Quiz 1: Plot the point $(-2, 1, 3)$ and find the distance to the following:

(a) xy-plane: 3

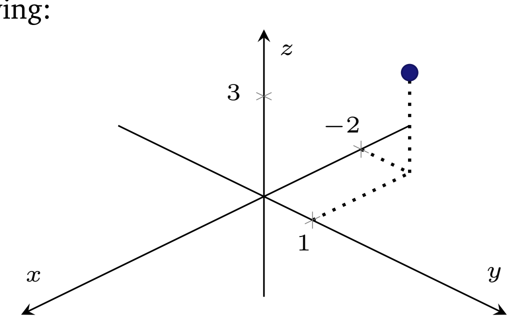
(b) yz-plane: 2

(c) xz-plane: 1

(d) x-axis: $\sqrt{1^2 + 3^2} = \sqrt{10}$

(e) y-axis: $\sqrt{(-2)^2 + 3^2} = \sqrt{13}$

(f) z-axis: $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$



Quiz 2: Let $\mathbf{u} = \langle 1, 0, -1, 2\rangle$ and $\mathbf{v} = \langle 1, 1, 1, 1\rangle$. Complete the following:

(a) Find $\mathbf{u} - 2\mathbf{v}$.

$$\mathbf{u} - 2\mathbf{v} = \langle 1, 0, -1, 2\rangle - 2\langle 1, 1, 1, 1\rangle = \langle 1, 0, -1, 2\rangle - \langle 2, 2, 2, 2\rangle = \langle -1, -2, -3, 0\rangle$$

(b) Find $\|\mathbf{v}\|$.

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$$

(c) Is $\mathbf{u}$ parallel to $\mathbf{v}$? Explain.

No, $\mathbf{u}$ is not parallel to $\mathbf{v}$. If $\mathbf{u}$ were parallel to $\mathbf{v}$, then there would be $c \in \mathbb{R}$ such that $\mathbf{u} = c\mathbf{v} = \langle c, c, c, c\rangle$. Comparing the first component, it is clear that we need $c = 1$. But $1\mathbf{v} = \mathbf{v} \neq \mathbf{u}$.

Quiz 3: Let $\mathbf{a} = \langle 1, -1, 2 \rangle$, $\mathbf{b} = \langle 3, -1, 2 \rangle$, $\mathbf{c} = \mathbf{i} + \mathbf{k}$, and $\mathbf{d} = 2\mathbf{j} - \mathbf{k}$.

(a) What is $\mathbf{a} \cdot \mathbf{b}$?

$$\mathbf{a} \cdot \mathbf{b} = \langle 1, -1, 2 \rangle \cdot \langle 3, -1, 2 \rangle = 1(3) - 1(-1) + 2(2) = 3 + 1 + 4 = 8$$

(b) Is $\mathbf{a} \perp \mathbf{b}$? Explain.

No, $\mathbf{a} \cdot \mathbf{b} \neq 0$, so that $\mathbf{a}$ cannot be perpendicular to $\mathbf{b}$.

(c) What is $\mathbf{c} \cdot \mathbf{d}$?

$$\mathbf{c} \cdot \mathbf{d} = \langle 1, 0, 1 \rangle \cdot \langle 0, 2, -1 \rangle = 1(0) + 0(2) + 1(-1) = 0 + 0 - 1 = -1$$

(d) If $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^3$, what is $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$?

$\mathbf{u} \times \mathbf{v}$ is a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Because $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$, we must have $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

x

Quiz 4: Find the vector and parametric forms of the line that passes through $(-1, 3, 5)$ and is parallel to the line $\mathbf{r}(t) = (2t + 1)\,\mathbf{i} - (t + 2)\,\mathbf{j} + 3\,\mathbf{k}$.

Solution. The line must have direction vector $\langle 2, -1, 0 \rangle$ and go through the point $(-1, 3, 5)$. Therefore, the vector form is $\ell(t) = \langle 2, -1, 0 \rangle t + (-1, 3, 5)$. Then we have $\ell(t) = (2t - 1, 3 - t, 5)$. This gives parametric form

$$\begin{cases} x = 2t - 1 \\ y = 3 - t \\ z = 5 \end{cases}$$

x

Quiz 5: Find the equation of the plane that passes through the point $(4, 5, 0)$ and is perpendicular to the line $\ell(t) = (5t + 1, 1 - 2t, 3t + 2)$.

Solution. If the line is perpendicular to the plane, then it points in the direction of a normal vector for the plane. But then $\langle 5, -2, 3 \rangle$ is a normal vector for the plane. The point $(4, 5, 0)$ is in the plane. Then

$$\begin{aligned} \langle 5, -2, 3 \rangle \cdot \langle x - 4, y - 5, z - 0 \rangle &= 0 \\ 5(x - 4) - 2(y - 5) + 3(z - 0) &= 0 \\ 5x - 20 - 2y + 10 + 3z &= 0 \\ 5x - 2y + 3z &= 10 \end{aligned}$$

Quiz 6: Find the point where the line $\ell(t) = \langle 2, -1, 1 \rangle\ t + (1, 1, 2)$ intersects the plane $x+y+z=6$.

Solution. The parametric form of the line is

$$\begin{cases} x = 2t + 1 \\ y = 1 - t \\ z = t + 2 \end{cases}$$

If the line and the plane intersect, the x, y, z in the plane satisfy these relations. Therefore,

$$\begin{aligned} x + y + z &= 6 \\ (2t+1) + (1-t) + (t+2) &= 6 \\ 2t + 4 &= 6 \\ 2t &= 2 \\ t &= 1 \end{aligned}$$

Given that each point on the line is of the form $(2t+1, 1-t, t+2)$, if $t = 1$, then the point is $(3, 0, 3)$.

x

Quiz 7: Is the point $(1, 2, 3)$ on the line $\ell(t) = \langle 2, -1, 1 \rangle t + (-3, 4, 1)$? Is the point $(1, 2, 3)$ on the plane $2x + y - z = 3$? Justify your answers.

Solution. The parametric form of the line is

$$\begin{cases} x = 2t - 3 \\ y = 4 - t \\ z = t + 1 \end{cases}$$

Then $(x, y, z) = (1, 2, 3)$ so that $(2t - 3, 4 - t, t + 1) = (1, 2, 3)$. Relating the z-component, we have $t + 1 = 3$ so that $t = 2$. Plugging in $t = 2$, we have $(2 \cdot 2 - 3, 4 - 2, 2 + 1) = (1, 2, 3)$. Then the point is on the line.

Now at the point $(1, 2, 3)$, we have $(x, y, z) = (1, 2, 3)$. Then we have

$$\begin{aligned} 2x + y - z &\stackrel{?}{=} 3 \\ 2(1) + 2 - 3 &\stackrel{?}{=} 3 \\ 2 + 2 - 3 &\stackrel{?}{=} 3 \\ 1 &\neq 3 \end{aligned}$$

Therefore, the point $(1, 2, 3)$ does not satisfy the relation $2x + y - z = 3$. The point $(1, 2, 3)$ is not on the plane.

Quiz 8: Is the line $\ell(t) = (t-2, 1-t, t-1)$ contained in the plane $3x + 9y + 7z = 0$? Explain.

Solution. The line has parametric form

$$\begin{cases} x = t - 2 \\ y = 1 - t \\ z = t - 1 \end{cases}$$

Therefore, if the line is contained in the plane then these x, y, z satisfy the relation $3x + 9y + 7z = 0$ for all x, y, z on the line, i.e. for all t. This means

$$\begin{aligned} 3x + 9y + 7z &= 0 \\ 3(t-2) + 9(1-t) + 7(t-1) &= 0 \\ 3t - 6 + 9 - 9t + 7t - 7 &= 0 \\ t - 4 &= 0 \end{aligned}$$

Notice while this does have a solution, it is not true *for all* t. Therefore, the line merely intersects the plane (when $t = 4$) but is not contained in the plane.

x

Quiz 9: Find parametrizations for the following:

(a) The line segment 'pointing' from $(1, 2, 3)$ to $(4, 5, 6)$.

$$\ell(t) = (1-t)(1, 2, 3) + t(4, 5, 6); \quad t \in [0, 1]$$

(b) The circle with radius 3 centered at $(4, 5)$, oriented counterclockwise.

$$r(t) = (3\cos t + 4, 3\sin t + 5)$$

(c) A helix with 'radius' 2, centered around $(1, 0, 1)$.

$$x(t) = (2\cos t + 1, t, 2\sin t + 1)$$

(d) The curve $y = e^x \sin x$.

$$C(t) = (t, e^t \sin t)$$

Quiz 10: Identify the following surfaces in $\mathbb{R}^3$:

(a) Paraboloid : $2x = 3y^3 + 4z^2$

(b) Plane : $x = y + z$

(c) Hyperboloid of Two Sheets : $z^2 = 1 + 2x^2 + 3y^2$

(d) Parabolic Cylinder : $y = x^2 + x + 1$

(e) Cone : $y^2 - z^2 - x^2 = 0$

(f) Hyperbolic Paraboloid : $y^2 = z + x^2$

(g) Ellipsoid : $\frac{1}{2}x^2 + y^2 + 2z^2 = 4$

(h) Hyperboloid of One Sheet : $z^2 + 4 = x^2 + y^2$

Quiz 11: Use limits along the x and y axes to show that the following limit does not exist: $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

Along x-axis, $y = 0$: $\lim_{x\to 0} \frac{x-0}{x+y} = \lim_{x\to 0} \frac{x}{x} = 1$

Along y-axis, $x = 0$: $\lim_{y\to 0} \frac{0-y}{0+y} = \lim_{y\to 0} -\frac{y}{y} = -1$

But as these limits are not the same, the $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ does not exist.

Quiz 12: Define $f(x,y) = \dfrac{x\cos(xy)}{y+1}$. Find the following:

$$\frac{\partial f}{\partial x} = \frac{1}{y+1}\big(\cos(xy) - xy\sin(xy)\big)$$

$$\frac{\partial f}{\partial y} = \frac{-x^2\sin(xy)(y+1) - x\cos(xy)}{(y+1)^2}$$

x

Quiz 13: Let $f(x,y) = y^x$.

(a) $f_x(x,y) = y^x \ln y$

(b) $f_y(x,y) = xy^{x-1}$

(c) $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0} 0^x = 0$

(d) $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = \lim\limits_{y\to 0} y^0 = 1$

(e) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \text{DNE}$

⋆ Note: This is why 0^0 is undefined. We want the values of y^x to be consistent (meaning continuous) near $(0,0)$. But from (c), (d), and (e), we can see there is no consistent definition of 0^0 to achieve this.

x

Quiz 14: Use the Chain Rule to find $\dfrac{\partial f}{\partial s}$, where $f(x,y,z) = x^2 + y + \cos z$, $x(s,t) = e^{st}$, $y(s,t) = \dfrac{s}{t}$, and $z(s,t) = t - s$.

Solution.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (2x+1)\cdot te^{st} + 1\cdot\frac{1}{t} + (-\sin z)\cdot(-1)$$

$$\left[= te^{st}(2e^{st}+1) + \frac{1}{t} + \sin(t-s)\right]$$

Quiz 15: Find the tangent plane to $x^2 + \cos y = 2e^{3-z}$ at the point $(1, 0, 3)$.

Solution. Let $F(x, y, z) = x^2 + \cos y - 2e^{3-z}$, so that the surfaces is the points where $F(x, y, z) = 0$. Then

$$\begin{aligned} \nabla F &= \langle 2x, -\sin y, 2e^{3-z} \rangle \\ \nabla F(1, 0, 3) &= \langle 2, 0, 2 \rangle \end{aligned}$$

Then the tangent plane is

$$\begin{aligned} \langle 2, 0, 2 \rangle \cdot \langle x - 1, y - 0, z - 3 \rangle &= 0 \\ 2\langle 1, 0, 1 \rangle \cdot \langle x - 1, y - 0, z - 3 \rangle &= 0 \\ \langle 1, 0, 1 \rangle \cdot \langle x - 1, y - 0, z - 3 \rangle &= 0 \\ 1(x - 1) + 0(y - 0) + 1(z - 3) &= 0 \\ x - 1 + 0 + z - 3 &= 0 \\ x + z &= 4 \end{aligned}$$

x

Quiz 16: Suppose $f(x, y) = 5xy$ measures the temperature at a point (x, y) on a flat surface.

(a) $\nabla f = \langle 5y, 5x \rangle$

(b) What is the rate of change for $f(x, y)$ at $(2, 1)$ in the direction $3\mathbf{i} + 4\mathbf{j}$?

$$\begin{aligned} \nabla f(2, 1) &= \langle 5(1), 5(2) \rangle = \langle 5, 10 \rangle \\ \|\mathbf{u}\| &= \|3\mathbf{i} + 4\mathbf{j}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \end{aligned}$$

$$D_{\mathbf{u}} f(2, 1) = \langle 5, 10 \rangle \cdot \frac{\langle 3, 4 \rangle}{5} = 5\langle 1, 2 \rangle \cdot \frac{\langle 3, 4 \rangle}{5} = \frac{5}{5} \cdot \big(1(3) + 2(4)\big) = 3 + 8 = 11$$

(c) What is the direction in which the temperature decreases fastest at $(2, 1)$?

$$-\nabla f(2, 1) - \langle 5, 10 \rangle = \langle -5, -10 \rangle$$

(d) At the point $(2, 1)$, name a direction in which one could move to keep the temperature constant.

Any direction perpendicular to $\pm\nabla f(2, 1)$ will suffice. For instance, $\pm\langle 2, -1 \rangle$.

Quiz 17: Integrate $\int_0^3 \int_1^{e^y} \frac{x+y}{x}\, dx\, dy$

Solution.

$$\begin{aligned}
\int_0^3 \int_1^{e^y} \frac{x+y}{x}\, dx\, dy &= \int_0^3 \int_1^{e^y} \left(1 + \frac{y}{x}\right)\, dx\, dy \\
&= \int_0^3 \left(\, x + y \ln|x| \,\right)\Big|_{x=1}^{x=e^y}\, dy \\
&= \int_0^3 \Big[\big(e^y + y \ln e^y\big) - \big(1 + y \ln 1\big)\Big]\, dy \\
&= \int_0^3 \big(e^y + y^2 - 1\big)\, dy \\
&= e^y + \frac{y^3}{3} - y\,\Bigg|_{y=0}^{y=3} \\
&= \left(e^3 + \frac{3^3}{3} - 3\right) - (e^0 + 0 - 0) \\
&= e^3 + 5
\end{aligned}$$

Quiz 18: Evaluate $\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$ by reversing the order of integration. Evaluate $\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$ by reversing the order of integration.

Solution.

$$\begin{aligned}
\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx &= \int_0^{\pi/2} \int_0^{y} \frac{\sin y}{y}\, dx\, dy \\
&= \int_0^{\pi/2} \frac{\sin y}{y} \int_0^{y} 1\, dx\, dy \\
&= \int_0^{\pi/2} \frac{\sin y}{y} \cdot x\,\Big|_0^{y}\, dy \\
&= \int_0^{\pi/2} \frac{\sin y}{y} \cdot (y - 0)\, dy \\
&= \int_0^{\pi/2} \frac{\sin y}{y} \cdot y\, dy \\
&= \int_0^{\pi/2} \sin y\, dy \\
&= -\cos y\,\Bigg|_0^{\pi/2} \\
&= -\cos(\pi/2) - (-\cos 0) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Quiz 19: Set up, but do not integrate, a triple integral that computes the volume bounded by $x^2 + z^2 = 1$, $y = 0$, and $y + z = 1$.

Solution.

$$V = \iiint_R dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-z} dy\, dz\, dx$$

x

Quiz 20: Make a change of variables to polar coordinates in $\int_0^{\sqrt{2}} \int_y^{\sqrt{2-y^2}} (x^2 + y^2)\, dx\, dy$. You do not need to integrate.

Solution.

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{2-y^2}} (x^2 + y^2)\, dx\, dy = \int_0^{\pi/4} \int_0^{\sqrt{2}} r^2 \cdot r\, dr\, d\theta = r^3\, dr\, d\theta$$

x

Quiz 21: Set up an integral, but do not integrate, using cylindrical coordinates that computes $\iiint_R y\, dV$, where R is the region bounded by $z = x^2 + y^2$ and $z = 3$.

Solution.

$$\iiint_R y\, dV = \int_0^{2\pi} \int_0^1 \int_0^3 r \sin\theta \cdot r\, dz\, dr\, d\theta$$

Quiz 22: Set up, but do not integrate, a spherical integral which computes $\iiint_R xe^{(x^2+y^2+z^2)^2}\, dV$, where R is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant.

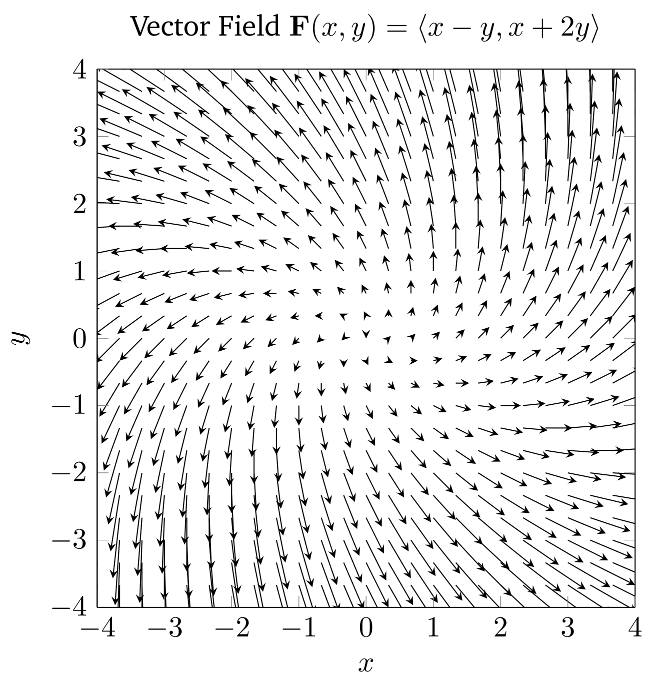
Solution.

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho\sin\phi\cos\theta\, e^{(\rho^2)^2}\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^3 e^{\rho^4}\sin^2\phi\cos\theta\, d\rho\, d\phi\, d\theta$$

x

Quiz 23: Sketch the vector field $\mathbf{F}(x,y)=\langle x-y, x+2y\rangle$.

Solution.



Quiz 24: Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y) = \langle y, -x \rangle$.

(a) Find div $\mathbf{F}$.

(b) Find curl $\mathbf{F}$.

(c) Is $\mathbf{F}$ incompressible? Is $\mathbf{F}$ irrotational?

Solution.

(a)

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0 + 0 = 0$$

(b)

$$\begin{aligned} \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} &= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & -x \end{vmatrix} \mathbf{k} \\ &= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-x) \right) \mathbf{i} - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) \right) \mathbf{k} \\ &= (0 - 0)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (-1 - 1)\,\mathbf{k} \\ &= -2\mathbf{k} \end{aligned}$$

(c) The vector field $\mathbf{F}$ is incompressible (or solenoidal or divergence free) because $\nabla \cdot \mathbf{F} = 0$. However, the vector field $\mathbf{F}$ is not irrotational (or a conservative vector field or a gradient field) because $\nabla \times \mathbf{F} \neq \mathbf{0}$.

x

Quiz 25: Compute $\int_C (x - y)\, ds$, where C is the line segment from $(-3, -5)$ to $(2, -4)$.

Solution. We parametrize the line segment:

$$\begin{aligned} \mathbf{m} &= (2, -4) - (-3, -5) = (2, -4) + (3, 5) = \langle 5, 1 \rangle \\ \mathbf{r}(t) &= \mathbf{m}t + P_0 = \langle 5, 1 \rangle t + (-3, -5) = \langle 5t - 3, t - 5 \rangle; \quad 0 \leq t \leq 1 \end{aligned}$$

Then we have $ds = |\mathbf{r}'(t)|\, dt = |\langle 5, 1 \rangle|\, dt = \sqrt{26}\, dt$. Therefore,

$$\begin{aligned} \int_C (x - y)\, ds &= \int_0^1 \Big((5t - 3) - (t - 5) \Big) \sqrt{26}\, dt \\ &= \sqrt{26} \int_0^1 (4t + 2)\, dt \\ &= \sqrt{26} \cdot \left[2t^2 + 2t \right]_0^1 \\ &= \sqrt{26}\, [(2 + 2) - 0] \\ &= 4\sqrt{26} \end{aligned}$$

Quiz 26: Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = -x\,\mathbf{i} + y\,\mathbf{j}$ and $\mathbf{r}(t)$ is the circle centered at $(0,0)$ with radius 2, oriented counterclockwise.

Solution. We parametrize the circle:

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle; \quad 0 \leq t \leq 2\pi$$
$$\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$$

Then we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -\cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \; dt \\
&= \int_0^{2\pi} \sin t \cos t + \sin t \cos t \; dt \\
&= \int_0^{2\pi} 2 \sin t \cos t \; dt \\
&= \int_0^{2\pi} \sin(2t) \; dt \\
&= -\frac{\cos 2t}{2} \Big|_0^{2\pi} \\
&= -\frac{1}{2}(\cos 4\pi - \cos 0) \\
&= -\frac{1}{2}(1 - 1) \\
&= 0
\end{aligned}$$

Quiz 27: Compute $\int_C (2x + y) \; dx + (y + z) \; dy + (x - z) \; dz$, where C is the line segment from $(1,1,2)$ to $(3,-1,3)$.

Solution. We parametrize the line segment:

$$\mathbf{m} = (3,-1,3) - (1,1,2) = (3,-1,3) + (-1,-1,-2) = \langle 2,-2,1 \rangle$$
$$\mathbf{r}(t) = \mathbf{m}t + P_0 = \langle 2,-2,1 \rangle t + (1,1,2) = \langle 2t+1, 1-2t, t+2 \rangle; \quad 0 \leq t \leq 1$$

Now we find the differentials:

$$dx = 2 \; dt$$
$$dy = -2 \; dt$$
$$dz = 1 \; dt$$

Then we have

$$\int_C (2x+y)\,dx + (y+z)\,dy + (x-z)\,dz =$$
$$\int_0^1 \Big(2(2t+1)+(1-2t)\Big)\cdot 2\,dt + \Big((1-2t)+(t+2)\Big)\cdot -2\,dt + \Big((2t+1)-(t+2)\Big)\cdot 1\,dt$$

This becomes

$$\begin{aligned}\int_0^1 (7t-1)\,dt &= \frac{7t^2}{2} - t\Big|_0^1 \\ &= \left(\frac{7}{2}-1\right) - 0 \\ &= \frac{7}{2} - \frac{2}{2} \\ &= \frac{5}{2}\end{aligned}$$

x

Quiz 28: Show that the vector field $\mathbf{F}(x,y) = \langle 2xy - y^2, x^2 - 2xy + 1\rangle$ is conservative. Find a potential function for $\mathbf{F}$. What is $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the line segment from $(1,0)$ to $(0,1)$? What about $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{r}(t)$ is the unit circle, oriented counterclockwise?

Solution. A vector field $\mathbf{F}$ is conservative if and only if curl $\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$.

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy - y^2 & x^2 - 2xy + 1 & 0 \end{vmatrix} \\ &= \left\langle \frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}(x^2-2xy+1), -\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}(2xy-y^2)\right), \frac{\partial}{\partial x}(x^2-2xy+1) - \frac{\partial}{\partial y}(2xy-y^2)\right\rangle \\ &= \langle 0-0, -(0-0), (2x-2y)-(2x-2y)\rangle \\ &= \mathbf{0}\end{aligned}$$

Equivalently because $\mathbf{F}$ is two-dimensional, $\mathbf{F} = \langle M, N\rangle$ is conservative if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Observe

$$\begin{aligned}\frac{\partial}{\partial y}(2xy - y^2) &\stackrel{?}{=} \frac{\partial}{\partial x}(x^2 - 2yx + 1) \\ 2x - 2y &= 2x - 2y\end{aligned}$$

In either case, we now know $\mathbf{F}$ is conservative. We need find a potential function for $\mathbf{F}$, i.e. a function $f(x,y)$ so that $\mathbf{F} = \nabla f$. Now $\mathbf{F} = \nabla f = \langle f_x, f_y\rangle$. So

$$f = \int \frac{\partial f}{\partial x}\,dx = \int (2xy - y^2)\,dx = x^2y - xy^2 + g(y)$$

where $g(y)$ is some function of y alone. But we know also

$$x^2 - 2xy + 1 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y - xy^2 + g(y)) = x^2 - 2xy + g'(y)$$

Therefore, $g'(y) = 1$ so that $g(y) = \displaystyle\int g'(y)\ dy = \int 1\ dy = y + C$, where C is a constant. Then we have found $f(x,y) = x^2y - xy^2 + y + C$. Now using the Fundamental Theorem of Calculus for Line Integrals, we know

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,1) - f(1,0) = (1 + C) - (0 + C) = 1$$

We know also that $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ because the path is closed and $\mathbf{F}$ is conservative.

x

Quiz 29: Use Green's Theorem to compute $\oint_C (y - 3x)\ dx + (2x - y)\ dy$, where C is the square with vertices $(0,0), (2,0), (2,2), (0,2)$, oriented counterclockwise.

Solution. From the integral, we know that $\mathbf{F} = \langle M, N \rangle = \langle y - 3x, 2x + y \rangle$. Then by Green's Theorem,

$$\begin{aligned}
\oint_C (y - 3x)\ dx + (2x - y)\ dy &= \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\ dA \\
&= \iint \left(\frac{\partial}{\partial x}(2x + y) - \frac{\partial}{\partial y}(y - 3x) \right)\ dA \\
&= \iint 1\ dA \\
&= \text{Area bounded by } C \\
&= 2 \cdot 2 = 4
\end{aligned}$$

Quiz 30: Find a parametrizations $\mathbf{r}(u,v)$ for the cylinder $x^2+z^2=4$ with $-1 \leq y \leq 1$ (both the sides and 'top'/'bottom'). Find $\mathbf{r}_u$, $\mathbf{r}_v$, and $\mathbf{r}_u \times \mathbf{r}_v$ for these parametrizations.

Solution.

'Top':

$$\begin{aligned} \mathbf{r}(r,t) &= \langle r\cos t, 1, r\sin t\rangle; \quad 0 \leq r \leq 2, 0 \leq t \leq 2\pi \\ \mathbf{r}_r &= \langle \cos t, 0, \sin t\rangle \\ \mathbf{r}_t &= \langle -r\sin t, 0, r\cos t\rangle \\ \mathbf{r}_r \times \mathbf{r}_t &= \langle 0, -r, 0\rangle \end{aligned}$$

Sides:

$$\begin{aligned} \mathbf{r}(y,t) &= \langle 2\cos t, y, 2\sin t\rangle; \quad -1 \leq y \leq 1, 0 \leq t \leq 2\pi \\ \mathbf{r}_y &= \langle 0, 1, 0\rangle \\ \mathbf{r}_t &= \langle -2\sin t, 0, 2\cos t\rangle \\ \mathbf{r}_y \times \mathbf{r}_t &= \langle 2\cos t, 0, 2\sin t\rangle \end{aligned}$$

'Bottom':

$$\begin{aligned} \mathbf{r}(r,t) &= \langle r\cos t, -1, r\sin t\rangle; \quad 0 \leq r \leq 2, 0 \leq t \leq 2\pi \\ \mathbf{r}_r &= \langle \cos t, 0, \sin t\rangle \\ \mathbf{r}_t &= \langle -r\sin t, 0, r\cos t\rangle \\ \mathbf{r}_r \times \mathbf{r}_t &= \langle 0, -r, 0\rangle \end{aligned}$$

x

Quiz 31: Find the surface area of the portion of the plane $2x+3y+z=6$ lying about the xy-plane in the first octant.

Solution. Observe we have $z = 6-2x-3y$. Then we can parametrize the surface as $z = g(x,y) := 6-2x-3y$, where $0 \leq x \leq 3, 0 \leq y \leq 2$. Then we have normal vector $\mathbf{n} = \langle -g_x, -g_y, 1\rangle = \langle 2,3,1\rangle$. Then $|\mathbf{n}| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$. Now

$$\begin{aligned} \text{SA} &= \iint_S 1\, dS \\ &= \int_0^2 \int_0^3 \sqrt{14}\, dx\, dy \\ &= \sqrt{14} \int_0^2 \int_0^3 1\, dx\, dy \\ &= 6\sqrt{14} \end{aligned}$$

Quiz 32: Use the Divergence Theorem to compute $\oiint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x, y, z \rangle$ and S is the sphere at $(0, 0, 0)$ with radius 2 with outward pointing normal.

Solution. By the Divergence Theorem (Gauss' Theorem), we have

$$\begin{aligned}\oiint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_R \nabla \cdot \mathbf{F}\, dV \\ &= \iiint_R (1 + 1 + 1)\, dV \\ &= 3 \iiint_R 1\, dV \\ &= 3 \cdot \text{volume } R \\ &= 3 \cdot \frac{4\pi}{3} r^3 \\ &= 3 \cdot \frac{4\pi}{3} \cdot 2^3 \\ &= 32\pi\end{aligned}$$

x

Quiz 33: Use Stokes' Theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle x^2 - z, 2x + y, z^2 + x \rangle$ and C is the square with vertices $(0, 0), (2, 0), (2, 2), (0, 2)$, oriented counterclockwise.

Solution. Note that C bounds a square in the plane $z = 0$ (our surface). Writing $z = g(x, y) := 0$, the surface has normal $\langle -g_x, -g_y, 1 \rangle = \langle 0, 0, 1 \rangle$. By Stokes' Theorem, we have

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_{\mathbf{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_R \langle 0, -2, 2 \rangle \cdot \langle 0, 0, 1 \rangle\, dA \\ &= 2 \iint_R 1\, dA \\ &= 8 \cdot \text{Square Area} \\ &= 8 \cdot (2 \cdot 2) \\ &= 8\end{aligned}$$