MAT 397: Exam 2
Spring – 2020
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180 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 14 pages (including this cover page) and 11 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	15	
2	16	
3	15	
4	15	
5	15	
6	16	
7	16	
8	16	
9	16	
10	15	
11	15	
Total:	170	

1. Determine the following limits. If the limit exists, compute its value. If the limit does not exist, prove it.

(a) (3 points) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x-2y}{2x^2+3y}$

$$\lim_{(x,y)\to(1,2)} \frac{x-2y}{2x^2+3y} = \frac{1-2(2)}{2+3(2)} = \frac{1-4}{2+6} = -\frac{3}{8}$$

(b) (4 points) $\displaystyle\lim_{(x,y,z)\to(2,-1,2)} \frac{x-z}{x^2+xy-xz-yz}$

$$\begin{aligned}\lim_{(x,y,z)\to(2,-1,2)} \frac{x-z}{x^2+xy-xz-yz} &= \lim_{(x,y,z)\to(2,-1,2)} \frac{x-z}{x(x+y)-z(x+y)} \\ &= \lim_{(x,y,z)\to(2,-1,2)} \frac{x-z}{(x-z)(x+y)} \\ &= \lim_{(x,y,z)\to(2,-1,2)} \frac{1}{x+y} \\ &= \frac{1}{2-1} \\ &= 1\end{aligned}$$

(c) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+2y^2}$

Along $y=0$*:* $\displaystyle\lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$

Along $x=0$*:* $\displaystyle\lim_{(0,y)\to(0,0)} \frac{0}{2y^2} = 0$

Along $y=x$*:* $\displaystyle\lim_{(x,x)\to(0,0)} \frac{3x^2}{x^2+2x^2} = \lim_{x\to 0} \frac{3x^2}{3x^2} = 1$

Therefore, the limit does not exist.

(d) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3 \sin x}{x^4+y^2}$

We know that $|\sin x| \leq 1$ *for all* x*. We also know that* $y^2 \leq x^4+y^2$*, so that* $\dfrac{1}{y^2} \geq \dfrac{1}{x^4+y^2}$*. But then we have*

$$\left|\frac{y^3\sin x}{x^4+y^2}\right| = \left|y^3 \cdot \sin x \cdot \frac{1}{x^4+y^2}\right| \leq \left|y^3 \cdot 1 \cdot \frac{1}{y^2}\right| = |y|$$

But $\displaystyle\lim_{(x,y)\to(0,0)} |y| = 0$*. Therefore by the Squeeze Theorem,*

$$\lim_{(x,y)\to(\frac{\pi}{4},0)} \frac{y^3\sin x}{x^4+y^2} = 0$$

2. Let $f(x,y,z) := \dfrac{xe^{xy} + y^x - z^2}{1-z}$. Find the following:

(a) (4 points) $f_x = \dfrac{e^{xy} + xye^{xy} + y^x \ln y}{1-z} = \dfrac{e^{xy}(1+xy) + y^x \ln y}{1-z}$

(b) (4 points) $f_y = \dfrac{x^2e^{xy} + xy^{x-1}}{1-z}$

(c) (4 points) $f_z = \dfrac{(1-z)(-2z) - (-1)(xe^{xy} + y^x - z^2)}{(1-z)^2} = \dfrac{-2z + 2z^2 + xe^{xy} + y^x - z^2}{(1-z)^2} = \dfrac{xe^{xy} + y^x + z^2 - 2z}{(1-z)^2}$

(d) (4 points) $f_{yz} = \dfrac{x^2e^{xy} + xy^{x-1}}{(1-z)^2}$

3. (15 points) Define the following functions:

$$f(x,y) = \frac{x^2}{y}$$

$$x(m) = \tan(\ln m) + 3 \qquad m(s,t) = 3 - t\sqrt{s}$$

$$y(n) = n^2 - \arctan n - 1 \qquad n(s,t) = 2 - te^{\sin(t-2s)}$$

Compute $\frac{\partial f}{\partial t}$ when $(s,t) = (1,2)$.

Using the Chain Rule, we have

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\,\frac{dx}{dm}\,\frac{\partial m}{\partial t} + \frac{\partial f}{\partial y}\,\frac{dy}{dn}\,\frac{\partial n}{\partial t}$$

If $s = 1$ *and* $t = 2$*, then* $m(1,2) = 3 - 2\sqrt{1} = 1$ *and* $n(1,2) = 2 - 2e^{\sin(2-2(1))} = 0$*. But then we have* $x(1) = \tan(\ln 1) + 3 = 3$ *and* $y(0) = 0^2 - \arctan 0 - 1 = -1$*. Therefore, we have points*

$$(x,y) = (3,-1)$$
$$(m,n) = (1,0)$$
$$(s,t) = (1,2)$$

Now taking derivatives, we find

$$\frac{\partial f}{\partial x} = \frac{2x}{y}\bigg|_{\substack{x=3\\y=-1}} = -6$$
$$\frac{dx}{dm} = \sec^2(\ln m)\cdot\frac{1}{m}\bigg|_{m=1} = 1$$
$$\frac{\partial m}{\partial t} = -\sqrt{s}\bigg|_{\substack{s=1\\t=2}} = -1$$
$$\frac{\partial f}{\partial y} = -\frac{x^2}{y^2}\bigg|_{\substack{x=3\\y=-1}} = -9$$
$$\frac{dy}{dn} = 2n - \frac{1}{1+n^2}\bigg|_{n=0} = -1$$
$$\frac{\partial n}{\partial t} = -e^{\sin(t-2s)} - te^{\sin(t-2s)}\cdot\cos(t-2s)\bigg|_{\substack{s=1\\t=2}} = -1 - 2\cdot 1\cdot 1 = -3$$

Therefore,

$$\frac{\partial f}{\partial t} = (-6)(1)(-1) + (-9)(-1)(-3) = 6 - 27 = -21$$

4. Let S be the surface defined by $z = x - y^2$.

(a) (10 points) Find the equation of the tangent plane to S at $(2, -1, 1)$.

Let $F(x, y, z) = z - x + y^2$*. Then the surface* S *is given by* $F(x, y, z) = 0$*. We have*

$$\nabla F = \langle -1, 2y, 1\rangle \Big|_{\substack{x=2\\ y=-1\\ z=1}} = \langle -1, -2, 1\rangle$$

Then the tangent plane is given by

$$\begin{aligned}\langle -1, -2, 1\rangle \cdot \langle x - 2, y - (-1), z - 1\rangle &= 0\\ -1(x - 2) - 2(y + 1) + 1(z - 1) &= 0\\ -x + 2 - 2y - 2 + z - 1 &= 0\\ -x - 2y + z - 1 &= 0\\ x + 2y - z &= -1\end{aligned}$$

(b) (1 point) Find a direction perpendicular to S at $(2, -1, 1)$.

If a surface S *is given by* $F = 0$*, then* ∇F *is a direction perpendicular to the surface. Therefore, a direction perpendicular to* S *at* $(2, -1, 1)$ *is* $\langle -1, -2, 1\rangle$*. Equivalently,* $\langle 1, 2, -1\rangle$ *is perpendicular to* S *at* $(2, -1, 1)$*, as is* $k\langle -1, -2, 1\rangle$ *for any nonzero* k*.*

(c) (4 points) Find the equation of the normal line to S at $(2, -1, 1)$.

$$\begin{aligned}\ell(t) &= \vec{m}t + \vec{b}\\ \ell(t) &= \langle -1, -2, 1\rangle t + \langle 2, -1, 1\rangle\\ \ell(t) &= \langle -t, -2t, t\rangle + \langle 2, -1, 1\rangle\\ \ell(t) &= \langle 2 - t, -2t - 1, t + 1\rangle\end{aligned}$$

5. Let $f(x,y,z) := x^{10} + 2\sqrt{y} - e^{4-z}$.

(a) (3 points) Compute $f(1,9,4)$.

$$f(1,9,4) = 1^{10} + 2\sqrt{9} - e^0 = 1 + 2(3) - 1 = 6$$

(b) (6 points) Find the total differential for $f(x,y,z)$.

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 10x^9\,dx + \frac{1}{\sqrt{y}}\,dy + e^{4-z}\,dz$$

(c) (6 points) Approximate $(0.99)^{10} + 2\sqrt{8.7} - e^{4.1}$.

$$df = 10(1)\cdot -0.01 + \frac{1}{\sqrt{9}}\cdot -0.3 + e^0 \cdot 0.1 = -0.1 - 0.1 + 0.1 = -0.1$$

Then we have

$$(0.99)^{10} + 2\sqrt{8.7} - e^{4.1} = f(0.99, 8.7, 4.1) \approx f(1,9,4) + df = 6 - 0.1 = 5.9$$

6. The temperature in the region of a research facility is given by the function

$$f(x,y) = \frac{x^2 + y + e^{x+2y}}{3}$$

where x is the number of miles East/West ($+\hat{x}/-\hat{x}$) from the facility and y is number of miles North/South ($+\hat{y}/-\hat{y}$) from the facility. Suppose you are at a point 4 miles East and 2 mile South of the facility, i.e. $(x,y) = (4,-2)$.

(a) (6 points) What is the rate of change in the temperature if you are hiking straight towards the facility?

This is $D_{\mathbf{u}}f(-2,1)$*, where* $\mathbf{u}$ *the direction towards the facility at* $(4,-2)$*. Now* $(0,0)-(4,-2) = \langle -4, 2\rangle$ *and* $\|\langle -4,2\rangle\| = \sqrt{(-4)^2+2^2} = \sqrt{20} = 2\sqrt{5}$*. Now we have gradient*

$$\nabla f(x,y) = \left\langle \frac{2x+e^{x+2y}}{3}, \frac{1+2e^{x+2y}}{3} \right\rangle \Bigg|_{\substack{x=4 \\ y=-2}} = \left\langle \frac{8+e^0}{3}, \frac{1+2e^0}{3} \right\rangle = \langle 3, 1\rangle$$

Then we have

$$D_{\mathbf{u}}f(-2,1) = \langle 3,1\rangle \cdot \frac{\langle -4,2\rangle}{2\sqrt{5}} = \frac{3(-4)+1(2)}{2\sqrt{5}} = \frac{-12+2}{2\sqrt{5}} = \frac{-10}{2\sqrt{5}} = \frac{-5}{\sqrt{5}} = -\sqrt{5}$$

(b) (2 points) At your current position, what direction does the temperature increase most rapidly?

$$\nabla f(4,-2) = \langle 3,1\rangle$$

(c) (2 points) At your current position, what direction does the temperature decrease most rapidly?

$$-\nabla f(4,-2) = -\langle 3,1\rangle = \langle -3,-1\rangle$$

(d) (3 points) Approximately how far must you travel in the direction you gave in (c) to see a decrease of $\frac{1}{\sqrt{5}}$ in the temperature?

$$\begin{aligned} \textit{Rate of Change} \cdot \textit{Distance} &= \Delta T \\ D_{\mathbf{u}} f(-2,1) \cdot d &= \Delta T \\ -\sqrt{5}\, d &= -\frac{1}{\sqrt{5}} \\ d &= \frac{1}{5} \\ \textit{Distance} &= 0.20 \textit{ miles} \end{aligned}$$

Therefore, you need to travel approximately 0.20 miles.

(e) (3 points) At your current position, what is a direction you can travel so that the temperature does not change?

There is no change at (x, y) in the direction $\mathbf{u}$ if $D_{\mathbf{u}} f(x, y) = 0$. We then have

$$0 = D_{\mathbf{u}} f(-2,1) = \nabla f(4,-2) \cdot \mathbf{u} = \langle 3, 1 \rangle \cdot \mathbf{u}$$

Therefore, any direction $\mathbf{u}$ (not necessarily a unit vector) perpendicular to $\nabla f(4,-2)$ will suffice. For example, $\langle 1, -3 \rangle$ is such a factor. Writing $\mathbf{u} = \langle a, b \rangle$, we have

$$0 = \langle 3, 1 \rangle \cdot \mathbf{u} = \langle 3, 1 \rangle \cdot \langle a, b \rangle = 3a + b$$

Then $b = -3a$ so that $\mathbf{u} = \langle a, b \rangle = \langle a, -3a \rangle = a\langle 1, -3 \rangle$ for any nonzero $a \in \mathbb{R}$ is the collection of all such vectors.

7. (16 points) Find and classify all critical points for the function $f(x, y) = e^{-x}(x^2+3y^2)$.

We have

$$\begin{aligned} f_x &= -e^{-x}(x^2+3y^2) + 2xe^{-x} = (2x - x^2 - 3y^2)e^{-x} \\ f_y &= 6ye^{-x} \\ f_{xx} &= -e^{-x}(2x - x^2 - 3y^2) + (2-2x)e^{-x} = e^{-x}(2 - 2x - 2x + x^2 + 3y^2) = e^{-x}(2 - 4x + x^2 + 3y^2) \\ f_{xy} = f_{yx} &= -6ye^{-x} \\ f_{yy} &= 6e^{-x} \end{aligned}$$

Now we set $f_x = 0$ and $f_y = 0$. From $f_y = 0$, we know that $6y = 0$ or $e^{-x} = 0$. But e^{-x} is never 0 so $6y = 0$, which implies $y = 0$. But then $y = 0$ in $f_x = 0$ so that we have $0 = (2x - x^2 - 3y^2)e^{-x} = (2x - x^2)e^{-x} = x(2-x)e^{-x}$. As $e^{-x} \neq 0$, then $x = 0$ or $2 - x = 0$. Therefore, $x = 0, 2$. Then the critical points are $(0, 0)$ and $(2, 0)$. We form the Hessian:

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} e^{-x}(2 - 4x + x^2 + 3y^2) & -6ye^{-x} \\ -6ye^{-x} & 6e^{-x} \end{pmatrix}$$

Now we examine sequence of principal minors for the critical points:

$(0, 0)$:

$$\begin{bmatrix} e^0(2 - 0 + 0 + 0) & 0 \\ 0 & 6e^0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

$$d_1 = 2 > 0, \quad d_2 = 2(6) - 0(0) = 12 > 0$$

$(2, 0)$:

$$\begin{bmatrix} e^{-2}(2 - 4(2) + 2^2 + 0) & 0 \\ 0 & 6e^{-2} \end{bmatrix} = \begin{bmatrix} -2e^{-2} & 0 \\ 0 & 6e^{-2} \end{bmatrix}$$

$$d_1 = -2e^{-2} < 0, \quad d_2 = -12e^{-4} < 0$$

Therefore, $(0, 0, 0)$ is a local minimum and $(2, 0, 4e^{-2})$ is a saddle point.

8. (16 points) An exoplanet's orbit about its star is given by $x^2+y^2=1$, i.e. the exoplanet has a circular orbit. The planet has 12 moons, and number of moons on the dark side of the planet at a point (x,y) in its orbit is given by $f(x,y)=x^2+4xy+y^2+4$. Find the greatest and fewest number of moons you could see at night on this exoplanet.

Let $g(x,y)=x^2+y^2-1$ so that the path of the planet is given by the curve $g(x,y)=0$. We then want to maximize $f(x,y)$ with respect to the constraint that $g(x,y)=0$. Now

$$\begin{aligned} \nabla f(x,y) &= \lambda \nabla g(x,y) \\ \langle 2x+4y, 4x+2y\rangle &= \lambda\langle 2x, 2y\rangle \end{aligned}$$

Equating components, we have a system of equations

$$\begin{aligned} 2x+4y &= 2\lambda x \\ 4x+2y &= 2\lambda y \\ x^2+y^2 &= 1 \end{aligned}$$

Note that neither x nor y can be 0. For example, if $x=0$, then from the second equation we find that $4x=0$ so that $x=0$. But then $x^2+y^2=1$. This follows mutatis mutandis for $y=0$. Then solving for λ in the first and second equation, we find

$$\lambda = \frac{x+2y}{x} = \frac{2x+y}{y}$$

Then cross multiplying we find

$$2x^2+xy = xy+2y^2$$

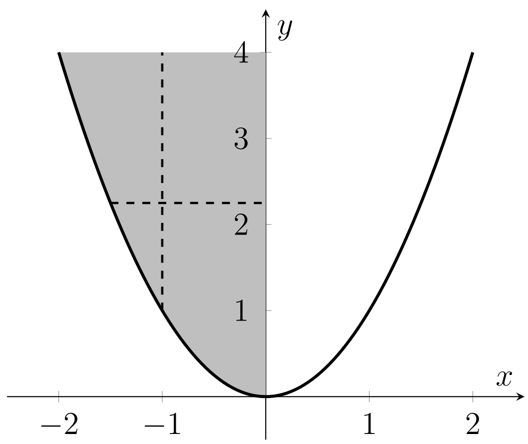
But then $2x^2=2y^2$, which implies $x^2=y^2$. Then $0=x^2-y^2=(x-y)(x+y)$. Therefore, $x=y$ or $x=-y$. Then $1=x^2+y^2=x^2+(\pm x)^2=2x^2$ so that $x=\pm 1/\sqrt{2}$. Now $y=\pm x$. If $x=1/\sqrt{2}$, then $y=\pm 1/\sqrt{2}$. If $x=-1/\sqrt{2}$, then $y=\mp 1/\sqrt{2}$. Then we have solutions $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, where the signs are taken independently. Finally, (taking the signs simultaneously)

$$\begin{aligned} f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) &= 3 \\ f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) &= 7 \end{aligned}$$

Therefore looking up at the night sky, you would see as few as three moons and as many as seven moons.

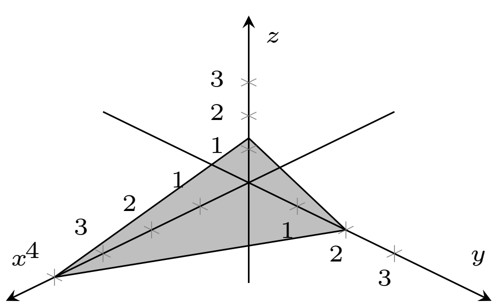
9. Complete the following parts:

(a) (8 points) $\int_{-2}^{0}\int_{x^2}^{4} x\sin(y^2)\;dy\;dx$



$$\begin{aligned}\int_{-2}^{0}\int_{x^2}^{4} x\sin y^2\;dy\;dx &= \int_{0}^{4}\int_{-\sqrt{y}}^{0} x\sin y^2\;dx\;dy\\ &= \int_0^4 \frac{x^2}{2}\sin y^2\Big|_{x=-\sqrt{y}}^{x=0}\;dx\;dy\\ &= -\frac{1}{2}\int_0^4 y\sin y^2\;dy\\ &= -\frac{1}{2}\cdot\frac{-\cos y^2}{2}\Big|_{y=0}^{y=4}\\ &= \frac{1}{4}(\cos 16-\cos 0)\\ &= \frac{\cos 16-1}{4}\end{aligned}$$

(b) (8 points) Compute the volume bounded by $x+2y+3z=4$ and the coordinate planes by using a triple integral.



$$\begin{aligned}V=\iiint_R dV &= \int_0^{4/3}\int_0^{(4-3z)/2}\int_0^{4-2y-3z} 1\;dx\;dy\;dz\\ &= \int_0^{4/3}\int_0^{(4-3z)/2} 4-2y-3z\;dy\;dz\\ &= \int_0^{4/3} 4y-y^2-3zy\Big|_{y=0}^{y=(4-3z)/2}\;dz\\ &= \int_0^{4/3} 4\left(\frac{4-3z}{2}\right)-\left(\frac{4-3z}{2}\right)^2-3z\left(\frac{4-3z}{2}\right)\;dz\\ &= \frac{1}{4}\int_0^{4/3} 16-24z+9z^2\;dz\\ &= \frac{1}{4}\cdot(16z-12z^2+3z^3)\Big|_{z=0}^{z=4/3}\\ &= \frac{1}{4}\cdot\left(16\cdot\frac{4}{3}-12\cdot\left(\frac{4}{3}\right)^2+3\cdot\left(\frac{4}{3}\right)^3\right)\\ &= \frac{1}{4}\cdot\frac{4}{3}\left(16-12\cdot\frac{4}{3}+3\cdot\frac{16}{9}\right)\\ &= \frac{1}{3}\left(\frac{48}{3}-\frac{48}{3}+\frac{16}{3}\right)=\frac{16}{9}\end{aligned}$$

This is more easily computed by noting $16-24z+9z^2=(4-3z)^2$.

10. Complete the following parts—you need not evaluate any of the integrals:

(a) (5 points) Change the following integral to polar coordinates: $\int_0^1 \int_y^1 \frac{y}{\sqrt{x^2+y^2}}\, dx\, dy$

$$\int_0^1 \int_y^1 \frac{y}{\sqrt{x^2+y^2}}\, dx\, dy = \int_0^{\pi/4} \int_0^{\sec\theta} \frac{r\sin\theta}{\sqrt{r^2}} \cdot r\, dr\, d\theta = \int_0^{\pi/4} \int_0^{\sec\theta} r\sin\theta\, dr\, d\theta$$

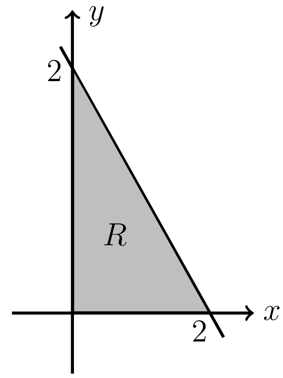
(b) (5 points) Change the following integral to cylindrical coordinates:
$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^2 xz\, dz\, dx\, dy$

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^2 xz\, dz\, dx\, dy = \int_0^{\pi} \int_0^2 \int_r^2 (r\cos\theta) z \cdot r\, dz\, dr\, d\theta = \int_0^{\pi} \int_0^2 \int_r^2 zr^2 \cos\theta\, dz\, dr\, d\theta$$

(c) (5 points) Change the following integral to spherical coordinates: $\iiint_R z\, dV$, where R is the bounded by $x = \sqrt{y^2+z^2}$ and the sphere $x^2+y^2+z^2 = 9$

$$\iiint_R z\, dV = \int_{\pi/4}^{3\pi/4} \int_{-\pi/4}^{\pi/4} \int_0^3 (\rho\cos\phi) \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \int_{\pi/4}^{3\pi/4} \int_{-\pi/4}^{\pi/4} \int_0^3 \rho^3 \sin\phi\cos\phi\, d\rho\, d\theta\, d\phi$$

11. (15 points) A steel plate has density function $\delta(x,y) = x + y$, and shape given by the region R given below:



Find the total mass and the center of mass for this steel plate.

$$M = \iint_R \delta(x,y)\,dA = \int_0^2 \int_0^{2-x} (x+y)\,dy\,dx = \int_0^2 \left(xy + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=2-x} dx = \int_0^2 x(2-x) + \frac{(2-x)^2}{2}\,dx$$
$$= \int_0^2 \left(2 - \frac{x^2}{2}\right) dx = \left(2x - \frac{x^3}{6}\right)\Bigg|_{x=0}^{x=2} = 2(2) - \frac{8}{6} = 4 - \frac{8}{6} = \frac{24}{6} - \frac{8}{6} = \frac{16}{6} = \frac{8}{3}$$

$$M_x = \iint_R x\delta(x,y)\,dA = \int_0^2 \int_0^{2-x} x(x+y)\,dy\,dx = \int_0^2 \int_0^{2-x} (x^2+xy)\,dy\,dx = \int_0^2 \left(x^2y + x\cdot\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=2-x} dx$$
$$= \int_0^2 \left(x^2(2-x) + x\cdot\frac{(2-x)^2}{2}\right) dx = \int_0^2 \left(2x - \frac{x^3}{2}\right) dx = \left(x^2 - \frac{x^4}{8}\right)\Bigg|_{x=0}^{x=2}$$
$$= 4 - \frac{16}{8} = 4 - 2 = 2$$

$$M_y = \iint_R y\delta(x,y)\,dA = \int_0^2 \int_0^{2-x} y(x+y)\,dy\,dx = \int_0^2 \int_0^{2-x} (xy+y^2)\,dy\,dx = \int_0^2 \left(x\cdot\frac{y^2}{2} + \frac{y^3}{3}\right)\Bigg|_{y=0}^{y=2-x} dx$$
$$= \int_0^2 \left(\frac{x(2-x)^2}{2} + \frac{(2-x)^3}{3}\right) dx = \int_0^2 \left(\frac{x^3}{6} - 2x + \frac{8}{3}\right) dx = \left(\frac{x^4}{24} - x^2 + \frac{8}{3}x\right)\Bigg|_{x=0}^{x=2}$$
$$= \frac{16}{24} - 4 + \frac{16}{3} = \frac{2}{3} - \frac{12}{3} + \frac{16}{3} = \frac{2-12+16}{3} = \frac{6}{3} = 2$$

Therefore, the total mass is $M = 8/3$ and the center of mass is

$$(\overline{x},\overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{2}{8/3}, \frac{2}{8/3}\right) = \left(\frac{3}{4}, \frac{3}{4}\right)$$

Note: The region is symmetric about $y = x$ and so is the function $\delta(x,y)$. Therefore, we know $M_x = M_y$, so only one need be computed.