MAT 397: Exam 1
Spring – 2020
02/12/2020
80 Minutes

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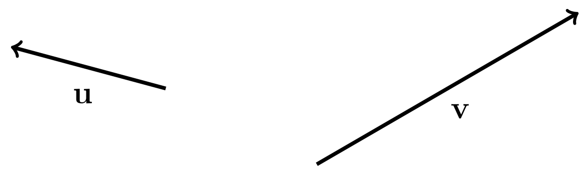
Write your name on the appropriate line on the exam cover sheet. This exam contains 10 pages (including this cover page) and 9 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

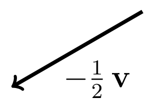
Question	Points	Score
1	10	
2	15	
3	17	
4	15	
5	12	
6	15	
7	16	
8	10	
9	10	
Total:	120	

1. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors shown below.

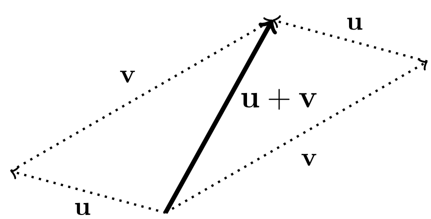


Sketch the following vectors.

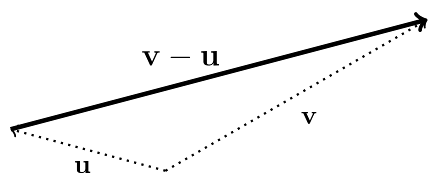
(a) (2 points) $-\frac{1}{2}\,\mathbf{v}$



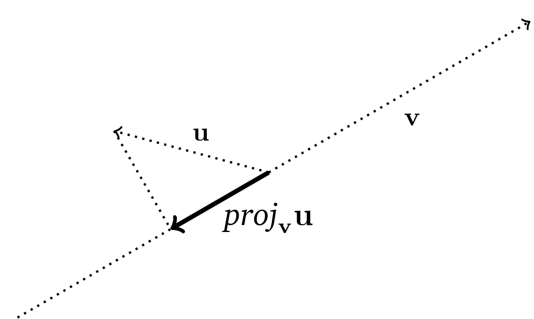
(b) (2 points) $\mathbf{u}+\mathbf{v}$



(c) (3 points) $\mathbf{v}-\mathbf{u}$



(d) (3 points) $\text{proj}_{\mathbf{v}}\mathbf{u}$



2. Let $\mathbf{u} = \langle 2, -2, 1\rangle$ and $\mathbf{v} = \langle 1, 1, 3\rangle$.

(a) (3 points) Compute $2\mathbf{v} - \mathbf{u}$.

$$2\mathbf{v} - \mathbf{u} = \langle 2, 2, 6\rangle - \langle 2, -2, 1\rangle = \langle 0, 4, 5\rangle$$

(b) (3 points) Are $\mathbf{u}$ and $\mathbf{v}$ parallel? Explain.

No. If $\mathbf{u}$ is parallel to $\mathbf{v}$, then there is a c so that $\mathbf{u} = c\mathbf{v}$. But then $\langle 2, -2, 1\rangle = \langle c, c, 3c\rangle$. Equating the first components, we find that $c = 2$. But $\langle 2, -2, 1\rangle \neq \langle 2, 2, 6\rangle$. Therefore, $\mathbf{u}$ is not parallel to $\mathbf{v}$.

(c) (5 points) Find a vector perpendicular to *both* $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 1 & 1 & 3 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} \\
&= \mathbf{i}\big(-2(3) - 1(1)\big) - \mathbf{j}\big(2(3) - 1(1)\big) + \mathbf{k}\big(2(1) - (-2)(1)\big) \\
&= \mathbf{i}(-6 - 1) - \mathbf{j}(6 - 1) + \mathbf{k}(2 + 2) \\
&= -7\,\mathbf{i} - 5\,\mathbf{j} + 4\,\mathbf{k} \\
&= \langle -7, -5, 4\rangle
\end{aligned}$$

(d) (4 points) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}
A &= \|\mathbf{u} \times \mathbf{v}\| \\
&= \|\langle -7, -5, 4\rangle\| \\
&= \sqrt{(-7)^2 + (-5)^2 + 4^2} \\
&= \sqrt{49 + 25 + 4} \\
&= \sqrt{90} \\
&= \sqrt{9 \cdot 10} \\
&= 3\sqrt{10}
\end{aligned}$$

3. Let $\mathbf{u} = \langle 2, -2, 1 \rangle$ and $\mathbf{v} = \langle 1, 1, 3 \rangle$.

(a) (4 points) Find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

$$\|\mathbf{u}\| = \|\langle 2, -2, 1 \rangle\| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\|\mathbf{v}\| = \|\langle 1, 1, 3 \rangle\| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{1 + 1 + 9} = \sqrt{11}$$

(b) (3 points) Find a unit vector which points in the same direction as $\mathbf{u}$.

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 2, -2, 1 \rangle}{3} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

(c) (3 points) Compute $\mathbf{u} \cdot \mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} = \langle 2, -2, 1 \rangle \cdot \langle 1, 1, 3 \rangle = 2(1) + (-2)(1) + 1(3) = 2 - 2 + 3 = 3$$

(d) (2 points) Are $\mathbf{u}$ and $\mathbf{v}$ perpendicular? Explain.

Because $\mathbf{u} \cdot \mathbf{v} \neq 0$, $\mathbf{u}$ and $\mathbf{v}$ are not perpendicular.

(e) (5 points) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta \\
3 &= 3 \cdot \sqrt{11} \cos\theta \\
\cos\theta &= \frac{1}{\sqrt{11}} \\
\theta &= \cos^{-1}\left(\frac{1}{\sqrt{11}}\right) \approx 72.4516^\circ
\end{aligned}$$

4. Complete the following parts:

   (a) (5 points) Find the line which passes through the point $(1, 3, -1)$ in the direction of $-\mathbf{i} + 2\mathbf{k}$.

$$\ell(t) = \langle -1, 0, 2 \rangle t + (1, 3, -1) = (-t, 0, 2t) + (1, 3, -1) = (1 - t, 3, 2t - 1)$$

   (b) (5 points) Find the equation of the plane with normal vector $\langle 3, -1, 5 \rangle$ and which contains the point $(-2, 9, 3)$.

$$\begin{aligned} \langle 3, -1, 5 \rangle \cdot \langle x - (-2), y - 9, z - 3 \rangle &= 0 \\ 3(x + 2) - (y - 9) + 5(z - 3) &= 0 \\ 3x + 6 - y + 9 + 5z - 15 &= 0 \\ 3x - y + 5z &= 0 \end{aligned}$$

   (c) (5 points) Find the line perpendicular to the plane $3x - 2y + z = 6$ and which passes through the point where the plane intersects the x-axis.

   The x-axis is where $y = z = 0$. But then using the equation of the plane, we find $3x = 6$ so that $x = 2$. Therefore, the plane intersects the x-axis at $(2, 0, 0)$. The direction vector for the line must be parallel to the normal vector for the plane, i.e. parallel to $\langle 3, -2, 1 \rangle$. Therefore, the line is

$$\ell(t) = \langle 3, -2, 1 \rangle t + (2, 0, 0) = (3t, -2t, t) + (2, 0, 0) = (3t + 2, -2t, t)$$

5. Consider the lines $r_1(t) = (1-t, t, t+1)$ and $r_2(t) = (-1-2t, 2t, 2t+5)$.

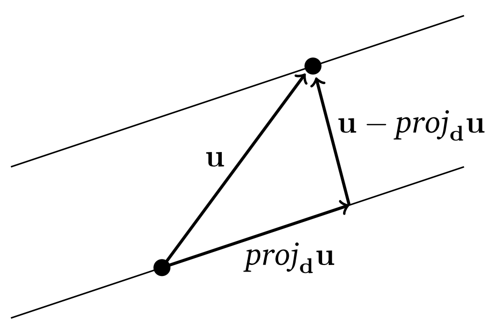
(a) (6 points) Show that $r_1(t)$ and $r_2(t)$ are parallel.

The direction vectors for the lines are $\langle -1,1,1\rangle$ and $\langle -2,2,2\rangle$, respectively. Because $\langle -2,2,2\rangle = 2\langle -1,1,1\rangle$, the vectors are parallel. To be sure that the lines are distinct, we need to show that there is a point on one line that is not on the other. Observe $r_2(0) = (-1,0,5)$. Then for some t, we must have $(1,0,3) = (1-t, t, t+1)$. Equating the second components, we find that $t = 0$. But $r_1(0) = (1,0,1) \neq (-1,0,5)$. Therefore, the lines are parallel.

(b) (6 points) Find the distance between $r_1(t)$ and $r_2(t)$.

Because the lines are parallel, the distances from any point on $r_1(t)$ to the line $r_2(t)$ are the same. We use the point $r_1(0) = (1,0,1)$. One point on $r_2(t)$ is $r_2(0) = (-1,0,5)$. Then let $\mathbf{u} = \langle -1,0,5\rangle - \langle 1,0,1\rangle = \langle -2,0,4\rangle$. We project $\mathbf{u}$ onto the direction vector for the lines, i.e. $\mathbf{d} = \langle -1,1,1\rangle$.

$$\begin{aligned} \textit{proj}_{\mathbf{d}}\mathbf{u} &= \frac{\mathbf{d}\cdot\mathbf{u}}{\mathbf{d}\cdot\mathbf{d}}\,\mathbf{d} \\ &= \frac{\langle -1,1,1\rangle\cdot\langle -2,0,4\rangle}{\langle -1,1,1\rangle\cdot\langle -1,1,1\rangle}\,\langle -1,1,1\rangle \\ &= \frac{2+0+4}{1+1+1}\,\langle -1,1,1\rangle \\ &= 2\langle -1,1,1\rangle \\ &= \langle -2,2,2\rangle \end{aligned}$$



Then the distance between the lines is the length of the vector $\mathbf{u} - \textit{proj}_{\mathbf{d}}\mathbf{u}$.

$$\begin{aligned} \|\mathbf{u} - \textit{proj}_{\mathbf{d}}\mathbf{u}\| &= \|\langle -2,0,4\rangle - \langle -2,2,2\rangle\| \\ &= \|\langle 0,-2,2\rangle\| \\ &= \sqrt{0^2 + (-2)^2 + 2^2} \\ &= \sqrt{0+4+4} \\ &= \sqrt{8} \\ &= \sqrt{4\cdot 2} \\ &= 2\sqrt{2} \end{aligned}$$

6. Suppose a subatomic particle has velocity vector $\mathbf{v}(t) = \left\langle \frac{1}{t+1},\ 2t,\ e^{-t} \right\rangle$.

(a) (4 points) Find $\mathbf{a}(t)$.

$$\mathbf{a}(t) = \frac{d}{dt}\,\mathbf{v}(t) = \left\langle \frac{-1}{(t+1)^2},\ 2, -e^{-t} \right\rangle$$

(b) (6 points) If $\mathbf{r}(0) = \langle 0, 1, 2 \rangle$, find $\mathbf{r}(t)$.

$$\mathbf{r}(t) = \int \left\langle \frac{1}{t+1},\ 2t,\ e^{-t} \right\rangle\, dt = \left\langle\, \ln|t+1|, t^2, -e^{-t} \,\right\rangle + \langle C_1, C_2, C_3 \rangle$$

$$\mathbf{r}(0) = \langle 0, 1, 2 \rangle = \langle 0, 0, -1 \rangle + \langle C_1, C_2, C_3 \rangle \Longrightarrow \langle C_1, C_2, C_3 \rangle = \langle 0, 1, 3 \rangle$$

$$\mathbf{r}(t) = \left\langle\, \ln|t+1|, t^2 + 1, 3 - e^{-t} \,\right\rangle$$

(c) (5 points) Set up completely, *but do not evaluate*, an integral which computes how far the particle travels between $t = 0$ and $t = 1$.

$$L = \int_0^1 \|\mathbf{r}'(t)\|\, dt$$

$$L = \int_0^1 \|\mathbf{v}(t)\|\, dt$$

$$L = \int_0^1 \left\| \left\langle \frac{1}{t+1},\ 2t,\ e^{-t} \right\rangle \right\|\, dt$$

$$L = \int_0^1 \sqrt{\frac{1}{(t+1)^2} + 4t^2 + e^{-2t}}\ \, dt$$

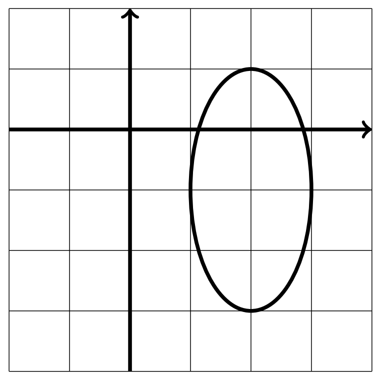
7. Find parametrizations for the following curves (specify the domain if necessary):

(a) (4 points) The line segment pointing from $(1, -1, 2)$ to $(3, 2, 1)$.

$$\begin{aligned}\mathbf{r}(t) &= (1-t)(1,-1,2) + t(3,2,1)\\ &= (1-t, t-1, 2-2t) + (3t, 2t, t)\\ &= (2t+1, 3t-1, 2-t)\end{aligned}$$

where $0 \leq t \leq 1$.

(b) (4 points) The curve shown below, oriented counterclockwise.



$$\mathbf{x}(t) = (\cos t + 2, 2\sin t - 1)$$

(c) (4 points) The curve given by $x = \cos y + e^{\sin y}$.

$$\mathbf{r}(t) = (\cos t + e^{\sin t}, t)$$

(d) (4 points) The curve formed by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x - 2y + z = 1$.

Suppose (x_0, y_0, z_0) *is in the intersection. The cylinder is parametrized by* $(2\cos t, 2\sin t, z)$*, so that* $(x_0, y_0, z_0) = (2\cos t, 2\sin t, z)$*. But the point is also on the plane, where* $x - 2y + z = 1$*, i.e.* $z = 1 + 2y - x$*. Then* $z_0 = 1 + 2y_0 - x_0$*. But* $x_0 = 2\cos t$ *and* $y_0 = 2\sin t$*. Then* $(x_0, y_0, z_0) = (2\cos t, 2\sin t, 1 + 4\sin t - 2\cos t)$*. Therefore,*

$$\mathbf{x}(t) = (2\cos t, 2\sin t, 1 + 4\sin t - 2\cos t)$$

8. (10 points) Identify the following curves/surfaces in $\mathbb{R}^3$:

(a) ______*Sphere*______ : $x^2 + y^2 + z^2 = 6$

(b) ______*Hyperboloid of Two Sheets*______ : $x^2 - 3 = 2y^2 + \frac{1}{3}z^2$

(c) ______*Cone*______ : $y^2 = 2x^2 + 3z^2$

(d) ______*Plane*______ : $y = x + z$

(e) ______*Hyperboloid of One Sheet*______ : $y^2 - 4 = z^2 - 9x^2$

(f) ______*Parabolic Cylinder*______ : $y = x^2 - 4$

(g) ______*Ellipsoid*______ : $2x^2 + 3y^2 + 4z^2 = 5$

(h) ______*Paraboloid*______ : $z = \frac{1}{2}x^2 + \frac{1}{3}y^2$

(i) ______*Hyperbolic Paraboloid*______ : $y + 2z^2 - x^2 = 0$

(j) ______*Line*______ : $(2t + 1, 1 - t, 4t)$

9. (10 points) Use the method of level curves, with slices/cross-sections in at least two directions, to sketch the surface $4x^2 + z^2 - y^2 = 1$.