MAT 295: Exam 2
Spring – 2021
04/06/2021
80 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 9 pages (including this cover page) and 7 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	15	
2	15	
3	25	
4	20	
5	15	
6	15	
7	20	
Total:	125	

1. (15 points) Compute the following derivatives:

(a) $\dfrac{d}{dx}\, x^5\, 3^x \arcsin x$

$$5x^4\, 3^x \arcsin x + x^5(3^x \ln 3) \arcsin x + \frac{x^5\, 3^x}{\sqrt{1-x^2}}$$

(b) $\dfrac{d}{dx}\, \dfrac{x^2+e^x}{1-\sin x}$

$$\frac{(1-\sin x)(2x+e^x)-(-\cos x)(x^2+e^x)}{(1-\sin x)^2}$$

(c) $\dfrac{d}{dx}\, \arctan\left(\dfrac{1}{1-x}\right)$

$$\frac{1}{1+\left(\dfrac{1}{1-x}\right)^2} \cdot \frac{-1}{(1-x)^2} \cdot -1$$

2. (15 points) Complete the following parts:

(a) Find $\dfrac{d^2}{dx^2}\ \ln(\sin x)$.

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\cdot\cos x = \frac{\cos x}{\sin x} = \cot x$$

$$\frac{d^2}{dx^2}\ \ln(\sin x) = \frac{d}{dx}\cot x = -\csc^2 x$$

(b) Find $\dfrac{d^{2021}}{dx^{2021}}\ \sin(2x)$.

$$\frac{d^{2021}}{dx^{2021}}\ \sin(2x) = 2^{2021}\cos(2x)$$

(c) If $f(x), g(x), h(x)$ are differentiable functions, find $\dfrac{d}{dx}\ f\big(g(x^2)\big) + f(-x)\big(h(x)\big)^2$.

$$f'\big(g(x^2)\big)\cdot g'(x^2)\cdot 2x + \Big(-f(-x)\big(h(x)\big)^2 + f(-x)\cdot 2h(x)h'(x)\Big)$$

3. (25 points) Complete the following parts:

(a) $\lim\limits_{x\to 0} \dfrac{\tan(2x)}{x+\sin x}$

$$\lim_{x\to 0} \frac{\tan(2x)}{x+\sin x} \overset{L.H.}{=} \lim_{x\to 0} \frac{2\sec^2(2x)}{1+\cos x} = \frac{2\sec^2(0)}{1+\cos 0} = \frac{2(1)}{1+1} = 1$$

(b) $\lim\limits_{x\to\infty} \dfrac{\ln(2+e^{5x})}{3x}$

$$\lim_{x\to\infty} \frac{\ln(2+e^{5x})}{3x} \overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{5e^{5x}}{2+e^{5x}}}{3} = \lim_{x\to\infty} \frac{5e^{5x}}{6+3e^{5x}} \overset{L.H.}{=} \lim_{x\to\infty} \frac{25e^{5x}}{15e^{5x}} = \frac{25}{15} = \frac{5}{3}$$

(c) $\lim\limits_{x\to\infty} \big(2\ln(x) - \ln(3x^2+5)\big)$

$$\lim_{x\to\infty} 2\ln(x) - \ln(3x^2+5) = \lim_{x\to\infty} \ln(x^2) - \ln(3x^2+5) = \lim_{x\to\infty} \ln\left(\frac{x^2}{3x^2+5}\right) = \ln(1/3) = -\ln 3$$

(d) $\lim\limits_{x \to 0} (1 + \sin 2x)^{1/x}$

$$\begin{aligned}
y &= \lim_{x \to 0} (1 + \sin 2x)^{1/x} \\
\ln y &= \lim_{x \to 0} \ln(1 + \sin 2x)^{1/x} \\
\ln y &= \lim_{x \to 0} \frac{\ln(1 + \sin 2x)}{x} \\
\ln y &\overset{L.H.}{=} \lim_{x \to 0} \frac{\frac{2\cos 2x}{1 + \sin 2x}}{1} \\
\ln y &= \lim_{x \to 0} \frac{2\cos 2x}{1 + \sin 2x} \\
\ln y &= \frac{2(1)}{1 + 0} \\
\ln y &= 2 \\
y &= e^2
\end{aligned}$$

(e) $\lim\limits_{x \to 0^+} \sqrt[3]{x} \ln(x)$

$$\lim_{x \to 0^+} \sqrt[3]{x} \ln(x) = \lim_{x \to 0^+} \frac{\ln x}{x^{-1/3}} \overset{L.H.}{=} \lim_{x \to 0^+} \frac{1/x}{-\frac{1}{3}x^{-4/3}} = \lim_{x \to 0^+} -3\sqrt[3]{x} = 0$$

4. (20 points) Let $\mathcal{C}$ be the curve formed by the points (x, y) satisfying the equation

$$e^{xy} = x^2 + y^2$$

(a) Find $\dfrac{dy}{dx}$ on $\mathcal{C}$.

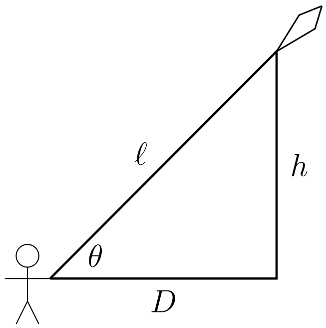
$$\begin{aligned}
e^{xy} &= x^2 + y^2 \\
\frac{d}{dx}e^{xy} &= \frac{d}{dx}\left(x^2 + y^2\right) \\
e^{xy}\left(y + xy'\right) &= 2x + 2yy' \\
ye^{xy} + xy'e^{xy} &= 2x + 2yy' \\
xy'e^{xy} - 2yy' &= 2x - ye^{xy} \\
y'(xe^{xy} - 2y) &= 2x - ye^{xy} \\
\frac{dy}{dx} &= \frac{2x - ye^{xy}}{xe^{xy} - 2y}
\end{aligned}$$

(b) Find the equation of the tangent line to $\mathcal{C}$ at $(0, 1)$.

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \left.\frac{2x - ye^{xy}}{xe^{xy} - 2y}\right|_{(0,1)} = \frac{0 - 1(1)}{0 - 2(1)} = \frac{-1}{-2} = \frac{1}{2}$$

$$\begin{aligned}
y &= y_0 + m(x - x_0) \\
y &= 1 + \frac{1}{2}(x - 0) \\
y &= \frac{1}{2}x + 1
\end{aligned}$$

5. (15 points) A child is flying a kite a mere 10 ft above the ground. The kite is moving away from the child at a speed of 1 foot per second. At what rate is the angle between the string and the horizontal decreasing when 20 ft of string has been left out?



We want $\frac{d\theta}{dt}$. We know at the time in question that $\ell = 20$ ft, $h = 10$ ft, and $\frac{dD}{dt} = 1$ ft/s. The side h is constant. And we have...

$$\cos\theta = \frac{D}{\ell} \text{ so that } \sec\theta = \frac{\ell}{D}$$

First, we will want to find D at the moment in question:

$$\begin{aligned} \ell^2 &= D^2 + h^2 \\ 20^2 &= D^2 + 10^2 \\ 400 &= D^2 + 100 \\ D^2 &= 300 \\ D &= 10\sqrt{3} \end{aligned}$$

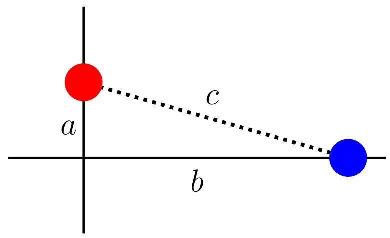
Then we have

$$\begin{aligned} \tan\theta &= \frac{h}{D} \\ \tan\theta &= \frac{10}{D} \\ \frac{d}{dt}\tan\theta &= \frac{d}{dt}\frac{10}{D} \\ \sec^2\theta\,\frac{d\theta}{dt} &= -\frac{10}{D^2}\frac{dD}{dt} \\ \left(\frac{20}{10\sqrt{3}}\right)^2\frac{d\theta}{dt} &= -\frac{10}{(10\sqrt{3})^2}\cdot 1 \\ \frac{d\theta}{dt} &= -\frac{10}{400}\cdot\frac{(10\sqrt{3})^2}{(10\sqrt{3})^2} \\ \frac{d\theta}{dt} &= -\frac{1}{40} \approx -0.025 \text{ rad/s} \end{aligned}$$

OR

One could also use the fact that $\tan\theta = \frac{h}{D}$ so that $D = h\cot\theta = 10\cot\theta$. Then $\frac{dD}{d\theta} = -10\csc^2\theta$. We know $\ell = 20$ so that $\csc\theta = \ell/h = 2$ so that $\frac{dD}{d\theta} = -40$. Finally, we know that $\frac{dD}{dt} = 1$. Then $\frac{dD}{dt} = \frac{dD}{d\theta}\frac{d\theta}{dt}$, so that $1 = -40\cdot\frac{d\theta}{dt}$, which gives the same solution.

6. (15 points) A runaway red bowling ball rolls down a lane at a rate of 3 feet per second directly towards a bowler. The bowler rolls a blue bowling ball in a direction perpendicular to the path of motion of the red ball at a rate of 5 feet per second. At what rate is the distance between the two balls changing when the red ball is 8 ft from the player and the blue ball is 4 ft from the player?



We want $\frac{dc}{dt}$. *We know at the time in question that* $a = 8$ *ft,* $b = 4$ *ft,* $\frac{da}{dt} = -3$ *ft/sec, and* $\frac{db}{dt} = 5$ *ft/sec.*

We first find c *at the moment in question:*

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 8^2 + 4^2 \\ c^2 &= 80 \\ c &= 4\sqrt{5} \end{aligned}$$

Now

$$\begin{aligned} c^2 &= a^2 + b^2 \\ \frac{d}{dt}c^2 &= \frac{d}{dt}\left(a^2 + b^2\right) \\ 2c\,\frac{dc}{dt} &= 2a\,\frac{da}{dt} + 2b\,\frac{db}{dt} \\ c\,\frac{dc}{dt} &= a\,\frac{da}{dt} + b\,\frac{db}{dt} \\ 4\sqrt{5}\,\frac{dc}{dt} &= 8(-3) + 4(5) \\ \frac{dc}{dt} &= \frac{-4}{4\sqrt{5}} \\ \frac{dc}{dt} &= -\frac{1}{\sqrt{5}} \approx -0.447214 \text{ ft/sec} \end{aligned}$$

7. (20 points) Consider the equation $x^3 = 2 - 6x^2$.

   (a) Use the Intermediate Value Theorem to show that this equation has a solution on the interval $[0, 1]$.

   A solution to this equation is equivalent to a solution to $x^3 + 6x^2 - 2 = 0$*, which is the same as the function* $f(x) = x^3 + 6x^2 - 2$ *having a root. The function* $f(x)$ *is everywhere continuous (it is a polynomial). Note that* $f(0) = -2 < 0$ *and* $f(1) = 5 > 0$*. Therefore by the Intermediate Value Theorem, there is a* $c \in (0, 1)$ *so that* $f(c) = 0$*, i.e.* $c^3 + 6c^2 - 2 = 0$ *so that* $c^3 = 2 - 6c^2$*.*

   (b) Use the Mean Value Theorem to show that the equation has a unique solution on the interval $[0, 1]$.

   If this equation had at least two distinct solutions on $[0, 1]$*, then* $f(x)$ *would have at least two distinct roots on* $[0, 1]$*, call two of them* r_1, r_2*. Now* $f(x)$ *is continuous on* $[0, 1]$ *and differentiable on* $(0, 1)$ *(note that* $f'(x) = 3x^2 + 12x$*). By the Mean Value Theorem, there is a* $c \in (0, 1)$ *so that*

$$\begin{aligned} f'(c) &= \frac{f(r_1) - f(r_2)}{r_1 - r_2} \\ 3c^2 + 12c &= \frac{0 - 0}{r_1 - r_2} \\ 3c^2 + 12c &= 0 \\ 3c(c + 4) &= 0 \end{aligned}$$

   But then $3c = 0$*, so that* $c = 0$ *or* $c + 4 = 0$ *so that* $c = -4$*. But neither of these solutions is in the interval* $(0, 1)$*. Therefore, there cannot be two distinct roots for* $f(x)$ *in* $[0, 1]$*, i.e. by using (a), the equation* $x^3 = 2 - 6x^2$ *has a unique solution on* $[0, 1]$*. Note this solution turns out to be* $c \approx 0.552475$*.*