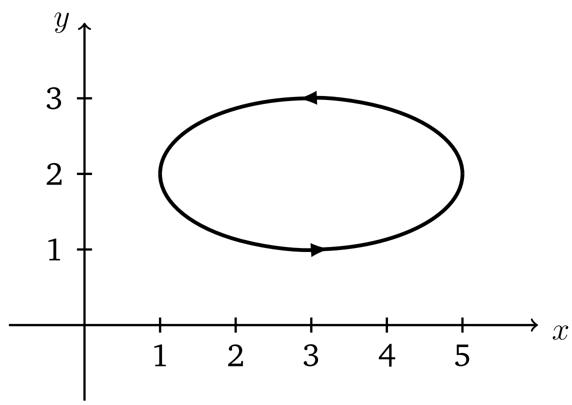
   We have

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\langle -2,3,0\rangle + t\langle 1,-1,2\rangle \\ &= \langle -2+2t, 3-3t, 0\rangle + \langle t, -t, 2t\rangle \\ &= \langle 3t-2, 3-4t, 2t\rangle; \quad 0 \le t \le 1\end{aligned}$$

   Equivalently, we have start point $(-2,3,0)$ *and direction* $(1,-1,2)-(-2,3,0) = \langle 3,-4,2\rangle$. *Then*

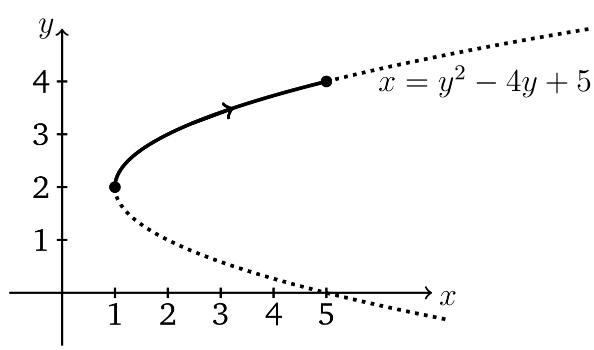
$$\begin{aligned}\mathbf{r}(t) &= \langle 3,-4,2\rangle t + \langle -2,3,0\rangle \\ &= \langle 3t-2, 3-4t, 2t\rangle; \quad 0 \le t \le 1\end{aligned}$$

   (b) (5 points) the ellipse shown below.



$$\mathbf{r}(t) = \langle 2\cos t + 3, \sin t + 2\rangle; \quad 0 \le t \le 2\pi$$

   (c) (5 points) the directed portion of the curve between the two given points shown below.



$$\mathbf{r}(t) = \langle t^2 - 4t + 5, t\rangle; \quad 2 \le t \le 4$$

5. Complete the following parts:

(a) (8 points) If $\mathbf{r}'(t) = 3t^2\,\mathbf{i} - e^{1-t}\,\mathbf{k}$ and $\mathbf{r}(1) = 2\,\mathbf{j} + \,\mathbf{k}$, find $\mathbf{r}(t)$.

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\;dt\\ &= \int \langle 3t^2, 0, -e^{1-t}\rangle\;dt\\ &= \langle t^3, 0, e^{1-t}\rangle + \langle C_1, C_2, C_3\rangle\end{aligned}$$

But $\mathbf{r}(1) = \langle 0, 2, 1\rangle$. *Then*

$$\begin{aligned}\langle 0,2,1\rangle &= \langle 1,0,1\rangle + \langle C_1, C_2, C_3\rangle\\ \langle 0,2,1\rangle &= \langle C_1+1, C_2, C_3+1\rangle\end{aligned}$$

so that $C_1 = -1$, $C_2 = 2$, *and* $C_3 = 0$. *Therefore,*

$$\mathbf{r}(t) = \langle t^3 - 1, 2, e^{1-t}\rangle$$

(b) (8 points) If $\mathbf{x}(t) = \langle 1, \sqrt{t}, \ln t\rangle$, find an integral which computes the length of the curve between $\mathbf{x}(1)$ and $\mathbf{x}(8)$. *Do not evaluate the integral.*

$$\mathbf{x}'(t) = \left\langle 0,\; \frac{1}{2\sqrt{t}}\,,\; \frac{1}{t}\right\rangle$$

$$\|\mathbf{x}'(t)\| = \sqrt{0^2 + \left(\frac{1}{2\sqrt{t}}\right)^2 + \left(\frac{1}{t}\right)^2} = \sqrt{\frac{1}{4t} + \frac{1}{t^2}}$$

Therefore,

$$L = \int_1^8 \sqrt{\frac{1}{4t} + \frac{1}{t^2}}\;dt = \frac{6\ln 2 + 7}{8} \approx 1.39486$$

6. Complete the following parts:

(a) (8 points) Evaluate the following limit:

$$\lim_{(x,y)\to(-2,1)} \frac{x^2+3xy+2y^2}{x^2-4y^2}$$

$$\begin{aligned}
\lim_{(x,y)\to(-2,1)} \frac{x^2+3xy+2y^2}{x^2-4y^2} &= \lim_{(x,y)\to(-2,1)} \frac{(x+y)(x+2y)}{(x-2y)(x+2y)} \\
&= \lim_{(x,y)\to(-2,1)} \frac{x+y}{x-2y} \\
&= \frac{-2+1}{-2-2} \\
&= \frac{-1}{-4} \\
&= \frac{1}{4}
\end{aligned}$$

(b) (8 points) Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^5+4y^5}$$

$$\begin{aligned}
x\textit{-axis } (y=0) : \lim_{x\to 0} \frac{x^2\cdot 0}{x^5+0} &= \lim_{x\to 0} \frac{0}{x^5} = 0 \\
y\textit{-axis } (x=0) : \lim_{y\to 0} \frac{0\cdot y^3}{0+4y^5} &= \lim_{y\to 0} \frac{0}{4y^5} = 0 \\
x=y : \lim_{y\to 0} \frac{y^2y^3}{y^5+4y^5} &= \lim_{y\to 0} \frac{y^5}{5y^5} = \frac{1}{5}
\end{aligned}$$

Because the limit as $(x,y)\to(0,0)$ is not the same in all directions, the limit does not exist.

7. Complete the following parts:

(a) (8 points) Find the equation of the line perpendicular to the plane $x+2y-z=6$ at the point where the plane intersects the y-axis.

To find the equation of the line, we need to find a point and a direction. First, we find the intersection of the plane and y-axis. Suppose (x_0, y_0, z_0) is the point of intersection. Because it is on the y-axis, $x_0 = z_0 = 0$. But because (x_0, y_0, z_0) is on the plane, we must also have $x + 2y - z = 6$, so that $0 + 2y_0 - 0 = 6$. Therefore, the point of intersection is $(x_0, y_0, z_0) = (0, 3, 0)$. To be perpendicular to the plane, the direction of the line must be perpendicular to the plane, but this is exactly the direction of the normal vector to the plane, i.e. $\langle 1, 2, -1\rangle$. Therefore, the equation of the line is

$$\begin{aligned} \ell(t) &= \langle 1, 2, -1\rangle t + \langle 0, 3, 0\rangle \\ &= \langle t, 2t+3, -t\rangle \end{aligned}$$

(b) (8 points) Find the equation of the plane containing the following two lines: $\ell_1(t) = (5t+4, t-1, 5t+2)$ and $\ell_2(t) = (3-5t, 2-t, 1-5t)$.

To find the equation of the plane, we need to find a point and a normal direction. Because the plane Π contains the line $\ell_1(t)$, it must contain the point $\ell_1(0) = (4, -1, 2)$. The direction $\mathbf{u} := \langle 5, 1, 5\rangle$ must be perpendicular to the normal vector to the plane because $\ell_1(t)$ is contained in Π. Now $\ell_2(0) = (3, 2, 1)$. But because Π contains $\ell_1(t)$ and $\ell_2(t)$, it must also contain the displacement vector $\mathbf{v} := \langle 4, -1, 2\rangle - \langle 3, 2, 1\rangle = \langle 1, -3, 1\rangle$. But then the normal vector is a direction perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Therefore, we can take the normal vector to be

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 5 & 1 & 5 \\ 1 & -3 & 1 \end{vmatrix} \\ &= (1 - (-15))\,\boldsymbol{i} - (5-5)\boldsymbol{j} + (-15-1)\,\boldsymbol{k} \\ &= (1+15)\,\boldsymbol{i} - 0\boldsymbol{j} - 16\,\boldsymbol{k} \\ &= \langle 16, 0, -16\rangle \end{aligned}$$

Because $\langle 16, 0, -16\rangle$ is parallel to $\langle 1, 0, -1\rangle$, we can use normal vector $\langle 1, 0, -1\rangle$. Therefore, the equation of the plane is

$$\begin{aligned} \langle 1, 0, -1\rangle \cdot \langle x-4, y-(-1), z-2\rangle &= 0 \\ (x-4) + 0 - (z-2) &= 0 \\ x - 4 - z + 2 &= 0 \\ x - z &= 2 \end{aligned}$$