Quiz 1: Let $\mathbf{u} = \langle\sqrt{3}, 1\rangle$ and $\mathbf{v} = \langle-\sqrt{3}, 2\rangle$

(a) Find $2\mathbf{u} - \mathbf{v}$.

$$2\mathbf{u} - \mathbf{v} = 2\langle\sqrt{3}, 1\rangle - \langle-\sqrt{3}, 2\rangle = \langle 2\sqrt{3}, 2\rangle + \langle\sqrt{3}, -2\rangle = \langle 3\sqrt{3}, 0\rangle$$

(b) Find $\|\mathbf{u}\|$.

$$\|\mathbf{u}\| = \|\langle\sqrt{3}, 1\rangle\| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

(c) Is $\mathbf{u}$ parallel to $\mathbf{v}$?

No, if $\mathbf{u}$ were parallel to $\mathbf{v}$, then there would be a c so that $\mathbf{v} = c\mathbf{u}$. But $c\mathbf{u} = \langle c\sqrt{3}, c\rangle$. Comparing the second component, we would have $c = 2$. But then $c\mathbf{u} = 2\langle\mathbf{u} = \langle 2\sqrt{3}, 2\rangle \neq \mathbf{v}$.

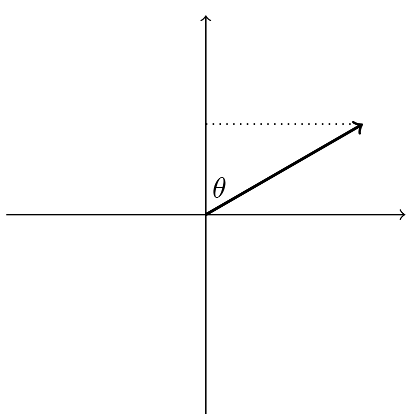
(d) Find a unit vector parallel to $\mathbf{u}$.

We know that for any nonzero vector $\mathbf{a}$, $\dfrac{\mathbf{a}}{\|\mathbf{a}\|}$ is always parallel to $\mathbf{a}$. Then the following vector is parallel to $\mathbf{u}$:

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle\sqrt{3}, 1\rangle}{2} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

Of course, $-\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \left\langle -\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2} \right\rangle$ is also a unit vector parallel to $\mathbf{u}$.

(e) What is the angle $\mathbf{u}$ makes with the $+y$-direction?



$$\begin{aligned} \tan\theta &= \frac{\text{opp}}{\text{adj}} \\ \tan\theta &= \frac{\sqrt{3}}{1} \\ \theta &= \arctan(\sqrt{3}) \\ \theta &= \frac{\pi}{3} \end{aligned}$$

$\square$

Quiz 2: Let $\mathbf{u} = \langle 1, -2, 1\rangle$ and $\mathbf{v} = \langle 1, -1, 3\rangle$.

(a) Find any nonzero vector perpendicular to $\mathbf{u}$.

Any nonzero vector $\mathbf{a} = \langle x, y, z\rangle$ *with* $\mathbf{u}\cdot\mathbf{a} = 0$ *is perpendicular to* $\mathbf{u}$. *But then* $0 = \mathbf{u}\cdot\mathbf{a} = x - 2y + z$. *Any choice of* x, y, z *that make this valid works. For instance,* $\langle 1, 0, -1\rangle$, $\langle -1, 0, 1\rangle$, $\langle 2, 1, 0\rangle$, $\langle 0, 1, 2\rangle$, $\langle 4, 1, 2\rangle$, *etc.*

(b) Is $\mathbf{u}$ perpendicular to $\mathbf{v}$?

$\mathbf{u}\cdot\mathbf{v} = \langle 1, -2, 1\rangle \cdot \langle 1, -1, 3\rangle = 1(1) + (-2)(-1) + 1(3) = 1 + 2 + 3 = 6 \neq 0$. *Therefore,* $\mathbf{u}$ *and* $\mathbf{v}$ *are not perpendicular.*

(c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \\
6 &= \sqrt{6}\cdot\sqrt{11}\cdot\cos\theta \\
\cos\theta &= \frac{6}{\sqrt{6}\sqrt{11}} \\
\cos\theta &= \sqrt{\frac{6}{11}} \\
\theta &= \cos^{-1}\left(\sqrt{\frac{6}{11}}\right) \approx 42.392^\circ
\end{aligned}$$

(d) Find $\mathbf{proj_v\, u}$.

$$\mathbf{proj_v\, u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6}{11}\langle 1, -1, 3\rangle = \left\langle \frac{6}{11}, -\frac{6}{11}, \frac{18}{11}\right\rangle$$

□

Quiz 3: Let $\mathbf{u} = 2\,\mathbf{i} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$.

(a) Find a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

(b) Find the area of the triangle that can be formed using $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$.

Solution.

(a) The cross product of vectors results in a vector perpendicular to them both:

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 1 & -3 & 1 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 0 & 1 \\ -3 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} \\
&= \mathbf{i}\big(0(1) - (-3)1\big) - \mathbf{j}\big(2(1) - 1(1)\big) + \mathbf{k}\big(2(-3) - 1(0)\big) \\
&= \mathbf{i}(0 + 3) - \mathbf{j}(2 - 1) + \mathbf{k}(-6 - 0) \\
&= 3\,\mathbf{i} - \mathbf{j} - 6\,\mathbf{k} = \langle 3, -1, -6\rangle
\end{aligned}$$

But then we need to make this into a unit vector. The length of $\mathbf{u}\times\mathbf{v}$ is $\|\mathbf{u}\times\mathbf{v}\| = \|\langle 3,-1,-6\rangle\| = \sqrt{3^2+(-1)^2+(-6)^2} = \sqrt{9+1+36} = \sqrt{46}$. Then

$$\frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|} = \frac{\langle 3,-1,-6\rangle}{\sqrt{46}} = \left\langle \frac{3}{\sqrt{46}}, -\frac{1}{\sqrt{46}}, -\frac{6}{\sqrt{46}} \right\rangle$$

is a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Furthermore, $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v})$ so that $\mathbf{v}\times\mathbf{u}$ points in the 'opposite' direction of $\mathbf{u}\times\mathbf{v}$. Therefore,

$$\frac{\mathbf{v}\times\mathbf{u}}{\|\mathbf{v}\times\mathbf{u}\|} = \frac{\langle -3,1,6\rangle}{\sqrt{46}} = \left\langle \frac{-3}{\sqrt{46}}, \frac{1}{\sqrt{46}}, \frac{6}{\sqrt{46}} \right\rangle$$

is also a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

(b) The area of the triangle 'spanned' by $\mathbf{u}$ and $\mathbf{v}$ is half the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$—which is $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{v}\times\mathbf{u}\|$. Therefore, the area is

$$A = \frac{\|\mathbf{u}\times\mathbf{v}\|}{2} = \frac{\sqrt{46}}{2} = \sqrt{\frac{23}{2}}$$

□

Quiz 4: Find the vector, parametric, and symmetric forms of the lines through the point $(6,-1,4)$ and parallel to the line $x(t) = t-1, y(t) = 2t+6, z(t) = 4-3t$.

Solution. The line must contain $(6,-1,4)$, and because the line must be parallel to the given line, the slope vector must be $\langle 1,2,-3\rangle$. Then the vector form of the line is $\ell(t) = \langle 1,2,-3\rangle t + \langle 6,-1,4\rangle = \langle t+6, 2t-1, 4-3t\rangle$. Then immediately gives the parametric form as

$$\begin{cases} x = t+6 \\ y = 2t-1 \\ z = 4-3t \end{cases}$$

Solving for t in each equation gives the symmetric form:

$$\frac{x-6}{1} = \frac{y+1}{2} = \frac{z-4}{-3}$$

□

Quiz 5: Find the equation of the plane through $(1,-1,1)$, $(1,0,1)$, and $(3,4,2)$.

Solution. We form vectors $\mathbf{u} = (1,-1,1)-(1,0,1) = \langle 0,-1,0\rangle$ and $\mathbf{v} = (3,4,2)-(1,0,1) = \langle 2,4,1\rangle$. These vectors lie in the plane. Therefore, a vector perpendicular to the plane is

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 0 \\ 2 & 4 & 1 \end{vmatrix} \\
&= \mathbf{i}\big(1(-1)-4(0)\big) - \mathbf{j}\big(0(1)-2(0)\big) + \mathbf{k}\big(0(4)-2(-1)\big) \\
&= -\mathbf{i} - 0\mathbf{j} + 2\mathbf{k} \\
&= \langle -1,0,2\rangle
\end{aligned}$$

The plane contains the point $(1,0,1)$. Therefore, the equation for the plane is

$$\begin{aligned}
\langle -1,0,2\rangle \cdot \langle x-1, y-0, z-1\rangle &= 0 \\
-1(x-1) + 0(y-0) + 2(z-1) &= 0 \\
-x+1+2z-2 &= 0 \\
-x+2z &= 1
\end{aligned}$$

□

Quiz 6: Identify the following surfaces in $\mathbb{R}^3$:

(a) Plane : $y = 2x - z$

(b) Hyperboloid of One Sheet : $x^2 + y^2 = \dfrac{z^2}{2} + 1$

(c) Cone : $\dfrac{x^2}{2} - 3y^2 - z^2 = 0$

(d) Parabolic Cylinder : $z = y^2 + 1$

(e) Sphere : $2x^2 + 2y^2 = 3 - 2z^2$

(f) Paraboloid : $y = \dfrac{x^2}{7} + z^2$

(g) Ellipsoid : $2x^2 + 3y^2 + 4z^2 = 5$

(h) Hyperbolic Paraboloid : $y = z^2 - x^2$

(i) Hyperboloid of Two Sheets : $y^2 - x^2 = z^2 + 4$

(j) Cylinder : $x^2 + y^2 = 1$

□

Quiz 7: Find parametrizations for the following geometric objects:

(a) the directed line segment from $(1,0,1)$ to $(-1,2,4)$.

(b) the circle with center $(-2,1)$ and radius 3, oriented counterclockwise.

(c) the portion of $y=x^2+1$ from $(0,1)$ to $(2,5)$.

(d) the curve resulting from intersecting $z=y^2$ and $x=e^y-\cos z$.

Solution.

(a) $\mathbf{m}=(-1,2,4)-(1,0,1)=\langle -2,2,3\rangle$. Then the line segment can be parametrized by

$$\ell(t)=\langle -2,2,3\rangle t+\langle 1,0,1\rangle=\langle 1-2t,2t,3t+1\rangle;\;\;0\le t\le 1$$

(b)

$$\mathbf{x}(t)=\langle 3\cos t-2,3\sin t+1\rangle;\;\;0\le t\le 2\pi$$

(c) Every point on the curve is of the form (x,y), but $y=x^2+1$, so every point is of the form $(x,y)=(x,x^2+1)$. We go from $x=0$ to $x=2$. Therefore, the curve can be parametrized by

$$\mathbf{r}(t)=\langle t,t^2+1\rangle;\;\;0\le t\le 2$$

(d) Every point on the curve is of the form (x,y,z). We know that $z=y^2$, so that we have $(x,y,z)=(x,y,y^2)$. Now $x=e^y-\cos x$ and $z=y^2$, so that we have $(x,y,z)=(x,y,y^2)=(e^y-\cos z,y,y^2)=(e^y-\cos(y^2),y,y^2)$. Therefore, we can parametrize the curve as

$$\mathbf{x}(t)=\langle e^t-\cos(t^2),t,t^2\rangle;\;\;t\in\mathbf{r}$$

□

Quiz 8: Find the length of the curve $\mathbf{x}(t)=\langle 2t,\frac{4}{3}t^{3/2},\frac{1}{2}t^2\rangle$, $0\le t\le 2$.

Solution.

$$\begin{aligned}
\mathbf{x}(t)&=\left\langle 2t,\frac{4}{3}t^{3/2},\frac{1}{2}t^2\right\rangle\\
\mathbf{x}'(t)&=\langle 2,2t^{1/2},t\rangle\\
\|\mathbf{x}'(t)\|&=\sqrt{2^2+(2t^{1/2})^2+t^2}=\sqrt{4+4t+t^2}=\sqrt{(t+2)^2}=t+2\\
L&=\int_a^b\|\mathbf{x}'(t)\|\,dt=\int_0^2(t+2)\,dt=\frac{t^2}{2}+2t\Big|_0^2=\left(\frac{4}{2}+2(2)\right)-0=6
\end{aligned}$$

□

Quiz 9: Show that the following limit does not exist by considering paths along the x-axis, y-axis, $y = x$, and the curve $x = y^2$. Would the curve $x = 1$ also work as one of the curve to show that the limit does not exist?

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$$

Solution.

$$\text{Along } x\text{-axis, } y = 0: \lim_{(x,0)\to(0,0)} \frac{x^4\cdot 0}{(x^2+0)^3} = \lim_{x\to 0} 0 = 0$$

$$\text{Along } y\text{-axis, } x = 0: \lim_{(0,y)\to(0,0)} \frac{0\cdot y^4}{(0+y^4)^3} = \lim_{y\to 0} 0 = 0$$

$$\text{Along } y = mx: \lim_{(x,x)\to(0,0)} \frac{x^4\cdot x^4}{(x^2+x^4)^3} = \lim_{x\to 0} \frac{x^8}{\left(x^2(1+x^2)\right)^3} = \lim_{x\to 0} \frac{x^8}{x^6(1+x^2)^3} = \lim_{x\to 0} \frac{x^2}{(1+x^2)^3} = 0$$

$$\text{Along } x = y^2: \lim_{(y^2,y)\to(0,0)} \frac{(y^2)^4y^4}{((y^2)^2+y^4)^3} = \lim_{y\to 0} \frac{y^8\cdot y^4}{(y^4+y^4)^3} = \lim_{y\to 0} \frac{y^{12}}{8y^{12}} = \lim_{y\to 0} \frac{1}{8} = \frac{1}{8}$$

Because the limit along the lines $y = mx$ and $x = y^2$ do not agree, the limit does not exist. Note that $x = 1$ would not be a possible curve because $(x, y) \to (0, 0)$, which is not possible if we fix $x = 1$! □

Quiz 10: Define $f(x, y) = \dfrac{ye^{xy}}{\ln x}$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution.

$$\frac{\partial f}{\partial x} = \frac{y^2e^{xy}\ln x - \dfrac{y}{x}\, e^{xy}}{(\ln x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{e^{xy} + xye^{xy}}{\ln x}$$

□

Quiz 11: Find the tangent plane to the surface $z = f(x, y)$ at $(x, y) = (1, -2)$, where $f(x, y) = x^2\cos(y + 2) + \dfrac{y}{x}$. Use this plane to approximate $f(1.1, -2.2)$.

Solution.

$$\begin{aligned} f(1,-2) &= -1 \\ \frac{\partial f}{\partial x} &= 2x\cos(y+2) - \frac{y}{x^2}\Big|_{(x,y)=(1,-2)} = 4 \\ \frac{\partial f}{\partial y} &= -x^2\sin(y+2) + \frac{1}{x}\Big|_{(x,y)=(1,-2)} = 1 \end{aligned}$$

Then the tangent plane is

$$\begin{aligned} z - z_0 &= f_x(1,-2)(x-1) + f_y(1,-2)(y-(-2)) \\ z - (-1) &= 4(x-1) + 1(y+2) \\ z + 1 &= 4(x-1) + 1(y+2) \\ z &= 4(x-1) + 1(y+2) - 1 \end{aligned}$$

Equivalently, the tangent plane is $z = 4x + y - 3$ or $4x + y - z = 3$. Then for points 'near' $(x, y, z) = (1, -2, -1)$, we know that $z \approx 4(x-1) + 1(y+2) - 1$. Then

$$\begin{aligned} f(1,-2) = z &\approx 4(x-1) + 1(y+2) - 1 \Big|_{x=1.1, y=-2.2} \\ &= 4(1.1-1) + 1(-2.2+2) - 1 \\ &= 4(0.1) - 0.2 - 1 \\ &= 0.4 - 0.2 - 1 \\ &= -0.8 \end{aligned}$$

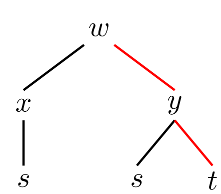
Therefore, $f(1.1, -2.2) \approx -0.8$. Note that $f(1.1, -2.2) = -0.814119$—meaning we have a 1.7% error! □

Quiz 12: Let $w(x, y) = 2^x \arctan y$, $x(s) = e^s$, and $y(s, t) = \tan(st)$. Use the Chain Rule to find $\dfrac{\partial w}{\partial t}$ in terms of x, y, s, t.

Solution.

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \\ &= 2^x \ln 2 \arctan y \cdot 0 + \frac{2^x}{1+y^2} \cdot s \sec^2(st) \\ &= \frac{s\, 2^x \sec^2(st)}{1+y^2} \end{aligned}$$

You may also use the 'chart' to help see what partials you will need (highlighted in red).



From this, we see that

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = \frac{2^x}{1+y^2} \cdot s \sec^2(st) = \frac{s\, 2^x \sec^2(st)}{1+y^2}$$

□

Quiz 13: Let $f(x,y) = \dfrac{x}{x+3y}$, and define $\mathbf{u} = \langle -3, 4 \rangle$.

(a) Find $D_{\mathbf{u}} f(-2,1)$.

(b) Find the direction of maximum increase for $f(x,y)$ at the point $(-2,1)$.

(c) Find the direction of maximum decrease for $f(x,y)$ at the point $(-2,1)$.

(d) Approximately what would be the change in the value for $f(x,y)$ if you traveled a 'distance' of 0.5 in the direction of $\mathbf{u}$?

Solution.

(a)

$$\nabla f(x,y) = \left\langle \frac{3y}{(x+3y)^2}, -\frac{3x}{(x+3y)^2} \right\rangle \Bigg|_{(x,y)=(-2,1)} = \langle 3, 6 \rangle$$
$$\|\mathbf{u}\| = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = 5$$
$$D_{\mathbf{u}} f(-2,1) = \langle 3, 6 \rangle \cdot \frac{\langle -3, 4 \rangle}{5} = \frac{1}{5} \cdot \big(3(-3) + 6(4)\big) = \frac{1}{5} \cdot 15 = 3$$

(b) The direction of maximum increase at $(-2,1)$ is the gradient at this point, i.e. $\langle 3, 6 \rangle$. Equivalently, you could use the direction $\langle 1, 2 \rangle$.

(c) The direction of maximum increase at $(-2,1)$ is the "opposite" direction from the gradient at this point, i.e. $\langle -3, -6 \rangle$. Equivalently, you could use the direction $\langle -1, -2 \rangle$.

(d) At $(-2,1)$, the rate of change in the direction of $\mathbf{u}$ is 3 because $D_{\mathbf{u}} f(-2,1) = 3$. If we travel a distance of $0.5 = \dfrac{1}{2}$, we should see a change of approximately $3 \cdot 1/2 = 3/2 = 1.50$, i.e. an increase of 1.50.

□

Quiz 14: Find and classify the extrema of $3x^2 + 2y^2 - 6x - 4y + 16$.

Solution. Let $f(x,y) = 3x^2 + 2y^2 - 6x - 4y + 16$. We have

$$\begin{aligned} f_x &= 6x - 6 = 6(x-1) \qquad & f_{xx} &= 6 \qquad & f_{xy} &= 0 \\ f_y &= 4y - 4 = 4(y-1) \qquad & f_{yy} &= 4 \qquad & f_{yx} &= 0 \end{aligned}$$

Setting $f_x = 0$ and $f_y = 0$, we find solution $(x,y) = (1,1)$. To classify the extrema, we use the Hessian

$$Hf(1,1) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$$

This gives sequence of principal minors $d_1 = 6 > 0$ and $d_2 = \begin{vmatrix} 6 & 0 \\ 0 & 4 \end{vmatrix} = 6(4) - 0(0) = 24 > 0$. Therefore, $(x,y) = (1,1)$ is a local minimum for $f(x,y)$. □

Quiz 15: Find and classify the critical points of $f(x, y, z) = x^2 - xy + z^2 - 2xz + 6z$.

Solution. We have

$$\begin{aligned} f_x &= 2x - y - 2z & f_{xy} &= -1 \\ f_y &= -x & f_{yx} &= -1 \\ f_z &= 2z - 2x & f_{xz} &= -2 \\ f_{xx} &= 2 & f_{zx} &= -2 \\ f_{yy} &= 0 & f_{yz} &= 0 \\ f_{zz} &= 2 & f_{zy} &= 0 \end{aligned}$$

We set $f_x = 0$, $f_y = 0$, and $f_z = 0$. From $f_y = 0$, we find that $x = 0$. Using this in $f_z = 0$, we find that $z = -3$. But then using both these in $f_x = 0$, we find that $y = 6$. Therefore, the only critical value is $(x, y, z) = (0, 6, -3)$. To classify this, we consider the Hessian,

$$Hf(x, y, z) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

The sequence of principal minors is then

$$\begin{aligned} d_1 &= 2 > 0 \\ d_2 &= \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = 2(0) - (-1)(-1) = 0 - 1 = -1 < 0 \\ d_3 &= \begin{vmatrix} 2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 2 \end{vmatrix} = (-1)(-1) \begin{vmatrix} -1 & -2 \\ 0 & 2 \end{vmatrix} + 0 - 0 = -2 - 0(-2) = -2 < 0 \end{aligned}$$

Therefore, $(0, 6, -3)$ is a saddle point for $f(x, y, z)$. □

Quiz 16: Find the maximum and minimum values of $f(x, y, z) = x + y - z$ if (x, y, z) must lie on the sphere $x^2 + y^2 + z^2 = 81$.

Solution. Letting $g(x, y, z) = x^2 + y^2 + z^2 - 81$. We have constraint $g(x, y, z) = 0$. Then $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, along with the constraint $x^2 + y^2 + z^2 = 81$ gives equations

$$\begin{aligned} 1 &= 2\lambda x \\ 1 &= 2\lambda y \\ -1 &= 2\lambda z \\ x^2 + y^2 + z^2 &= 81 \end{aligned}$$

Comparing the first two equations, we have $2\lambda x = 2\lambda y$ so that $x = y$. But observe $2\lambda z = -1 = -(1) = -(2\lambda x) = -2\lambda x$ so that $z = -x$. Then we have

$$81 = x^2 + y^2 + z^2 = x^2 + x^2 + (-x)^2 = 3x^2$$

From this we find that $x = \pm 3\sqrt{3}$. This gives extremum at $(3\sqrt{3}, 3\sqrt{3}, -3\sqrt{3})$ and $(-3\sqrt{3}, -3\sqrt{3}, 3\sqrt{3})$. Now

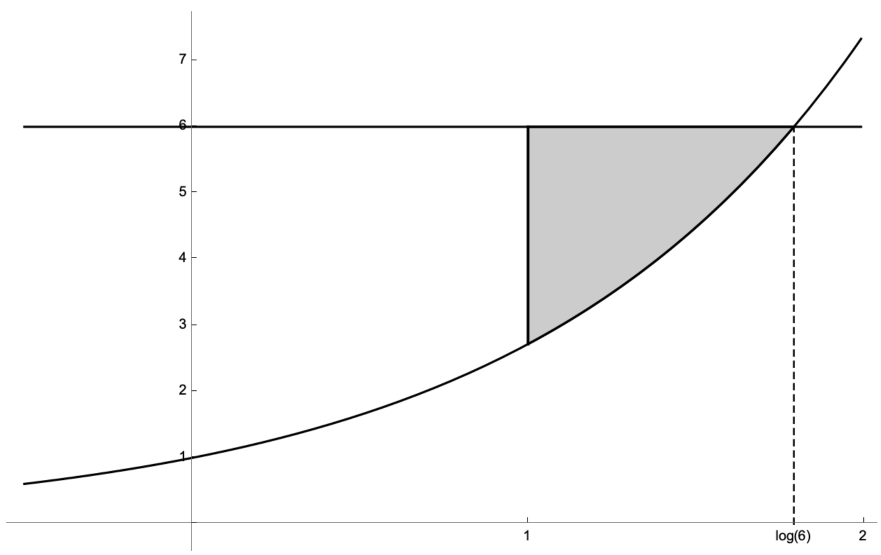
$$f(3\sqrt{3}, 3\sqrt{3}, -3\sqrt{3}) = 3\sqrt{3} + 3\sqrt{3} - (-3\sqrt{3}) = 9\sqrt{3}$$
$$f(-3\sqrt{3}, -3\sqrt{3}, 3\sqrt{3}) = -3\sqrt{3} - 3\sqrt{3} - 3\sqrt{3} = -9\sqrt{3}$$

Therefore, the maximum value of $f(x, y, z)$ on the sphere $x^2+y^2+z^2 = 81$ is $9\sqrt{3}$ and the minimum value is $-9\sqrt{3}$. □

Quiz 17: Sketch the region of integration for the following integral. In addition, evaluate the integral.

$$\int_1^{\ln 6} \int_{e^x}^{6} \frac{1}{y^2}\, dy\, dx$$

Solution.



$$\begin{aligned}
\int_1^{\ln 6} \int_{e^x}^{6} \frac{1}{y^2}\, dy\, dx &= \int_1^{\ln 6} -\frac{1}{y}\bigg|_{y=e^x}^{y=6} dx \\
&= \int_1^{\ln 6} \left(-\frac{1}{6} - \frac{-1}{e^x}\right) dx \\
&= \int_1^{\ln 6} \left(-\frac{1}{6} + e^{-x}\right) dx \\
&= -\frac{x}{6} - e^{-x}\bigg|_{x=1}^{x=\ln 6}
\end{aligned}$$

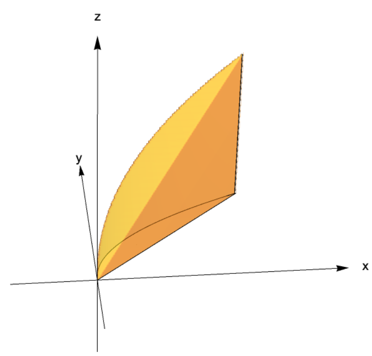
$$
\begin{aligned}
&= \left(-\frac{\ln 6}{6} - e^{-\ln 6}\right) - \left(-\frac{1}{6} + e^{-1}\right) \\
&= -\frac{\ln 6}{6} - \frac{1}{6} + \frac{1}{6} - \frac{1}{e} \\
&= \frac{1}{e} - \frac{\ln 6}{6}
\end{aligned}
$$

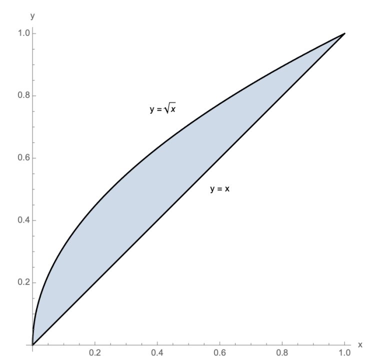
$\square$

Quiz 18: Let R be the region bounded by $x = y^2$, $y = z$, $x = y$, and $z = 0$. Evaluate the following integral:

$$\iiint_R (2x - y)\ dV$$

Solution.





$$
\begin{aligned}
\iiint_R (2x - y)\ dV &= \int_0^1 \int_{y^2}^{y} \int_0^y (2x - y)\ dz\ dx\ dy \\
&= \int_0^1 \int_{y^2}^{y} z(2x - y)\Big|_{z=0}^{z=y}\ dx\ dy \\
&= \int_0^1 \int_{y^2}^{y} \left[y(2x - y) - 0\right]\ dx\ dy \\
&= \int_0^1 \int_{y^2}^{y} (2xy - y^2)\ dx\ dy \\
&= \int_0^1 (x^2 y - xy^2)\Big|_{x=y^2}^{x=y}\ dy \\
&= \int_0^1 \left[(y^3 - y^3) - (y^5 - y^4)\right]\ dy \\
&= \int_0^1 (y^4 - y^5)\ dy
\end{aligned}
$$

$$= \left(\frac{y^5}{5} - \frac{y^6}{6}\right)\Bigg|_{y=0}^{y=1}$$
$$= \left(\frac{1}{5} - \frac{1}{6}\right) - 0$$
$$= \frac{6}{30} - \frac{5}{30}$$
$$= \frac{1}{30}$$

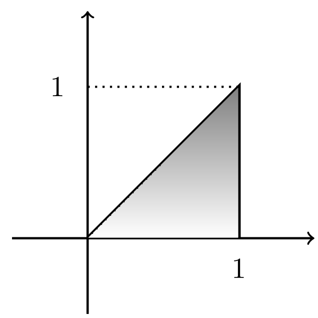
□

Quiz 19: Change the order of integration and evaluate the integral.

$$\int_0^1 \int_y^1 x^2 \sin xy \; dx \; dy$$

Solution.

$$\begin{aligned}\int_0^1 \int_y^1 x^2 \sin xy \; dx \; dy &= \int_0^1 \int_0^x x^2 \sin xy \; dy \; dx \\ &= \int_0^1 -x\cos xy\Big|_{y=0}^{y=x} \; dx \\ &= \int_0^1 \left[-x\cos x^2 - (-x\cos 0)\right] \; dx \\ &= \int_0^1 (x - x\cos x^2) \; dx \\ &= \frac{x^2}{2} - \frac{\sin x^2}{2}\Bigg|_{x=0}^{x=1} \\ &= \left(\frac{1-\sin 1}{2}\right) - (0-0) \\ &= \frac{1-\sin 1}{2}\end{aligned}$$



□

Quiz 20: Consider the following integral:

$$\int_0^2 \int_{x/2}^{x/2+1} x^5(2y-x)e^{(2y-x)^2} \; dy \; dx$$

Set-up (but do not evaluate) an integral in terms of u, v, where $u = x$ and $v = 2y - x$.

Solution. We have $x = u$ so that $v = 2y - x = 2y - u$. But then $y = \dfrac{u+v}{2}$. Now we need to find the Jacobian of the transformation:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix}1 & 0\\ 1/2 & 1/2\end{pmatrix}\right| = |1/2 - 0| = \frac{1}{2}$$

Now we need find the bounds for the new integral:

$$\begin{aligned} x = 0 &\iff u = 0 \\ x = 2 &\iff u = 2 \\ y = x/2 &\iff 2y = x \iff 2y - x = 0 \iff v = 0 \\ y = x/2 + 1 &\iff 2y = x + 2 \iff 2y - x = 2 \iff v = 2 \end{aligned}$$

Clearly, the region of integration is a rectangle with $0 \leq u \leq 2, 0 \leq v \leq 2$. Therefore,

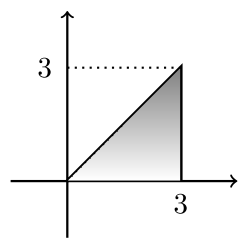
$$\int_0^2 \int_{x/2}^{x/2+1} x^5 (2y - x) e^{(2y-x)^2} \, dy \, dx = \int_0^2 \int_0^2 u^5 v e^{v^2} \cdot \frac{1}{2} \, du \, dv = \frac{1}{2} \int_0^2 \int_0^2 u^5 v e^{v^2} \, du \, dv = \frac{8}{3}(e^4 - 1)$$

□

Quiz 21: Evaluate the following:

$$\int_0^\pi \int_0^3 \int_0^x \frac{dy \, dx \, dz}{\sqrt{x^2 + y^2}}$$

Solution. We make a change to cylindrical coordinates. We have $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and Jacobian r. Drawing the projection of our region to the plane, we have



Given θ, r varies from 0 to the distance where r 'hits' the vertical portion of the triangle. This gives another, smaller triangle with hypotenuse r and sides $3, y$. Then we know that $\cos\theta = \frac{3}{r}$. Then $r = 3\sec\theta$. The smallest angle choice is 0 and the largest is $\pi/4$. Then we have

$$\begin{aligned} \int_0^\pi \int_0^3 \int_0^x \frac{dy \, dx \, dz}{\sqrt{x^2 + y^2}} &= \int_0^\pi \int_0^{\pi/4} \int_0^{3\sec\theta} \frac{1}{\sqrt{r^2}} \cdot r \, dr \, d\theta \, dz \\ &= \int_0^\pi \int_0^{\pi/4} \int_0^{3\sec\theta} dr \, d\theta \, dz \\ &= \int_0^\pi \int_0^{\pi/4} 3\sec\theta \, d\theta \, dz \\ &= \int_0^\pi 3\ln|\sec\theta + \tan\theta| \Big|_{\theta=0}^{\theta=\pi/4} dz \\ &= \int_0^\pi 3\ln|\sqrt{2} + 1| - 3\ln|1 + 0| \, dz \\ &= 3\ln(1 + \sqrt{2}) \int_0^\pi dz \\ &= 3\pi \ln(1 + \sqrt{2}) \end{aligned}$$

□

Quiz 22: Let R be the region bounded by the two sphere $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 5$. Evaluate the following

$$\iiint_R \frac{dV}{\sqrt{x^2+y^2+z^2}}$$

Solution. We use spherical coordinates: $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$, and Jacobian $\rho^2 \sin\phi$. Then

$$\begin{aligned}
\iiint_R \frac{dV}{\sqrt{x^2+y^2+z^2}} &= \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{5}} \frac{1}{\sqrt{\rho^2}} \cdot \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{5}} \rho \sin\phi \; d\rho \; d\phi \; d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{5}} \frac{\rho^2}{2} \sin\phi \Big|_{\rho=1}^{\rho=5} d\phi \; d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \frac{\sin\phi}{2}(5-1) \; d\phi \; d\theta \\
&= 2\int_0^{2\pi}\int_0^{\pi} \sin\phi \; d\phi \; d\theta \\
&= 2\int_0^{2\pi} -\cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta \\
&= 2\int_0^{2\pi} -\cos(\pi) - (-\cos 0) \; d\theta \\
&= 2\int_0^{2\pi} -(-1) + 1 \; d\theta \\
&= 2\int_0^{2\pi} 2 \; d\theta \\
&= 8\int_0^{2\pi} d\theta \\
&= 8\pi
\end{aligned}$$

□

Quiz 23: Find the center of mass of a lamina given by the region $\{(x, y) : 0 \le y \le \sqrt{x}, 0 \le x \le 9\}$ with density varying as xy. [You may use an integration calculator for the integrals.]

Solution.

$$\begin{aligned}
M &= \iint \rho(x,y) \; dA = \int_0^9 \int_0^{\sqrt{x}} (xy) \; dy \; dx = \frac{243}{2} \\
M_x &= \iint y\rho(x,y) \; dA = \int_0^9 \int_0^{\sqrt{x}} (xy^2) \; dy \; dx = \frac{1458}{7} \\
M_y &= \iint x\rho(x,y) \; dA = \int_0^9 \int_0^{\sqrt{x}} (x^2 y) \; dy \; dx = \frac{6561}{8}
\end{aligned}$$

$$\overline{x} = \frac{M_y}{M} = \frac{6561/8}{243/2} = \frac{27}{4}$$
$$\overline{y} = \frac{M_x}{M} = \frac{1458/7}{243/2} = \frac{12}{7}$$

$$(\overline{x}, \overline{y}) = (27/4, 12/7) \approx (6.75, 1.71)$$

□

Quiz 24: Let R be the region under the plane $z = 1 + x + y$ and above the region lying in the xy-plane bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$. Evaluate the following:

$$\iiint_R 3xy\ dV$$

Solution. Observe z varies from $z = 0$ up to the plane $z = 1 + x + y$. In the plane, if we 'slice' in x, y varies from $y = 0$ to $y = \sqrt{x}$. We can choose any x from $x = 0$ to $x = 1$. This gives the integral as

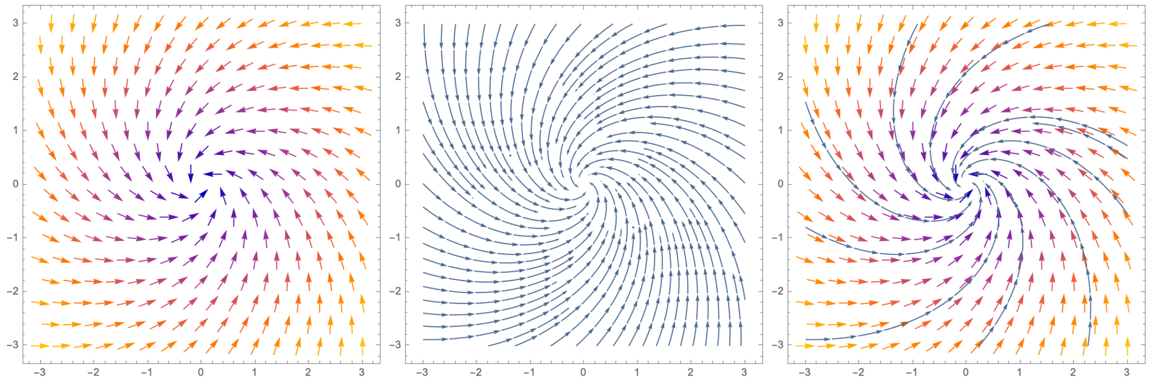
$$\begin{aligned}
\iiint_R 3xy\ dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 3xy\ dz\ dy\ dx \\
&= \int_0^1 \int_0^{\sqrt{x}} 3xyz \Big|_{z=0}^{z=1+x+y} dy\ dx \\
&= \int_0^1 \int_0^{\sqrt{x}} 3xy\big((1 + x + y) - 0\big)\ dy\ dx \\
&= \int_0^1 \int_0^{\sqrt{x}} (3xy + 3x^2y + 3xy^2)\ dy\ dx \\
&= \int_0^1 \left(\frac{3xy^2}{2} + \frac{3x^2y^2}{2} + \frac{3xy^3}{3} \right) \Big|_{y=0}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left(\frac{3x^2}{2} + \frac{3x^3}{2} + \frac{3x^{5/2}}{3} \right) - 0\ dx \\
&= \frac{1}{6} \int_0^1 (9x^2 + 9x^3 + 6x^{5/2})\ dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} \left(3x^3 + \frac{9x^4}{4} + \frac{12x^{7/2}}{7} \right) \Big|_{x=0}^{x=1} \\
&= \frac{1}{6} \left[\left(3 + \frac{9}{4} + \frac{12}{7} \right) - 0 \right] \\
&= \frac{1}{6} \left(\frac{84 + 63 + 48}{28} \right) \\
&= \frac{1}{6} \cdot \frac{195}{28} \\
&= \frac{65}{56}
\end{aligned}$$

□

Quiz 25: Sketch the vector field $\mathbf{F}(x,y) = -(x+y)\,\mathbf{i} + (x-y)\,\mathbf{j}$. On your vector plot, sketch a few streamlines.

Solution.



$\square$

Quiz 26: Find the divergence and curl of the vector field $\mathbf{F}(x,y) = \langle x^2 y, x\cos y\rangle$.

Solution.

$$\text{div }\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(x\cos y) = 2xy - x\sin y$$

$$\begin{aligned}
\text{curl }\mathbf{F} &= \nabla\times\mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & x\cos y & 0 \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x\cos y)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2y)\right) + \mathbf{k}\left(\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(x^2y)\right) \\
&= 0\,\mathbf{i} - 0\,\mathbf{j} + (\cos y - x^2)\,\mathbf{k} \\
&= \langle 0, 0, \cos y - x^2\rangle
\end{aligned}$$

$\square$

Quiz 27: Let C be the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + (2-t)\,\mathbf{j}$ for $0 \le t \le 2$. Compute the following

$$\int_C 3(x-y)\,ds$$

Solution.

$$\begin{aligned}
\mathbf{r}(t) &= \langle t, 2-t\rangle \\
\mathbf{r}'(t) &= \langle 1, -1\rangle \\
|\mathbf{r}'(t)| &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
x(t) &= t \\
y(t) &= 2 - t
\end{aligned}$$

Then we have

$$\begin{aligned}\int_C 3(x-y)\,ds &= \int_0^2 3\big(t-(2-t)\big)\cdot\sqrt{2}\,dt \\ &= 3\sqrt{2}\int_0^2 (2t-2)\,dt \\ &= 3\sqrt{2}\cdot\big(t^2-2t\big)\Big|_0^2 \\ &= 3\sqrt{2}\cdot((4-4)-0) \\ &= 0\end{aligned}$$

□

Quiz 28: Let C be the curve given by $y^2 = x^3$ from $(1,-1)$ to $(1,1)$. Evaluate the following

$$\int_C x^2y\,dx - xy\,dy$$

Solution. We can parametrize this curve by $\mathbf{r}(t) = \langle t^2, t^3\rangle$, $-1 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 2t, 3t^2\rangle$. Therefore,

$$\begin{aligned}\int_C x^2y\,dx - xy\,dy &= \int_{-1}^1 \big((t^2)^2t^3\big)\cdot 2t\,dt - \big(t^2\cdot t^3\big)\cdot 3t^2\,dt \\ &= \int_{-1}^1 2t^8 - 3t^7\,dt \\ &= \left(\frac{2t^9}{9} - \frac{3t^8}{8}\right)\Big|_{-1}^1 \\ &= \left(\frac{2}{9}-\frac{3}{8}\right) - \left(-\frac{2}{9}-\frac{3}{8}\right) \\ &= \frac{2}{9}-\frac{3}{8}+\frac{2}{9}+\frac{3}{8} \\ &= \frac{4}{9}\end{aligned}$$

□

Quiz 29: Let C be the curve given by $\mathbf{r}(t) = \dfrac{t^3e^{t(3-t)}}{3}\,\mathbf{i} + \dfrac{10\sin(\pi t/6)\cos(2\pi t)}{1+t^2}\,\mathbf{j}$, $0 \le t \le 3$. Evaluate the following integral

$$\int_C (2xy - y)\,dx + (x^2 - x + 1)\,dy$$

Solution. Observe that

$$\begin{aligned}\frac{\partial N}{\partial x} &\overset{?}{=} \frac{\partial M}{\partial y} \\ 2x-1 &= 2x-1\end{aligned}$$

Therefore, the vector field $\mathbf{F}(x, y) = \langle 2xy - y, x^2 - x + 1 \rangle$ is conservative.

$$\int (x^2 - x + 1)\ dy = x^2y - xy + y + g(x)$$
$$\frac{\partial}{\partial x}(x^2y - xy + y + g(x)) = 2xy - y + g'(x)$$

$$\begin{aligned} 2xy - y + g'(x) &= 2xy - y \\ g'(x) &= 0 \\ \int g'(x)\ dx &= \int 0\ dx \\ g(x) &= C \end{aligned}$$

Therefore, $f(x, y) = x^2y - xy + y + C$ is a function such that $\nabla f(x, y) = \mathbf{F}$. Now $\mathbf{r}(3) = \langle 9, 1\rangle$ and $\mathbf{r}(0) = \langle 0, 0\rangle$. Then

$$\int_C (2xy - y)\ dx + (x^2 - x + 1)\ dy = f(\mathbf{r}(3)) - f(\mathbf{r}(0)) = f(9, 1) - f(0, 0) = 73 - 0 = 73$$

□

Quiz 30: Use Green's Theorem to evaluate the line integral

$$\oint_C x^2y^2\ dx + x^3y\ dy$$

where C is the triangle with vertices $(0, 0), (1, 0), (1, 3)$, oriented counterclockwise.

Solution. Using Green's Theorem,

$$\begin{aligned} \oint_C x^2y^2\ dx + x^3y\ dy &= \iint_R \frac{\partial}{\partial x}(x^3y) - \frac{\partial}{\partial y}(x^2y^2)\ dA \\ &= \iint_R (3x^2y - 2x^2y)\ dA \\ &= \int_0^1 \int_0^{3x} x^2y\ dy\ dx \end{aligned}$$

$$\begin{aligned} &= \int_0^1 \frac{x^2y^2}{2}\bigg|_{y=0}^{y=3x} \\ &= \frac{9}{2}\int_0^1 x^4\ dx \\ &= \frac{9}{2} \cdot \frac{x^5}{5}\bigg|_0^1 \\ &= \frac{9}{2} \cdot \frac{1}{5} \\ &= \frac{9}{10} \end{aligned}$$

□

Quiz 31: Let $\mathbf{F}(x,y) = e^x \sin y\,\mathbf{i} + (e^x \cos y + 2y)\,\mathbf{j}$, and C be the line segment from $(1,0)$ to $(0,\pi/2)$. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Solution. Observe that

$$\frac{\partial N}{\partial x} \stackrel{?}{=} \frac{\partial M}{\partial y}$$
$$e^x \cos y = e^x \cos y$$

Therefore, $\mathbf{F}$ is a conservative vector field. Now

$$\int (e^x \cos y + 2y)\ dy = e^x \sin y + y^2 + g(x)$$

$$\frac{\partial}{\partial x}(e^x \sin y + y^2 + g(x)) = e^x \sin y + g'(x)$$

$$\begin{aligned} e^x \sin y + g'(x) &= e^x \sin y \\ g'(x) &= 0 \\ \int g'(x)\ dx &= \int 0\ dx \\ g(x) &= C \end{aligned}$$

Therefore, $f(x,y) = e^x \sin y + y^2 + C$ is a function such that $\nabla f(x,y) = \mathbf{F}$. Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f(x,y) \cdot d\mathbf{r} \\ &= f(0,\pi/2) - f(1,0) \\ &= \left(1 + \frac{\pi^2}{4} + C\right) - C \\ &= 1 + \frac{\pi^2}{4} \end{aligned}$$

□

Quiz 32: Parametrize the part of the cylinder $x^2 + z^2 = 4$ between $y = -1$ and $y = 3$, and find $\mathbf{N}$ for this surface.

Solution.

$$\begin{aligned}
\mathbf{X}(s,t) &= \langle 2\cos t, s, 2\sin t\rangle; \quad -1 \le s \le 3, 0 \le t \le 2\pi \\
\mathbf{T}_s(s,t) &= \langle 0,1,0\rangle \\
\mathbf{T}_t(s,t) &= \langle -2\sin t, 0, 2\cos t\rangle \\
\mathbf{N}(s,t) &= \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -2\sin t & 0 & 2\cos t \end{vmatrix} \\
&= \langle 2\cos t, 0, 2\sin t\rangle
\end{aligned}$$

□

Quiz 33: Let S be the surface with bottom $z = 0$, top $z = 4$, and sides $x^2 + y^2 = 9$, oriented outward normals. Evaluate

$$\iint_S z\, dS$$

Solution. We parametrize the top, bottom, and sides of the cylinder.

$$\begin{aligned}
X_{\text{top}}(s,t) &= \langle s\cos t, s\sin t, 4\rangle; \quad 0 \le s \le 3, 0 \le t \le 2\pi \\
X_{\text{bottom}}(s,t) &= \langle s\cos t, s\sin t, 0\rangle; \quad 0 \le s \le 3, 0 \le t \le 2\pi \\
X_{\text{sides}}(s,t) &= \langle 3\cos t, 3\sin t, s\rangle; \quad 0 \le s \le 4, 0 \le t \le 2\pi \\
\mathbf{N}_{\text{top}}(s,t) &= \langle 0,0,s\rangle \\
\mathbf{N}_{\text{bottom}}(s,t) &= \langle 0,0,-s\rangle \\
\mathbf{N}_{\text{side}}(s,t) &= \langle 3\cos t, 3\sin t\rangle \\
\|\mathbf{N}_{\text{top}}(s,t)\| &= s \\
\|\mathbf{N}_{\text{bottom}}(s,t)\| &= s \\
\|\mathbf{N}_{\text{side}}(s,t)\| &= 3
\end{aligned}$$

Then we have

$$\begin{aligned}
\iint_S z\, dS &= \iint_{\text{top}} z\, dS + \iint_{\text{bottom}} z\, dS + \iint_{\text{side}} z\, dS \\
&= \int_0^{2\pi}\int_0^3 4s\, ds\, dt + \int_0^{2\pi}\int_0^3 0\, ds\, dt + \int_0^{2\pi}\int_0^4 3s\, ds\, st \\
&= 36\pi + 0 + 48\pi \\
&= 84\pi
\end{aligned}$$

□

Quiz 34: Let $\mathbf{F}(x, y, z) = \langle 2x, 2y, z^2\rangle$, and define S to be the portion of the cone $x^2 + y^2 = z^2$ between the planes $z = -2$ and $z = 1$, oriented outwards. Find the value of the following:

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

Solution. We can parametrize the surface by $\mathbf{X}(s, t) = \langle s\cos t, s\sin t, s\rangle$, where $-2 \le s \le 1$, $0 \le t \le 2\pi$. Then we have

$$\begin{aligned}
\mathbf{T}_s &= \langle \cos t, \sin t, 1\rangle \\
\mathbf{T}_t &= \langle -s\sin t, s\cos t, 0\rangle \\
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 1 \\ -s\sin t & s\cos t & 0 \end{vmatrix} \\
&= \langle -s\cos t, -s\sin t, s\rangle
\end{aligned}$$

Note that this $\mathbf{N}$ is not the desired one. We want an upward normal but this points downward as it points upward when $z = s > 0$ and downward when $z = s < 0$. Therefore, we use $-\mathbf{N} = \langle s\cos t, s\sin t, -s\rangle$. Then

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_{-2}^{1} \langle 2s\cos t, 2s\sin t, s^2\rangle \cdot \langle s\cos t, s\sin t, -s\rangle \, ds \, dt \\
&= \int_0^{2\pi} \int_{-2}^{1} (2s^2 \sin^2 t + 2s^2 \cos^2 t - s^3) \, ds \, dt \\
&= \int_0^{2\pi} \int_{-2}^{1} (2s^2 - s^3) \, ds \, dt \\
&= \left(\int_0^{2\pi} dt \right) \left(\int_{-2}^{1} (2s^2 - s^3) \, ds \right) \\
&= 2\pi \cdot \left(\frac{2s^3}{3} - \frac{s^4}{4} \right) \Bigg|_{-2}^{1} \\
&= 2\pi \left[\left(\frac{2}{3} - \frac{1}{4} \right) - \left(-\frac{16}{3} - 4 \right) \right] \\
&= 2\pi \cdot \frac{8 - 3 + 64 + 48}{12} \\
&= 2\pi \cdot \frac{117}{12} \\
&= \frac{39\pi}{2}
\end{aligned}$$

□

Quiz 35: Let S be the surface given by the four sides and the bottom of the cube with vertices $(\pm 1, \pm 1, \pm 1)$. Orient S with outward-pointing normals. Let $\mathbf{F}(x, y, z) = x^2yz^3\,\mathbf{i} + x^2y\,\mathbf{j} + xe^x \sin yz\,\mathbf{k}$. Compute

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

Solution. Stoke's Theorem implies

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}'$$

where $\tilde{S}$ is the top face ($z = 1$) of the cube, oriented with downward normal $-\mathbf{k}$.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \langle xze^x \cos yz, 3x^2yz - (1 + x)e^x \sin yz, 2xy - x^2z^3 \rangle \\
\tilde{\mathbf{S}}(s, t) &= (s, t, 1); \;\; 0 \leq s, t, \leq 1 \\
\mathbf{N}(s, t) &= -\mathbf{k} \\
\iint_{\tilde{S}} \nabla \times \mathbf{F} \cdot d\tilde{\mathbf{S}} &= \int_{-1}^{1} \int_{-1}^{1} \langle se^s \cos t, 3s^2t - (1 + s)e^s \sin t, 2st - s^2 \rangle \cdot \langle 0, 0, -1 \rangle \; ds \; dt \\
&= -\int_{-1}^{1} \int_{-1}^{1} (2st - s^2) \; ds \; dt \\
&= -\int_{-1}^{1} \left(s^2t - \frac{s^3}{3} \right) \Bigg|_{s=-1}^{s=1} dt \\
&= -\int_{-1}^{1} \left[\left(t - \frac{1}{3} \right) - \left(t - \frac{-1}{3} \right) \right] dt \\
&= -\int_{-1}^{1} \frac{-2}{3} \; dt \\
&= \frac{2}{3} \int_{-1}^{1} dt \\
&= \frac{2}{3} \cdot 2 \\
&= \frac{4}{3}
\end{aligned}$$

□