Problem 1: What is the geometric meaning of $\mathbf{u} \cdot \mathbf{v} = 0$?

Problem 2: Is it possible that $\mathbf{u} \cdot \mathbf{u} < 0$? What does it mean if $\mathbf{u} \cdot \mathbf{u} = 0$?

Problem 3: Suppose $\mathbf{u} = \langle 3, -1, 0 \rangle$, $\mathbf{v} = \langle 2, 5, 3 \rangle$, $\mathbf{a} = 2\,\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\,\mathbf{j}$, $\mathbf{c} = \mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$. Compute the following dot products:

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $\mathbf{a} \cdot \mathbf{b}$

(c) $\mathbf{b} \cdot \mathbf{c}$

Problem 4: Suppose $\mathbf{a} = \langle 12, 24, 48 \rangle$ and $\mathbf{b} = \langle -20, -5, 10 \rangle$. What is $\mathbf{a} \cdot \mathbf{b}$?

Problem 5: Use the dot product to find the length of the vector $\langle 1, 3, -2 \rangle$.

Problem 6: If $\mathbf{u} = \langle 1, 4, -2 \rangle$ and $\mathbf{v} = \langle -1, 1, 1 \rangle$, find the angle between $\mathbf{u}$ and $\mathbf{v}$.

Problem 7: If $|\mathbf{u}| = 2$, $|\mathbf{v}| = 6$, and the angle between then is $60°$, what is $\mathbf{u} \cdot \mathbf{v}$?

Problem 8: Find any (nonzero) vector perpendicular to $\langle 4, 7, 3 \rangle$.

Problem 9: Find the acute angle between $y = (x+1)^2$ and $y = x^2 - 3x - 29$ at their point(s) of intersection.

Problem 10: Compute the following determinant:

$$\begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix}$$

Problem 11: Compute the following determinant:

$$\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix}$$

Problem 12: Compute the following determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -2 & 3 \end{vmatrix}$$

Problem 13: Compute the following determinant:

$$\begin{vmatrix} 4 & 0 & -1 \\ 1 & 2 & 1 \\ -3 & 0 & 5 \end{vmatrix}$$

Problem 14: Compute the following determinant:

$$\begin{vmatrix} \pi & 2 & 3 \\ 1 & 3e^2 & 2 \\ 0 & 2 & 2 \end{vmatrix}$$

Problem 15: Compute the following determinant:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 0 & 6 & 1 \\ 1 & -2 & 3 & 1 \\ 1 & 0 & 1 & 3 \end{vmatrix}$$

Problem 16: If $\mathbf{u} = \langle 1, 1, 1 \rangle$ and $\mathbf{v} = \langle 2, -1, 5 \rangle$, compute $\mathbf{u} \times \mathbf{v}$. Describe the result geometrically.

Problem 17: If $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle 1, -1 \rangle$, compute $\mathbf{u} \times \mathbf{v}$. Does the final component have to be nonzero?

Problem 18: Use the cross product to find the angle between $\mathbf{u} = \langle 1, 0, -1 \rangle$ and $\mathbf{v} = \langle 2, 0, 1 \rangle$.

Problem 19: Is $\langle 4, -2, 6 \rangle$ parallel to $\langle 2, -1, 3 \rangle$? Use the cross product to explain your answer.

Problem 20: Is $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$? Explain.

Problem 21: Find the area of the triangle spanned by the vectors $\mathbf{u} = \langle 1, 1 \rangle$ and $\mathbf{v} = \langle 1, -1 \rangle$.

Problem 22: Find the area of the triangle with vertices $(1, 0)$, $(-1, 0)$, and $(0, 1)$.

Problem 23: What is $\mathbf{i} \times \mathbf{j}$? What about $\mathbf{k} \times \mathbf{i}$ and $\mathbf{k} \times \mathbf{j}$?

Problem 24: Find the volume of the parallelepiped spanned by $\mathbf{u} = \langle 1, 1, 1 \rangle$, $\mathbf{v} = \langle 1, -1, 1 \rangle$, and $\mathbf{w} = \langle -1, 1, 1 \rangle$.

Problem 25: Is there a plane containing all the vectors $\mathbf{u} = \langle 1, 0, 1 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$, and $\mathbf{w} = \langle 1, 1, 1 \rangle$, i.e. are the vectors coplanar?

Problem 26: Find the projection of $\mathbf{u} = \langle 1, 1 \rangle$ onto $\mathbf{v} = \langle 1, -1 \rangle$. What is the length of this projection? Then what are the lengths of the side of the right triangle formed by $\mathbf{u}$ and the projection?

Problem 27: Find the projection of $\mathbf{u} = \langle 1, -1, 1 \rangle$ onto $\mathbf{v} = \langle 2, 0, -3 \rangle$. What is the length of this projection? What are the lengths of the side of the right triangle formed by $\mathbf{u}$ and the projection?

Solutions

Problem 1: What is the geometric meaning of $\mathbf{u} \cdot \mathbf{v} = 0$?

Solution. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} \perp \mathbf{v}$. This works both ways, so if $\mathbf{u} \perp \mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

Problem 2: Is it possible that $\mathbf{u} \cdot \mathbf{u} < 0$? What does it mean if $\mathbf{u} \cdot \mathbf{u} = 0$?

Solution. Recall that $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$, where $\|\mathbf{u}\|$ is the length of $\mathbf{u}$. But length is always 0 or bigger. [In fact, the length of a vector is 0 if and only if $\mathbf{u}$ is the zero vector, i.e. $\mathbf{u} = \mathbf{0}$. This is also written $\mathbf{u} = \vec{0} = \langle 0, 0, \ldots, 0 \rangle$.] But then $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 \geq 0$, so that it is impossible that $\mathbf{u} \cdot \mathbf{u} < 0$. If $\mathbf{u} \cdot \mathbf{u} = 0$, then $\|\mathbf{u}\|^2 = 0$ so that $\|\mathbf{u}\| = 0$. But then $\mathbf{u}$ is the zero vector. Note it is possible that $\mathbf{u} \cdot \mathbf{v} < 0$ if $\mathbf{v} \neq \mathbf{u}$. For example, if $\mathbf{u} = \langle 1, 0 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$, then $\mathbf{u} \cdot \mathbf{v} = 1(-1) + 0(2) = -1 + 0 = -1 < 0$.

Problem 3: Suppose $\mathbf{u} = \langle 3, -1, 0 \rangle$, $\mathbf{v} = \langle 2, 5, 3 \rangle$, $\mathbf{a} = 2\,\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\,\mathbf{j}$, $\mathbf{c} = \mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$. Compute the following dot products:

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $\mathbf{a} \cdot \mathbf{b}$

(c) $\mathbf{b} \cdot \mathbf{c}$

Solution.

(a) $\mathbf{u} \cdot \mathbf{v} = \langle 3, -1, 0 \rangle \cdot \langle 2, 5, 3 \rangle = 3(2) - 1(5) + 0(3) = 6 - 5 + 0 = 1$

(b) $\mathbf{a} \cdot \mathbf{b} = (2\,\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} + 3\,\mathbf{j}) = \langle 2, -1 \rangle \cdot \langle 1, 3 \rangle = 2(1) - 1(3) = -1$

(c) $\mathbf{b} \cdot \mathbf{c} = (\mathbf{i} + 3\,\mathbf{j}) \cdot (\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}) = \langle 1, 3, 0 \rangle \cdot \langle 1, -1, 2 \rangle = 1(1) + 3(-1) + 2(0) = 1 - 3 + 2 = 0$

Problem 4: Suppose $\mathbf{a} = \langle 12, 24, 48 \rangle$ and $\mathbf{b} = \langle -20, -5, 10 \rangle$. What is $\mathbf{a} \cdot \mathbf{b}$?

Solution. You can compute this directly, but it is easier to use the following: if c is a scalar, then $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ and $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$. Now $\mathbf{a} = \langle 12, 24, 48 \rangle = 12\langle 1, 2, 4 \rangle$ and $\mathbf{b} = \langle -20, -5, 10 \rangle = -5\rangle 4, 1, -2 \rangle$. Then

$$\mathbf{a} \cdot \mathbf{b} = 12\langle 1, 2, 4 \rangle \cdot -5\rangle 4, 1, -2 \rangle = (12 \cdot -5)\big(1(4) + 2(1) + 4(-2)\big) = -60 \cdot (4 + 2 - 8) = -60 \cdot -2 = 120$$

Problem 5: Use the dot product to find the length of the vector $\langle 1, 3, -2 \rangle$.

Solution. We know $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$. Now

$$\begin{aligned} \|\langle 1, 3, -2 \rangle\|^2 &= \langle 1, 3, -2 \rangle \cdot \langle 1, 3, -2 \rangle \\ &= 1(1) + 3(3) - 2(-2) \\ &= 1 + 9 + 4 \\ &= 14 \end{aligned}$$

Then $\|\langle 1, 3, -2 \rangle\|^2 = 14$ so that $\|\langle 1, 3, -2 \rangle\| = \sqrt{14}$.

Problem 6: If $\mathbf{u} = \langle 1, 4, -2\rangle$ and $\mathbf{v} = \langle -1, 1, 1\rangle$, find the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution. Recall $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$.

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{1^2+4^2+(-2)^2} = \sqrt{1+16+4} = \sqrt{21}\\ \|\mathbf{v}\| &= \sqrt{(-1)^2+1^2+1^2} = \sqrt{1+1+1} = \sqrt{3}\\ \mathbf{u}\cdot\mathbf{v} &= \langle 1,4,-2\rangle\cdot\langle -1,1,1\rangle = 1(-1)+4(1)-2(1) = -1+4-2 = 1\end{aligned}$$

But then we have

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta\\ 1 &= \sqrt{21}\sqrt{3}\cos\theta\\ 1 &= \sqrt{3\cdot 7\cdot 3}\cos\theta\\ 1 &= 3\sqrt{7}\cos\theta\\ \theta &= \cos^{-1}\left(\frac{1}{3\sqrt{7}}\right) \approx 82.76^\circ\end{aligned}$$

Problem 7: If $|\mathbf{u}| = 2$, $|\mathbf{v}| = 6$, and the angle between then is 60°, what is $\mathbf{u}\cdot\mathbf{v}$?

Solution. We know that $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$, so that

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta = 2\cdot 6\cdot\cos\left(\frac{\pi}{3}\right) = 2\cdot 6\cdot\frac{1}{2} = 6$$

Problem 8: Find any (nonzero) vector perpendicular to $\langle 4, 7, 3\rangle$.

Solution. A simple way is to make one component zero, then 'switch and negate.' For example, $\langle 7, -4, 0\rangle$ and $\langle -7, 4, 0\rangle$ are both perpendicular to $\langle 4, 7, 3\rangle$. Generally, if $\langle x, y, z\rangle$ is perpendicular to $\langle 4, 7, 3\rangle$, then $0 = \langle 4, 7, 3\rangle\cdot\langle x, y, z\rangle = 4x + 7y + 3z$, so that $4x + 7y + 3z = 0$. [This is the equation of a plane, not surprising because a plane is defined in terms of a normal vector.] Then one can simply plug for any two of the variables and solve for the third. For example, let $x = 1, y = -2$, then $0 = 4(1) + 7(-2) + 3z = 4 - 14 + 3z = -10 + 3z$ so that $z = 3/10$. Then $\langle 1, -2, 3/10\rangle$ is perpendicular to $\langle 4, 7, 3\rangle$.

Problem 9: Find the acute angle between $y = (x + 1)^2$ and $y = x^2 - 3x - 29$ at their point(s) of intersection.

Solution. The angle between curves is the angle between the direction vectors at their point(s) of intersection. First, we find where the curves intersect.

$$\begin{aligned}(x+1)^2 &= x^2 - 3x - 29\\ x^2 + 2x + 1 &= x^2 - 3x - 29\\ 5x + 30 &= 0\\ 5x &= -30\\ x &= -6\end{aligned}$$

Then $y = (x+1)^2 = (-6+1)^2 = 25$ so that the curves intersect at $(-6, 25)$. Because there is only one point of intersection, we have only one angle to compute. Now we need to find the direction vector, i.e. the direction the tangent line points, for each curve at the point of intersection.

$$y' = 2(x+1)\Big|_{x=-6} = -10$$
$$y' = 2x - 3\Big|_{x=-6} = -15$$

Then the direction vectors are $\langle 1, -10 \rangle$ and $\langle 1, -15 \rangle$, respectively. [Notice that the $\frac{\Delta y}{\Delta x}$ matches the slope of the tangent line.] Now we use $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta$ to compute the angle between the curves:

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta \\
\langle 1, -10 \rangle \cdot \langle 1, -15 \rangle &= \|\langle 1, -10 \rangle\| \, \|\langle 1, -15 \rangle\| \, \cos\theta \\
1(1) - 10(-15) &= \sqrt{1^2 + (-10)^2} \cdot \sqrt{1^2 + (-15)^2} \cdot \cos\theta \\
1 + 150 &= \sqrt{101} \cdot \sqrt{226} \cos\theta \\
151 &= \sqrt{22826} \cos\theta \\
\theta &= \cos^{-1}\left(\frac{151}{\sqrt{22826}} \right) \approx 1.897^\circ
\end{aligned}$$

Problem 10: Compute the following determinant:

$$\begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix} = 1(5) - 4(-3) = 5 + 12 = 17$$

Problem 11: Compute the following determinant:

$$\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix}$$

Solution.

$$\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} = 6(1) - 2(3) = 6 - 6 = 0$$

Problem 12: Compute the following determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -2 & 3 \end{vmatrix}$$

Solution. We can expand along any row or column. We choose the first row.

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -2 & 3 \end{vmatrix} &= 1 \begin{vmatrix} 5 & 6 \\ -2 & 3 \end{vmatrix} - (-2) \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 1 & -2 \end{vmatrix} \\
&= 1\big(5(3) - 6(-2)\big) + 2\big(4(3) - 1(6)\big) + 3\big(4(-2) - 5(1)\big) \\
&= (15 + 12) + 2(12 - 6) + 3(-8 - 5) \\
&= 27 + 12 - 39 \\
&= 0
\end{aligned}$$

Problem 13: Compute the following determinant:

$$\begin{vmatrix} 4 & 0 & -1 \\ 1 & 2 & 1 \\ -3 & 0 & 5 \end{vmatrix}$$

Solution. We can expand along any row or column. We choose the second column because it has the most zero entries.

$$\begin{aligned} \begin{vmatrix} 4 & 0 & -1 \\ 1 & 2 & 1 \\ -3 & 0 & 5 \end{vmatrix} &= -0 \begin{vmatrix} 1 & 1 \\ -3 & 5 \end{vmatrix} + 2 \begin{vmatrix} 4 & -1 \\ -3 & 5 \end{vmatrix} - 0 \begin{vmatrix} 4 & -1 \\ 1 & 1 \end{vmatrix} \\ &= 0 + 2\big(4(5) - (-3)(-1)\big) - 0 \\ &= 20 - 3 \\ &= 17 \end{aligned}$$

Problem 14: Compute the following determinant:

$$\begin{vmatrix} \pi & 2 & 3 \\ 1 & 3e^2 & 2 \\ 0 & 2 & 2 \end{vmatrix}$$

Solution. We can expand along any row or column. We choose the third row because it has the most zero entries.

$$\begin{aligned} \begin{vmatrix} \pi & 2 & 3 \\ 1 & 3e^2 & 2 \\ 0 & 2 & 2 \end{vmatrix} &= 0 \begin{vmatrix} 2 & 3 \\ 3e^2 & 2 \end{vmatrix} - 2 \begin{vmatrix} \pi & 3 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} \pi & 2 \\ 1 & 3e^2 \end{vmatrix} \\ &= 0 - 2\big(2\pi - 3\big) + 2\big(3\pi e^2 - 2\big) \\ &= -4\pi + 6 + 6\pi e^2 - 4 \\ &= 2 - 4\pi + 6\pi e^2 \end{aligned}$$

Problem 15: Compute the following determinant:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 0 & 6 & 1 \\ 1 & -2 & 3 & 1 \\ 1 & 0 & 1 & 3 \end{vmatrix}$$

Solution. We can expand along any row or column. We choose the second column because it has the most zero entries.

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 0 & 6 & 1 \\ 1 & -2 & 3 & 1 \\ 1 & 0 & 1 & 3 \end{vmatrix} &= -2 \begin{vmatrix} 4 & 6 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 3 & 4 \\ 4 & 6 & 1 \\ 1 & 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 & 4 \\ 4 & 6 & 1 \\ 1 & 3 & 1 \end{vmatrix} \\ &= -2\left(4 \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 6 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}\right) + 0 + 2\left(1 \begin{vmatrix} 6 & 1 \\ 1 & 3 \end{vmatrix} - 3 \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} + 4 \begin{vmatrix} 4 & 6 \\ 1 & 1 \end{vmatrix}\right) + 0 \end{aligned}$$

$$
\begin{aligned}
&= -2\Big(4\big(3(3)-1(1)\big)-6\big(1(3)-1(1)\big)+1\big(1(1)-1(3)\big)\Big)+ \\
&\qquad\qquad 2\Big(1\big(6(3)-1(1)\big)-3\big(4(3)-1(1)\big)+4\big(4(1)-1(6)\big)\Big) \\
&= -2\Big(4(9-1)-6(3-1)+1(1-3)\Big)+2\Big(1(18-1)-3(12-1)+4(4-6)\Big) \\
&= -2\Big(4(8)-6(2)+1(-2)\Big)+2\Big(1(17)-3(11)+4(-2)\Big) \\
&= -2\big(32-12-2\big)+2\big(17-33-8\big) \\
&= -2(18)+2(-24) \\
&= -36-48 \\
&= -84
\end{aligned}
$$

Problem 16: If $\mathbf{u} = \langle 1,1,1 \rangle$ and $\mathbf{v} = \langle 2,-1,5 \rangle$, compute $\mathbf{u} \times \mathbf{v}$. Describe the result geometrically.

Solution.

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -1 & 5 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 1 & 1 \\ -1 & 5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\
&= \mathbf{i}\big(1(5)-1(-1)\big) - \mathbf{j}\big(1(5)-1(2)\big) + \mathbf{k}\big(1(-1)-2(1)\big) \\
&= \mathbf{i}(5+1) - \mathbf{j}(5-2) + \mathbf{k}(-1-2) \\
&= 6\,\mathbf{i} - 3\mathbf{j} - 3\,\mathbf{k} \\
&= \langle 6,-3,-3 \rangle
\end{aligned}
$$

The vector $\mathbf{u} \times \mathbf{v}$ is a vector which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Problem 17: If $\mathbf{u} = \langle 2,3 \rangle$ and $\mathbf{v} = \langle 1,-1 \rangle$, compute $\mathbf{u} \times \mathbf{v}$. Does the final component have to be nonzero?

Solution.

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 1 & -1 & 0 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 3 & 0 \\ -1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \\
&= \mathbf{i}\big(3(0)-0(-1)\big) - \mathbf{j}\big(2(0)-1(0)\big) + \mathbf{k}\big(2(-1)-3(1)\big) \\
&= \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(-2-3) \\
&= -5\,\mathbf{k} \\
&= \langle 0,0,-5 \rangle
\end{aligned}
$$

Because $\mathbf{u}$ and $\mathbf{v}$ lie in the xy-plane, we expect the cross product to point straight up/down. Then the last component must be nonzero with the other components zero, which is the form our answer takes.

Problem 18: Use the cross product to find the angle between $\mathbf{u} = \langle 1, 0, -1\rangle$ and $\mathbf{v} = \langle 2, 0, 1\rangle$.

Solution. We know that $|\mathbf{u} \times \mathbf{v}| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta$. Now

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2} \\
\|\mathbf{v}\| &= \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5} \\
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 2 & 0 & 1 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \\
&= \mathbf{i}(0-0) - \mathbf{j}(1+2) + \mathbf{k}(0-0) \\
&= -3\,\mathbf{j} \\
&= \langle 0, -3, 0\rangle \\
|\mathbf{u} \times \mathbf{v}| &= \sqrt{0^2 + (-3)^2 + 0^2} = \sqrt{9} = 3
\end{aligned}$$

Then we have

$$\begin{aligned}
|\mathbf{u} \times \mathbf{v}| &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta \\
3 &= \sqrt{2}\sqrt{5}\sin\theta \\
\theta &= \sin^{-1}\left(\frac{3}{\sqrt{10}}\right) \approx 71.57^\circ
\end{aligned}$$

Notice this is a computationally slower method than using the dot product to compute the angle via $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$.

Problem 19: Is $\langle 4, -2, 6\rangle$ parallel to $\langle 2, -1, 3\rangle$? Use the cross product to explain your answer.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix}$$

Solution. Observe if $\langle 4, -2, 6\rangle = c\langle 2, -1, 3\rangle = \langle 2c, -c, 3c\rangle$, then equating the first components, $4 = 2c$ so that $c = 2$. Now $2\langle 2, -1, 3\rangle = \langle 4, -2, 6\rangle$. Then the vectors are parallel. Using the cross product,

$$\begin{aligned}
\langle 4, -2, 6\rangle \times \langle 2, -1, 3\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & 6 \\ 2 & -1 & 3 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} -2 & 6 \\ -1 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 4 & -2 \\ 2 & -1 \end{vmatrix} \\
&= \mathbf{i}(-6+6) - \mathbf{j}(12-12) + \mathbf{k}(-4+4) \\
&= \mathbf{0}
\end{aligned}$$

Because $\langle 4, -2, 6\rangle \times \langle 2, -1, 3\rangle = \mathbf{0}$, the vectors must be parallel. Notice this method is computationally slower than the ‘scaling’ method.

Problem 20: Is $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$? Explain.

Solution. This is one of the properties of the cross product. Intuitively, reversing the order of the cross product flips your hand upside-down in the right hand rule (RHR). The negative reverses the vector back to the original RHR direction.

Problem 21: Find the area of the triangle spanned by the vectors $\mathbf{u} = \langle 1, 1 \rangle$ and $\mathbf{v} = \langle 1, -1 \rangle$.
Solution. The area of the triangle is half the area of the parallelogram spanned by the two vectors. The area of the parallelogram spanned by two vectors is $|\mathbf{u} \times \mathbf{v}|$.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\
&= \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(-1-1) \\
&= -2\,\mathbf{k} \\
&= \langle 0, 0, -2 \rangle \\
|\mathbf{u} \times \mathbf{v}| &= \sqrt{0^2 + 0^2 + (-2)^2} = \sqrt{4} = 2
\end{aligned}$$

Then the area of the triangle is $A = \frac{1}{2} \cdot 2 = 1$.

Problem 22: Find the area of the triangle with vertices $(1, 0)$, $(-1, 0)$, and $(0, 1)$.

Solution. First, we form the displacement vectors from $(1, 0)$ to $(-1, 0)$ and $(0, 1)$. [These will be vectors that form two sides of the triangle.]

$$\begin{aligned}
\mathbf{u} &:= (-1, 0) - (1, 0) = \langle -2, 0 \rangle \\
\mathbf{v} &:= (0, 1) - (1, 0) = \langle -1, 1 \rangle
\end{aligned}$$

The area of the triangle will be half the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$, which is $|\mathbf{u} \times \mathbf{v}|$.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 0 \\ -1 & 1 & 0 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & 0 \\ -1 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 0 \\ -1 & 1 \end{vmatrix} \\
&= \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(-2-0) \\
&= -2\,\mathbf{k} \\
&= \langle 0, 0, -2 \rangle \\
|\mathbf{u} \times \mathbf{v}| &= \sqrt{0^2 + 0^2 + (-2)^2} = \sqrt{4} = 2
\end{aligned}$$

Then the area of the triangle is $A = \frac{1}{2} \cdot 2 = 1$.

Problem 23: What is $\mathbf{i} \times \mathbf{j}$? What about $\mathbf{k} \times \mathbf{i}$ and $\mathbf{k} \times \mathbf{j}$?

Solution. Using the 'circle diagram', we know $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, and (noticing we have to go counterclockwise) $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$.

Problem 24: Find the volume of the parallelepiped spanned by $\mathbf{u} = \langle 1, 1, 1 \rangle$, $\mathbf{v} = \langle 1, -1, 1 \rangle$, and $\mathbf{w} = \langle -1, 1, 1 \rangle$.

Solution. The volume of the parallelepiped spanned by the three vectors is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. Using the determinant 'trick', we have

$$\begin{aligned}
V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \\
&= \left| \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \right| \\
&= \left| 1 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \right| \\
&= |1(-1-1) - 1(1-(-1)) + 1(1-1)| \\
&= |-2-2| \\
&= |-4| \\
&= 4
\end{aligned}$$

Problem 25: Is there a plane containing all the vectors $\mathbf{u} = \langle 1, 0, 1 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$, and $\mathbf{w} = \langle 1, 1, 1 \rangle$, i.e. are the vectors coplanar?

Solution. The vectors lie in the same plane if and only if the volume of the parallelepiped formed by the vectors is zero, i.e. the parallelepiped lies flat so that all the vectors are in a plane.

$$\begin{aligned}
V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \\
&= \left| \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \right| \\
&= \left| 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \right| \\
&= |1(1-0) - 0(0-0) + 1(0-1)| \\
&= |1-1| \\
&= |0| \\
&= 0
\end{aligned}$$

Therefore, there is a plane containing all three vectors.

Problem 26: Find the projection of $\mathbf{u} = \langle 1, 1 \rangle$ onto $\mathbf{v} = \langle 1, -1 \rangle$. What is the length of this projection? Then what are the lengths of the side of the right triangle formed by $\mathbf{u}$ and the projection?

Solution.

$$\mathbf{proj_v\, u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \left(\frac{1(1) + 1(-1)}{1(1) + (-1)(-1)} \right) \langle 1, -1 \rangle = \frac{0}{2} \langle 1, -1, \rangle = \langle 0, 0 \rangle$$

Then $\|\mathbf{proj_v\, u}\| = \|\mathbf{0}\| = 0$. Note that this means that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, i.e. $\mathbf{u}$ 'casts no shadow' onto $\mathbf{v}$. We can also see this from the fact that $\mathbf{u} \cdot \mathbf{v} = 0$ from the computation of $\mathbf{proj_v\, u}$ so that $\mathbf{u} \perp \mathbf{v}$. The legs of the right triangle formed by $\mathbf{u}$ and $\mathbf{v}$ are the vectors themselves. These have lengths $\|\mathbf{u}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Then using the Pythagorean Theorem, i.e. $a^2 + b^2 = c^2$, we have $c^2 = \sqrt{2}^2 + \sqrt{2}^2 = 2 + 2 = 4$ so that $c = \sqrt{4} = 2$.

Problem 27: Find the projection of $\mathbf{u} = \langle 1, -1, 1 \rangle$ onto $\mathbf{v} = \langle 2, 0, -3 \rangle$. What is the length of this projection? What are the lengths of the side of the right triangle formed by $\mathbf{u}$ and the projection?

Solution.

$$\mathbf{proj_v\, u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \left(\frac{1(2) + (-1)0 + 1(-3)}{2(2) + 0(0) + (-3)(-3)} \right) \langle 2, 0, -3 \rangle = \frac{2 + 0 - 3}{4 + 0 + 9} \langle 2, 0, 3 \rangle = -\frac{1}{13} \langle 2, 0, -3 \rangle$$

Then $\|\mathbf{proj_v\, u}\| = \|\frac{-1}{13} \langle 2, 0, 3 \rangle\| = \frac{1}{13} \sqrt{2^2 + 0^2 + 3^2} = \frac{1}{13} \sqrt{13} = \frac{1}{\sqrt{13}}$. The right triangle formed by $\mathbf{u}$ and the projection are $\mathbf{u}$, $\mathbf{proj_v\, u}$, and $\mathbf{u} - \mathbf{proj_v\, u}$. These have lengths

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \\
\|\mathbf{proj_v\, u}\| &= \frac{1}{\sqrt{13}} \\
\|\mathbf{u} - \mathbf{proj_v\, u}\| &= \|\langle 1, -1, 1 \rangle - \frac{-1}{13} \langle 2, 0, -3 \rangle\| \\
&= \left\| \left\langle \frac{15}{13}, -1, \frac{10}{13} \right\rangle \right\| \\
&= \|\frac{1}{13} \langle 15, -13, -10 \rangle\| \\
&= \frac{1}{13} \sqrt{15^2 + (-13)^2 + 10^2} \\
&= \frac{\sqrt{494}}{13} = \frac{\sqrt{2 \cdot 13 \cdot 19}}{13} = \sqrt{\frac{38}{13}}
\end{aligned}$$

Note, you could also have found the lengths of any two sides of the triangle—for instance $\mathbf{u}$ and $\mathbf{proj_v\, u}$—then used the Pythagorean Theorem. For example, $\|\mathbf{u}\|^2 = \|\mathbf{proj_v\, u}\|^2 + \|\mathbf{u} - \mathbf{proj_v\, u}\|^2$ so that $3 = 1/13 + \|\mathbf{u} - \mathbf{proj_v\, u}\|^2$, and one again finds $\|\mathbf{u} - \mathbf{proj_v\, u}\| = \sqrt{\frac{38}{13}}$.