MAT 295: Exam 1
Spring – 2021
03/02/2021
80 Minutes

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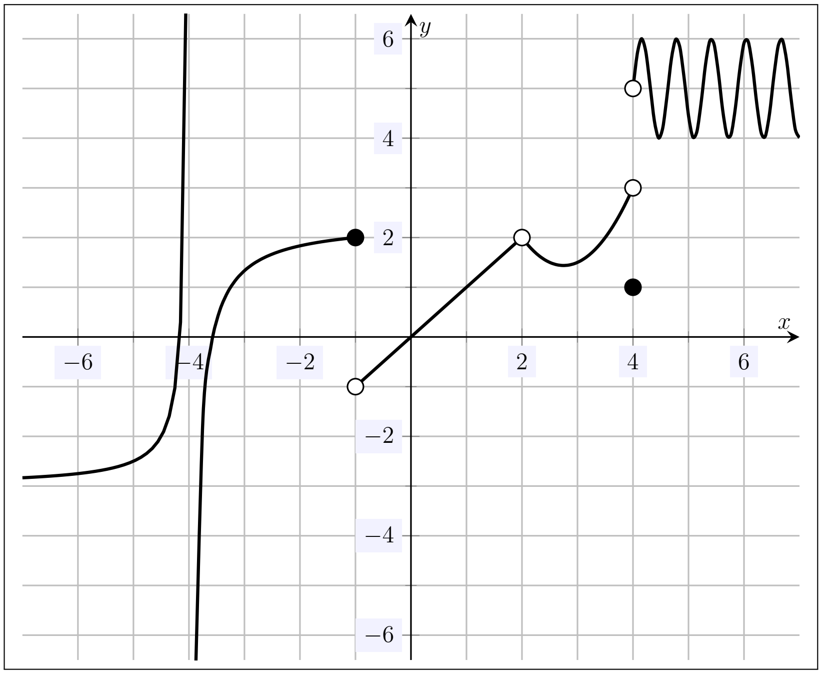
Write your name on the appropriate line on the exam cover sheet. This exam contains 8 pages (including this cover page) and 7 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	20	
2	20	
3	20	
4	10	
5	10	
6	10	
7	10	
Total:	100	

1. (20 points) For the function $f(x)$, whose graph is shown in the figure below, compute the following limits. If the limit does not exist, write 'DNE.'



(a) $\lim_{x \to -4^-} f(x) = \infty$

(b) $\lim_{x \to -4^+} f(x) = -\infty$

(c) $\lim_{x \to -4} f(x) =$ DNE

(d) $f(-4) =$ Undefined

(e) $\lim_{x \to -1^-} f(x) = 2$

(f) $\lim_{x \to -1^+} f(x) = -1$

(g) $\lim_{x \to -1} f(x) =$ DNE

(h) $f(-1) = 2$

(i) $\lim_{x \to 2^-} f(x) = 2$

(j) $\lim_{x \to 2^+} f(x) = 2$

(k) $\lim_{x \to 2} f(x) = 2$

(l) $f(2) =$ Undefined

(m) $\lim_{x \to 4^-} f(x) = 3$

(n) $\lim_{x \to 4^+} f(x) = 5$

(o) $\lim_{x \to 4} f(x) =$ DNE

(p) $f(4) = 1$

(q) $\lim_{x \to -\infty} f(x) = -3$

(r) $\lim_{x \to \infty} f(x) =$ DNE

2. (20 points) Compute the following limits. Be sure to show your work.

(a) $\lim_{x \to -1} \dfrac{x+1}{x-1} = \dfrac{-1+1}{-1-1} = \dfrac{0}{-2} = 0$

(b) $\lim_{x \to 2} \dfrac{x^2+3x-10}{x^2-4} = \lim_{x \to 2} \dfrac{(x-2)(x+5)}{(x-2)(x+2)} = \lim_{x \to 2} \dfrac{x+5}{x+2} = \dfrac{7}{4}$

(c) $\lim_{x \to 0} \dfrac{(4+x)^2-16}{x} = \lim_{x \to 0} \dfrac{(16+x^2+8x)-16}{x} = \lim_{x \to 0} \dfrac{x^2+8x}{x} = \lim_{x \to 0}(x+8) = 8$

(d) $\lim_{x \to 0} \dfrac{\dfrac{1}{x+2} - \dfrac{1}{2}}{x} = \lim_{x \to 0} \dfrac{\dfrac{2-(x+2)}{2(x+2)}}{x} = \lim_{x \to 0} \dfrac{-x}{2x(x+2)} = \lim_{x \to 0} \dfrac{-1}{2(x+2)} = -\dfrac{1}{4}$

3. (20 points) Compute the following limits. Be sure to show your work.

(a) $\lim_{x\to 0} \dfrac{\sin(5x)}{3x} = \lim_{x\to 0} \dfrac{\sin(5x)}{3x} \cdot \dfrac{5}{5} = \lim_{x\to 0} \dfrac{\sin(5x)}{5x} \cdot \dfrac{5}{3} = 1 \cdot \dfrac{5}{3} = \dfrac{5}{3}$

(b) $\lim_{x\to\infty} \left(1 + \dfrac{1}{3x}\right)^{2x} = \lim_{x\to\infty} \left(1 + \dfrac{1}{3x}\right)^{2x\cdot 3/3} = \lim_{x\to\infty} \left[\left(1 + \dfrac{1}{3x}\right)^{3x}\right]^{2/3} = e^{2/3}$

(c) $\lim_{x\to 4^+} \dfrac{x-6}{x-4} = -\infty$

(d) $\lim_{x\to 0} \dfrac{\tan(6x)}{\sin(6x)} = \lim_{x\to 0} \dfrac{\sin(6x)}{\cos(6x)} \cdot \dfrac{1}{\sin(6x)} = \lim_{x\to 0} \dfrac{1}{\cos(6x)} = 1$

4. (10 points) Find the following limit. You must show your work to justify your answer completely. [Note: Using 'growth rates' will only receive partial points. You must "*rigorously*" find the limit.]

$$\lim_{x\to\infty} \frac{2x^2+4x-5}{3x^2-x+7}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2+4x-5}{3x^2-x+7} &= \lim_{x\to\infty} \frac{2x^2+4x-5}{3x^2-x+7} \cdot \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to\infty} \frac{\dfrac{2x^2}{x^2}+\dfrac{4x}{x^2}-\dfrac{5}{x^2}}{\dfrac{3x^2}{x^2}-\dfrac{x}{x^2}+\dfrac{7}{x^2}} \\
&= \lim_{x\to\infty} \frac{2+4/x-5/x^2}{3-1/x+7/x^2} \\
&= \frac{2+0-0}{3-0+0} \\
&= \frac{2}{3}
\end{aligned}$$

5. (10 points) Use the Squeeze Theorem to prove the following limit is correct:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$$

We know that $-1 \leq \sin x \leq 1$ *for all* x*. Therefore,* $-1 \leq \sin\left(\frac{1}{\sqrt[3]{x}}\right) \leq 1$*. But then we have*

$$-x^2 = x^2 \cdot -1 \leq x^2 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \leq x^2 \cdot 1 = x^2$$

We know that

$$\lim_{x \to 0} -x^2 = 0$$
$$\lim_{x \to 0} x^2 = 0$$

Therefore by the Squeeze Theorem,

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$$

6. Let $f(x)$ be the following function:

$$f(x) = \begin{cases} x^2 + 4, & x \leq 1 \\ 1 - cx, & x > 1 \end{cases}$$

(a) (5 points) Explain why $f(x)$ is continuous for $x < 1$ and $x > 1$.

For $x < 1$ and $x > 1$, $f(x)$ is given by a polynomial. Polynomials are continuous everywhere. Therefore, $f(x)$ is continuous for $x < 1$ and $x > 1$.

(b) (5 points) Find c so that $f(x)$ is continuous at $x = 1$. Be sure to justify that the function is continuous at $x = 1$.

$$\begin{aligned} f(1) &= 1 + 4 = 5 \\ \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (x^2 + 4) = 1 + 4 = 5 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (1 - cx) = 1 - c \end{aligned}$$

For $f(x)$ to be continuous at $x = 1$, we need $f(1) = \lim_{x \to 1} f(x)$. Therefore, we need

$$\begin{aligned} f(1) &= \lim_{x \to 1} f(x) \\ 5 &= 1 - c \\ c &= -4 \end{aligned}$$

7. Let $f(x) = \sqrt{x}$.

(a) (5 points) Use the definition of the derivative to show that $f'(4) = \dfrac{1}{4}$.

$$
\begin{aligned}
f'(4) &:= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \\
&= \lim_{h\to 0} \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h} + 2)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} \\
&= \frac{1}{\sqrt{4+0} + 2} \\
&= \frac{1}{2+2} \\
&= \frac{1}{4}
\end{aligned}
$$

(b) (5 points) Use (a) to find the tangent line to $f(x)$ at $x = 4$.

The tangent line is...

$$
\begin{aligned}
y &= f(4) + f'(4)(x-4) \\
y &= 2 + \frac{1}{4}(x-4) \\
y &= 2 + \frac{1}{4}x - 1 \\
y &= 1 + \frac{1}{4}x
\end{aligned}
$$