MAT 397: Exam 2
Fall – 2020
10/19/2020
150 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 12 pages (including this cover page) and 10 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total:	200	

1. Consider the surface given by $y^2 + xy + 2 = x^3 - y^4 + z^5$.
   (a) (10 points) Find the equation of the tangent plane to this surface at the point $(1, 0, 1)$.

$$y^2 + xy + 2 = x^3 - y^4 + z^5$$
$$\underbrace{y^2 + xy + 2 - x^3 + y^4 - z^5}_{f(x,y,z)} = 0$$

$$\nabla f(x, y, z) = \langle y - 3x^2, 2y + x + 4y^3, -5z^4 \rangle$$
$$\nabla f(1, 0, 1) = \langle 0 - 3(1), 2(0) + 1 + 4(0), -5(1) \rangle = \langle -3, 1, -5 \rangle$$

Therefore, the tangent plane is

$$\begin{aligned} \nabla f(1,0,1) \cdot \langle x - 1, y - 0, z - 1 \rangle &= 0 \\ \langle -3, 1, -5 \rangle \cdot \langle x - 1, y - 0, z - 1 \rangle &= 0 \\ -3(x - 1) + 1(y - 0) - 5(z - 1) &= 0 \\ -3x + 3 + y - 5z + 5 &= 0 \\ -3x + y - 5z &= -8 \end{aligned}$$

   (b) (10 points) Find the parametric equations for the normal line to the surface at the point $(1, 0, 1)$.

The line will have direction $\nabla f(1, 0, 1) = \langle -3, 1, -5 \rangle$. *The line passes through the point* $(1, 0, 1)$. *Therefore, the line is*

$$\begin{cases} x = 1 - 3t \\ y = t \\ z = 1 - 5t \end{cases}$$

2. Let $f(x, y, z) = x \cos y + 3z$.

(a) (4 points) Find $f(2, 0, 1)$.

$$f(2, 0, 1) = 2\cos 0 + 3(1) = 2(1) + 3 = 2 + 3 = 5$$

(b) (8 points) Find the total differential of $f(x, y, z)$.

$$\begin{aligned} df &= f_x\ dx + f_y\ dy + f_z\ dz \\ &= (\cos y)\ dx + (-x \sin y)\ dy + 3\ dz \\ &= \cos y\ dx - x \sin y\ dy + 3\ dz \end{aligned}$$

(c) (8 points) Use parts (a) and (b) to approximate $f(2.1, -0.1, 0.8)$.

$$\begin{aligned} df &= (\cos y)\ dx + (-x \sin y)\ dy + 3\ dz \\ &= (\cos 0) \cdot 0.1 + (-2 \sin 0) \cdot -0.1 + 3 \cdot -0.2 \\ &= 0.1 + 0 - 0.6 \\ &= -0.5 \end{aligned}$$

Then we have

$$f(2.1, -0.1, 0.8) \approx f(2, 0, 1) + df = 5 - 0.5 = 4.5$$

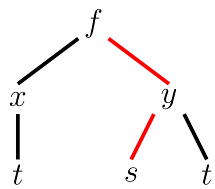
3. Define the following functions:

$$f(x,y) = xy + y^3$$
$$x(t) = \sqrt[3]{t^2}$$
$$y(s,t) = e^{t/s}$$

Use the multivariable chain rule to find the following in terms of x, y, s, t:

(a) (10 points) $\dfrac{\partial f}{\partial s}$

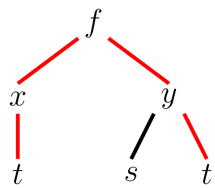
$$f_x = y$$
$$f_y = x + 3y^2$$
$$x_t = \frac{2}{3\sqrt[3]{t}}$$
$$y_s = -\frac{te^{t/s}}{s^2}$$
$$y_t = \frac{e^{t/s}}{s}$$



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s}$$
$$= (x + 3y^2) \cdot -\frac{te^{t/s}}{s^2}$$
$$= -\frac{te^{t/s}(x + 3y^2)}{s^2}$$

(b) (10 points) $\dfrac{\partial f}{\partial t}$

$$f_x = y$$
$$f_y = x + 3y^2$$
$$x_t = \frac{2}{3\sqrt[3]{t}}$$
$$y_s = -\frac{te^{t/s}}{s^2}$$
$$y_t = \frac{e^{t/s}}{s}$$



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t}$$
$$= y \cdot \frac{2}{3\sqrt[3]{t}} + (x + 3y^2) \cdot \frac{e^{t/s}}{s}$$
$$= \frac{2y}{3\sqrt[3]{t}} + \frac{e^{t/s}(x + 3y^2)}{s}$$

4. Let $f(x, y) = xe^{x-2y}$ and $\mathbf{u} = 2\,\mathbf{i} + \mathbf{j}$.

(a) (8 points) Find $D_{\mathbf{u}}f(2, 1)$.

$$\nabla f(x, y) = \langle e^{x-2y} + xe^{x-2y}, -2xe^{x-2y} \rangle \Big|_{(x,y)=(2,1)} = \langle 1 + 2(1), -2(2)1 \rangle = \langle 3, -4 \rangle$$

$$\|\mathbf{u}\| = \sqrt{2^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5}$$

$$D_{\mathbf{u}}f(2, 1) = \nabla f(2, 1) \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} = \langle 3, -4 \rangle \cdot \frac{\langle 2, 1 \rangle}{\sqrt{5}} = \frac{1}{\sqrt{5}}\big(3(2) + (-4)1\big) = \frac{6 - 4}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

(b) (3 points) What is the direction of the maximum rate of increase for $f(x, y)$ at $(2, 1)$? What is the rate of change in that direction?

We know that the direction of max increase is $\nabla f(2, 1) = \langle 3, -4 \rangle$. *The rate of max increase is* $\|\nabla f(2, 1)\| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

(c) (3 points) What is the direction of the maximum rate of decrease for $f(x, y)$ at $(2, 1)$? What is the rate of change in that direction?

We know that the direction of max decrease is $-\nabla f(2, 1) = -\langle 3, -4 \rangle = \langle -3, 4 \rangle$. *The rate of max decrease is* $-\|\nabla f(2, 1)\| = -\sqrt{3^2 + (-4)^2} = -\sqrt{9 + 16} = -\sqrt{25} = -5$.

(d) (3 points) Estimate the change in $f(2,1)$ if you travel a distance of 0.5 in the direction of $\mathbf{u}$.

We use the familiar concept that Total Change = Rate of Change·Amount of Change.

$\Delta f \approx$ *rate of change in the direction · distance* $= D_{\mathbf{u}} f(2,1) \cdot 0.5 = \dfrac{2}{\sqrt{5}} \cdot \dfrac{1}{2} = \dfrac{1}{\sqrt{5}} \approx 0.447214$

(e) (3 points) At the point $(2,1)$, give a direction in which $f(x,y)$ does not change, i.e. a direction for which $D_{\mathbf{v}} f(2,1) = 0$.

There is no change at $(2,1)$ *in the direction* $\mathbf{v}$ *if* $D_{\mathbf{v}} f(2,1) = 0$. *But then we have*

$$0 = D_{\mathbf{v}} f(2,1) = \nabla f(2,1) \cdot \mathbf{v} = \langle 3, -4 \rangle \cdot \mathbf{v}$$

Therefore, any direction $\mathbf{v}$ *(not necessarily a unit vector) perpendicular to* $\nabla f(2,1)$ *will suffice, i.e. a direction* $\mathbf{v}$ *so that the dot product is 0. For example,* $\langle 4, 3 \rangle$ *and* $\langle -4, -3 \rangle$ *are such a vector. If we want to find all the possible directions, writing* $\mathbf{v} = \langle a, b \rangle$*, we have*

$$0 = \langle 3, -4 \rangle \cdot \mathbf{v} = \langle 3, -4 \rangle \cdot \langle a, b \rangle = 3a - 4b$$

Then $3a = 4b$ *so that* $b = \frac{3}{4}a$. *Therefore,* $\mathbf{v} = \langle a, b \rangle = \langle a, \frac{3}{4}a \rangle = a\langle 1, \frac{3}{4} \rangle$ *for any nonzero* $a \in \mathbb{R}$ *is the collection of all such vectors. The choices above simply correspond to choosing* $a = 4$ *and* $a = -4$*, respectively.*

5. (20 points) Show that the only critical values for $f(x,y,z) = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$ are $(0,0,0)$ and $(2,0,0)$. Classify these two critical values.

First, we first the partials:

$$\begin{array}{lll} f_x = 3x^2 + z^2 - 6x & f_{xx} = 6x - 6 & f_{xy} = f_{yx} = 0 \\ f_y = 2y & f_{yy} = 2 & f_{xz} = f_{zx} = 2z \\ f_z = 2xz + 4z & f_{zz} = 4 & f_{yz} = f_{zy} = 0 \end{array}$$

To find the critical values set $f_x = f_y = f_z = 0$. From $f_y = 0$, we know that $2y = 0$ so that $y = 0$. From $f_z = 0$, we know that $0 = 2xz + 4z = 2z(x+2)$. But then either $z = 0$ or $x + 2 = 0$. If $z = 0$, from the first equation we know that $3x^2 - 6x = 0$. But then $3x(x-2) = 0$ so that $x = 0$ or $x = 2$. Then we have critical values $(0,0,0)$ and $(2,0,0)$. Now if $x + 2 = 0$, then $x = -2$. Then using the first equation, we have $0 = 3(-2)^2 + z^2 - 6(-2) = 12 + z^2 + 12 = 24 + z^2$. But then $z^2 = -24$, which is impossible. Now we classify the critical points $(0,0,0)$ and $(2,0,0)$ by using the Hessian:

$$Hf(x,y,z) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 6(x-1) & 0 & 2z \\ 0 & 2 & 0 \\ 2z & 0 & 2(x+2) \end{pmatrix}$$

Now we examine sequence of principal minors for the critical points:

$$Hf(0,0,0) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$(0,0,0):$

$$d_1 = -6 < 0, \quad d_2 = -6(2) = -12 < 0, \quad d_3 = -6(2)(4) = -48 < 0$$

$$Hf(2,0,0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$(2,0,0):$

$$d_1 = 6 > 0, \quad d_2 = 6(2) = 12 > 0, \quad d_3 = 6(2)8 = 96 > 0$$

Therefore, $(0,0,0)$ is a saddle and $(2,0,0)$ is a minimum. At these values, we have $f(0,0,0) = 0$ and $f(2,0,0) = -4$.

6. (20 points) Use the method of Lagrange Multipliers to find the points on the ellipse $3x^2 - 4xy + 3y^2 = 50$ nearest and farthest from the origin. *[Hint: Work instead with the square of the distance. You should have four points in total.]*

If (x, y, z) is on the ellipse, its distance to the origin is $d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$. Therefore, $d^2 = x^2 + y^2$. If the distance is maximized/minimized, then the square of the distance is maximized/minimized, respectively, and vice versa.

We use Lagrange Multipliers. If $3x^2 - 4xy + 3y^2 = 50$, then $3x^2 - 4xy + 3y^2 - 50 = 0$. Define $g(x, y) = 3x^2 - 4xy + 3y^2 - 50$ and $f(x, y) = d^2(x, y)$. So we want $\nabla f = \lambda \nabla g$ and $g(x, y) = 0$. Then we have the system of equations

$$\begin{cases} 2x = \lambda(6x - 4y) \\ 2y = \lambda(6y - 4x) \\ 3x^2 - 4xy + 3y^2 - 50 = 0 \end{cases}$$

Solving for λ in the first two equations, we have

$$\lambda = \frac{2x}{6x - 4y} = \frac{2y}{6y - 4x}$$

Then we have

$$\begin{aligned} \frac{2x}{6x - 4y} &= \frac{2y}{6y - 4x} \\ 2x(6y - 4x) &= 2y(6x - 4y) \\ 12xy - 8x^2 &= 12xy - 8y^2 \\ -8x^2 &= -8y^2 \\ x^2 &= y^2 \\ x &= \pm y \end{aligned}$$

If $x = y$, using $g(x, y) = 0$, we have $0 = 3x^2 - 4x^2 + 3x^2 - 50 = 2x^2 - 50$ so that $2x^2 = 50$ and then $x^2 = 25$ so that $x = \pm 5$. Because $x = y$, this gives points $(5, 5)$ and $(-5, -5)$.

If $x = -y$, we have $0 = 3x^2 + 4x^2 + 3x^2 - 50 = 10x^2 - 50$ so that $10x^2 = 50$. But then $x^2 = 5$ so that $x = \pm\sqrt{5}$. Because $x = -y$, this gives points $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$.

The squares of the distances are then

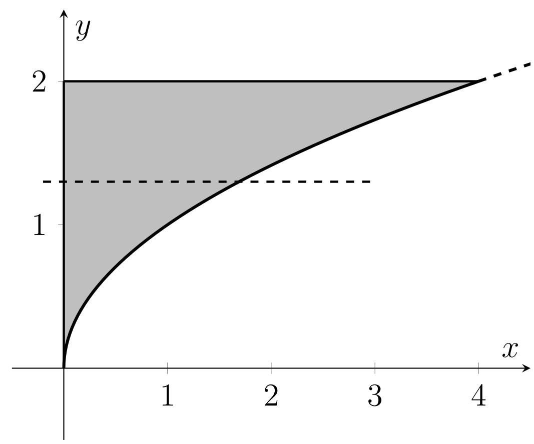
$$f(5, 5) = f(-5, -5) = 25 + 25 = 50$$
$$f(\sqrt{5}, -\sqrt{5}) = f(-\sqrt{5}, \sqrt{5}) = 5 + 5 = 10$$

Therefore, $(5, 5)$ and $(-5, -5)$ are the points on the ellipse farthest from the origin and $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$ are the points on the ellipse closest to the origin.

7. (20 points) Integrate the following:

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\; dy\; dx$$

As written, the function 'cannot' be integrated. The only hope is to change the order of integration. We sketch the region of integration. The boundary of the region is given by $y = \sqrt{x}$, $y = 3$, $x = 0$, *and* $x = 4$.

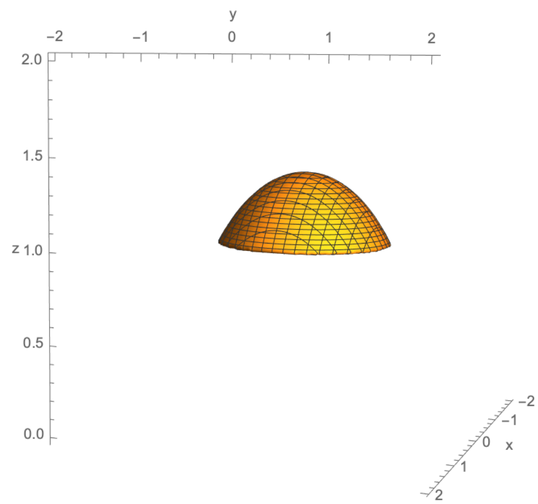


To reverse the order, we slice in y *and see how* x *varies. Observe that for each* y*-slice,* x *goes from* $x = 0$ *to* $y = \sqrt{x} \Leftrightarrow x = y^2$. *We can choose any* $0 \leq y \leq 2$. *Therefore, we have*

$$\begin{aligned}
\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\; dy\; dx &= \int_0^2 \int_0^{y^2} e^{y^3}\; dx\; dy \\
&= \int_0^2 x e^{y^3} \Big|_{x=0}^{x=y^2}\; dy \\
&= \int_0^2 (y^2 - 0)e^{y^3}\; dy \\
&= \int_0^2 y^2 e^{y^3}\; dy \\
&= \frac{1}{3} e^{y^3} \Big|_{y=0}^{y=2} \\
&= \frac{1}{3}\left(e^8 - e^0\right) \\
&= \frac{e^8 - 1}{3} \approx 993.319
\end{aligned}$$

8. (20 points) Write down an integral in cylindrical coordinates to compute the volume of the region inside the sphere centered at the origin with radius $\sqrt{2}$ and above the plane $z = 1$. *You do not need to evaluate the integral.*

The region is the 'cap' of the sphere, which we have plotted below.



Observe that for each z-slice, the intersection with this 'cap' is a circle (or a point at the very top of the 'cap'). So we will use cylindrical coordinates.

The smallest z-slice we can choose is $z = 1$, and the largest z-slice we can choose is at the top of the sphere (where $x = y = 0$), so that $z^2 = 2$, i.e. $z = \sqrt{2}$ because $z > 0$.

The equation of the sphere is $x^2 + y^2 + z^2 = 2$. For each z-slice, the intersection of $z = z_0$ with the 'cap' is $x^2 + y^2 = 2 - z_0^2$, i.e. a circle with radius $r = \sqrt{2 - z_0^2}$. We need to integrate over the entirety of each one of these circles, i.e. $0 \leq r \leq \sqrt{2 - z_0^2}$ and $0 \leq \theta \leq 2\pi$. Alternatively, if you slice in r, θ, then z varies from $z = 1$ to $\sqrt{2 - r^2}$ (because $r^2 + z^2 = 2$), and we can choose $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$ (notice the max r occurs when $x^2 + y^2 + 1^2 = 2$, i.e. where the sphere intersects the plane).

$$
\begin{aligned}
V &= \iiint_R dV \\
&= \iiint_R r \, dr \, d\theta \, dz \\
&= \int_1^{\sqrt{2}} \int_0^{2\pi} \int_0^{\sqrt{2-z^2}} r \, dr \, d\theta \, dz \\
&= \frac{\pi(4\sqrt{2} - 5)}{3} \approx 0.687856
\end{aligned}
$$

OR

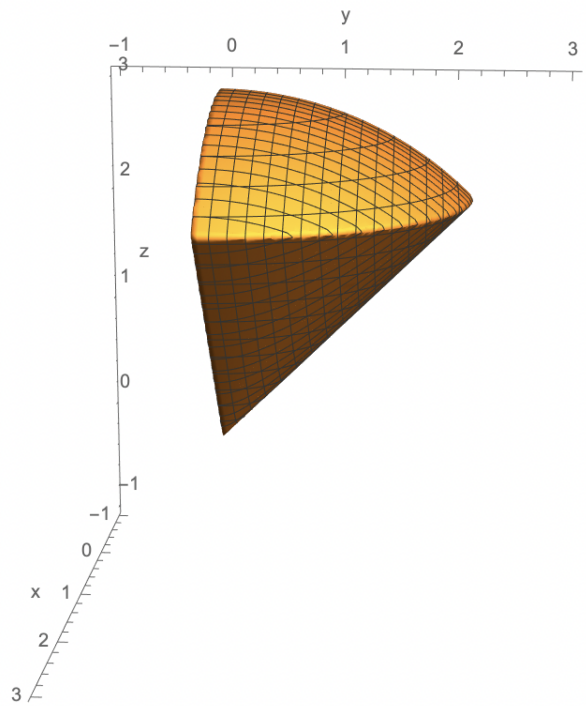
$$
\begin{aligned}
V &= \iiint_R dV \\
&= \iiint_R r \, dr \, d\theta \, dz \\
&= \int_0^{2\pi} \int_0^{1} \int_1^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta \\
&= \frac{\pi(4\sqrt{2} - 5)}{3} \approx 0.687856
\end{aligned}
$$

9. (20 points) Consider the following integral:

$$\iiint_R (x+z)\, dV$$

where R is the region in the first octant bounded above by the sphere centered at the origin with radius 3 and below by the cone $z^2 = x^2 + y^2$. Set-up this integral in spherical coordinates. *You do not need to evaluate the integral.*

The region is the 'sliced ice cream cone' plotted below.



Because we are in the first octant, we will clearly have $0 \leq \theta \leq \dfrac{\pi}{2}$. *For each slice in* θ *and* ϕ, ρ *varies from* 0 *to 3 because we go from the origin out to the sphere.*

Then we need only see how ϕ *varies. Clearly, we begin at* $\phi = 0$. *We need to find the maximum* ϕ. *This can be done many ways. A simple way is to observe we 'see' the maximum* ϕ *for each* $0 \leq x \leq \sqrt{3}$. *So we can choose* $x = 0$. *Using* $z^2 = x^2 + y^2$, *we then have* $z^2 = y^2$. *Because we are in the first octant, we must then have* $z = y$. *Then using the right triangle in the plane* $x = 0$, *i.e. the* yz*-plane, we have* $\tan\phi = \dfrac{y}{z} = \dfrac{y}{y} = 1$, *so that* $\phi = \frac{\pi}{4}$. *Therefore,*

$$\iiint_R (x+z)\, dV = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^3 (\rho\sin\phi\cos\theta + \rho\cos\phi)\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{81(\pi-1)}{16} \approx 10.8418$$

10. Consider an object whose shape is given by the region in Quadrants I & II bounded by the curves $x^2+y^2=1$ and $x^2+y^2=5$, and whose mass density at a point (x,y) is given by $D(x,y)=\dfrac{1}{x^2+y^2}$. Let $(\overline{x},\overline{y})$ denote the center of mass.

(a) (4 points) Explain why $\overline{x}=0$.

Observe that the region is symmetric about the y-axis, and we have $D(x,y)=D(-x,y)=\dfrac{1}{x^2+y^2}$, so that $D(x,y)$ is also symmetric about the y-axis. Therefore, the mass 'on the left' is equal to the mass 'on the right'. So we must have $\overline{x}=0$.

(b) (8 points) Write down an integral in polar coordinates that gives the total mass of the object. *You do not need to evaluate the integral.*

$$M=\iint_R \rho(x,y)\,dA=\int_0^{\pi}\int_1^{\sqrt{5}}\frac{1}{r^2}\cdot r\,dr\,d\theta=\int_0^{\pi}\int_1^{\sqrt{5}}\frac{1}{r}\,dr\,d\theta=\frac{\pi\ln 5}{2}\approx 2.5281$$

(c) (8 points) Write down an integral in polar coordinates that gives $\overline{y}$. *You do not need to evaluate the integral.*

$$\overline{y}=\frac{1}{M}\iint_R y\,\rho(x,y)\,dA=\frac{1}{M}\int_0^{\pi}\int_1^{\sqrt{5}}\frac{r\sin\theta}{r^2}\cdot r\,dr\,d\theta=\frac{1}{M}\int_0^{\pi}\int_1^{\sqrt{5}}\sin\theta\,dr\,d\theta=\frac{2(\sqrt{5}-1)}{M}\approx 0.9777864$$