MAT 295: Exam 3
Spring – 2021
05/04/2021
80 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 8 pages (including this cover page) and 6 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

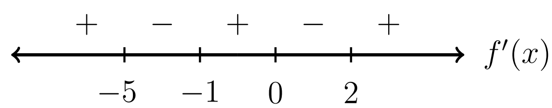
Question	Points	Score
1	20	
2	10	
3	20	
4	15	
5	20	
6	15	
Total:	100	

1. (20 points) Suppose there existed a function[1] $f(x)$ with...

$$f(x) = \frac{3x}{x+1}, \qquad f'(x) = \frac{3x(x+5)}{(x+1)(x-2)}, \qquad f''(x) = \frac{x+1}{x(x+4)}$$

(a) Find the intervals where $f(x)$ is increasing and decreasing.

We look for where $f'(x) = 0$ or is undefined. Clearly, $f'(x)$ is undefined at $x = -1$ and $x = 2$ (where the denominator is 0). Setting $f'(x) = 0$, we find $3x(x+5) = 0$, so that $x = 0$ or $x = -5$. So the critical numbers are $x = -5, -1, 0, 2$. We can then create a 'number line chart' for $f'(x)$:

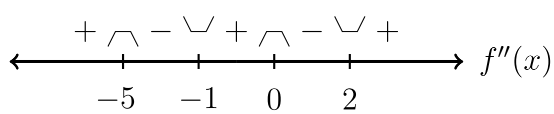


Then we have the following behavior for $f(x)$:

Increasing: $(-\infty, -5) \cup (-1, 0) \cup (2, \infty)$
Decreasing: $(-5, -1) \cup (0, 2)$

(b) Find and classify the local maxima/minima values for $f(x)$.

We can use the number line above to classify the local maxima/minima for $f(x)$. Note, the domain for $f(x)$ is all $x \in \mathbb{R}$ with $x \neq -1$.



Because $f(x)$ is not defined at $x = -1$, this cannot correspond to a maxima or minima. Now at $x = -5, 0$ we have local maxima, and at $x = 2$ we have a local minima. We plug these into $f(x)$ to find the values.

Local Maxima: $\frac{15}{4}$ *at* $x = -5$, 0 *at* $x = 0$
Local Minima: 2 *at* $x = 2$

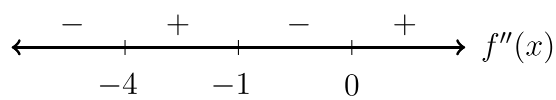
[1] There does not exist such a function, but for sake of argument, say that there does.

For convenience, we restate $f(x)$, $f'(x)$, and $f''(x)$ here:

$$f(x) = \frac{3x}{x+1}, \qquad f'(x) = \frac{3x(x+5)}{(x+1)(x-2)}, \qquad f''(x) = \frac{x+1}{x(x+4)}$$

(c) Find the intervals where $f(x)$ is concave up or concave down.

We look for where $f''(x) = 0$ or is undefined. Clearly, $f''(x)$ is undefined at $x = 0$ and $x = -4$. Setting $f''(x) = 0$, we find $x + 1 = 0$ so that $x = -1$. We then create a 'number line chart' for $f''(x)$:



Then we have the following behavior for $f(x)$:

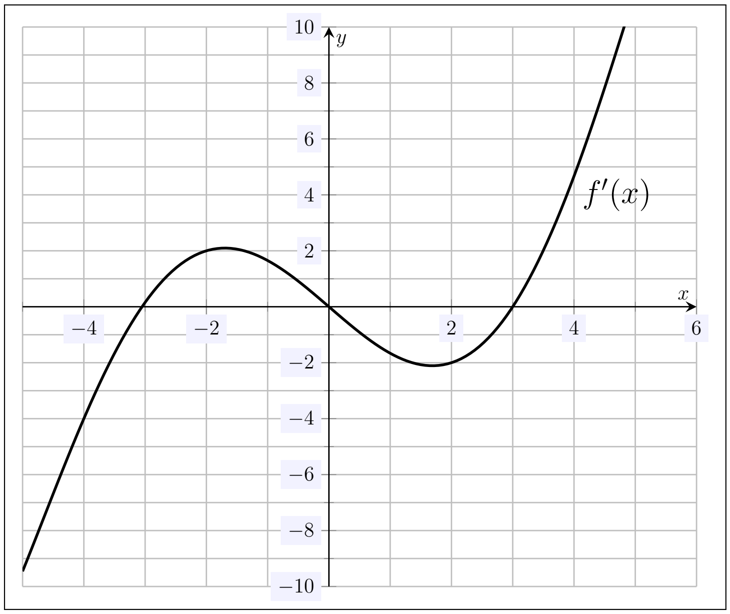
Concave Up: $(-4, -1) \cup (0, \infty)$
Concave Down: $(-\infty, -4) \cup (-1, 0)$

(d) Find any points of inflection on $f(x)$.

Because $f(x)$ is not defined at $x = -1$, we only need consider $f''(x)$ at the values $x = -4, 0$. At both of $x = -4, 0$, $f''(x)$ switches sign. So we evaluate $f(x)$ at these values to find the points of inflection:

Points of Inflection: $(-4, 4)$ *and* $(0, 0)$

2. (10 points) For some function $f(x)$, a graph of $f'(x)$ is given below.



(a) Find all the intervals on which $f(x)$ is increasing or decreasing.

We know $f(x)$ is increasing if $f'(x) > 0$ and is decreasing if $f'(x) < 0$.

Increasing: $(-3, 0) \cup (3, \infty)$
Decreasing: $(-\infty, -3) \cup (0, 3)$

(b) Find all classify all the x-values of local maxima/minima for $f(x)$.

The only critical values for $f(x)$, i.e. where $f'(x) = 0$ are $x = -3, 0, 3$. Using the information on where $f(x)$ is increasing/decreasing, $x = -3, 3$ are local minima and $x = 0$ is a local maxima.

(c) Approximate all the intervals on which $f(x)$ is concave up and down.

We know $f(x)$ is concave up if $f''(x) > 0$, in which case $f'(x)$ is increasing, and concave down if $f''(x) < 0$, in which case $f'(x)$ is decreasing.

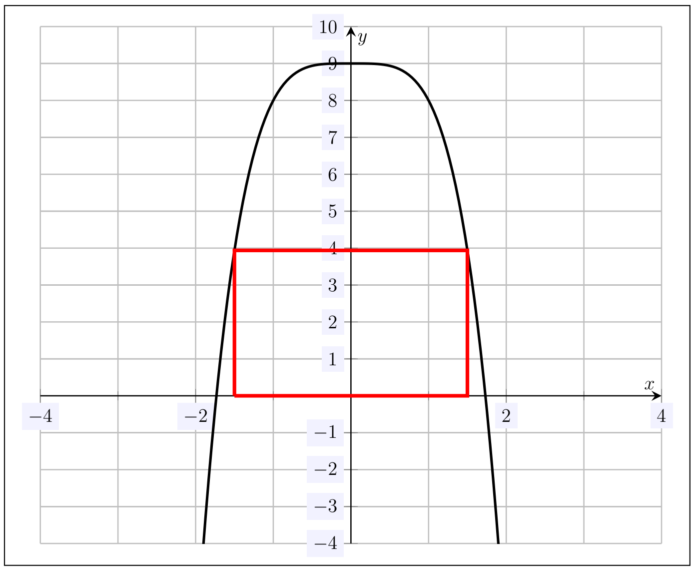
Concave Up: $(-\infty, -1.8) \cup (1.6, \infty)$
Concave Down: $(-1.8, 1.6)$

(d) Does it appear as though $f(x)$ has a point of inflection? Explain.

Yes. A point of inflection is where $f''(x)$ would change sign. But then $f'(x)$ would have a local maxima or minima. There are two such values on $f'(x)$.

3. (20 points) Let $f(x)$ be the 'parabola-like' function $f(x) = 9 - x^4$. Suppose that a rectangle is constructed so that one of its sides lies along the x-axis with two of its vertices lying on the portion of the function $f(x)$ where $f(x) \geq 0$. Of all such rectangles, what are the dimensions of the rectangle with the largest possible area? For this problem, be sure to show your dimensions give the maximal area, and state the interval on which you are optimizing.

First, we sketch the function, along with a possible rectangle.



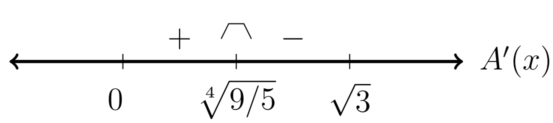
Let x be the distance from the point $(0,0)$ to the bottom right vertex of the rectangle. Then the rectangle has width $2x$ and height $f(x)$, i.e.

$$A = wh = (2x)f(x) = 2x(9 - x^4) = 2(9x - x^5)$$

Clearly, the largest value for x is when $9 - x^4 = 0$, which implies $x = 9^{1/4} = \sqrt{3}$. Then we know $x \in [0, \sqrt{3}]$. We need to maximize the function $A(x)$.

$$A'(x) = 2(9 - 5x^4)$$

Setting this to 0, we find that $9 - 5x^4 = 0$ so that $x^4 = 9/5$, which implies (noting $x \geq 0$) that $x = \sqrt[4]{9/5} < \sqrt[4]{9} = \sqrt{3}$. We show that this is a maximum:



Then $A'(x)$ is a maximum. If $x = \sqrt[4]{9/5}$, then we have $w(x) = 2\sqrt[4]{9/5}$ and $f(\sqrt[4]{9/5}) = 9 - 9/5 = 36/5$. [In addition, we have $A(\sqrt[4]{9/5}) = (72\sqrt{3})/(5\sqrt[4]{5}) \approx 16.6794$.] The dimensions are

$$2\sqrt[4]{9/5} \times 36/5 \approx 2.3166 \times 7.2$$

4. (15 points) Given that the following fact from a table of integrals:

$$\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right),$$

compute the following integral:

$$\int \frac{2^x}{2^x\sqrt{4^x+5}}\,dx$$

We rewrite the integral to fit the form of the given integral:

$$\int \frac{2^x}{2^x\sqrt{4^x+5}}\,dx = \int \frac{2^x}{2^x\sqrt{(2^2)^x+\sqrt{5}^2}}\,dx = \int \frac{2^x}{2^x\sqrt{(2^x)^2+(\sqrt{5})^2}}\,dx$$

Now we let $u = 2^x$ so that $du = 2^x \ln 2\; dx$. But then we have $dx = \dfrac{du}{2^x \ln 2}$. Using this in the integral, and using the given integral (with $a = \sqrt{5}$), we have

$$\begin{aligned}
\int \frac{2^x}{2^x\sqrt{4^x+5}}\,dx &= \int \frac{2^x}{2^x\sqrt{(2^x)^2+(\sqrt{5})^2}}\,dx \\
&= \int \frac{\cancel{2^x}}{u\sqrt{u^2+(\sqrt{5})^2}}\,\frac{du}{\cancel{2^x}\ln 2} \\
&= \frac{1}{\ln 2}\int \frac{du}{u\sqrt{u^2+(\sqrt{5})^2}}\,du \\
&= -\frac{1}{\sqrt{5}\ln 2}\ln\left(\frac{\sqrt{5}+\sqrt{u^2+\sqrt{5}^2}}{u}\right)+C \\
&= -\frac{1}{\sqrt{5}\ln 2}\ln\left(\frac{\sqrt{5}+\sqrt{(2^x)^2+5}}{2^x}\right)+C \\
&= -\frac{1}{\sqrt{5}\ln 2}\ln\left(\frac{\sqrt{5}+\sqrt{4^x+5}}{2^x}\right)+C
\end{aligned}$$

5. (20 points) Compute the following:

(a) $\displaystyle\int \left(\frac{1-x}{x}\right)^2\,dx$

$$\int \left(\frac{1-x}{x}\right)^2 dx = \int \frac{x^2-2x+1}{x^2}\,dx = \int \left(1-\frac{2}{x}+\frac{1}{x^2}\right) dx = x-2\ln|x|-\frac{1}{x}+C$$

(b) $\displaystyle\int (\sin x - 2^x + \sec^2 x)\,dx$

$$\int (\sin x - 2^x + \sec^2 x)\,dx = -\cos x - \frac{2^x}{\ln 2} + \tan x + C$$

(c) $\displaystyle\int_0^1 (4x-1)\,dx$

$$\int_0^1 (4x-1)\,dx = 2x^2 - x\,\Big|_{x=0}^{x=1} = (2-1)-(0-0) = 1$$

(d) $\displaystyle\frac{d}{dx}\int_0^{2x} e^{-x^3}\,dx$

$$\frac{d}{dx}\int_0^{2x} e^{-x^3} = e^{-(2x)^3}\cdot 2 = 2e^{-8x^3}$$

6. (15 points) Suppose that you are given the following information about a function $v(x)$:

x	0	1	2	3	4	5	6	7	8	9	10
$v(x)$	-2	8	10	0	-7	4	2	6	3	9	-3

(a) Using a right-hand sum, estimate $\int_0^{10} v(x)\,dx$ using 5 equal width rectangles.

First, we find our widths: $\Delta x = \dfrac{b-a}{5} = \dfrac{10-0}{5} = 2$. *Then*

$$\begin{aligned}\int_0^{10} v(x)\,dx &\approx \sum f(x_i)\Delta x \\ &= 2\big(f(2)+f(4)+f(6)+f(8)+f(10)\big) \\ &= 2(10-7+2+3-3) \\ &= 2(5) \\ &= 10\end{aligned}$$

(b) Suppose you were told that $\int_0^{10} v(x)\,dx = 13.6$ and $\int_1^{10} v(x)\,dx = 8.8$, find $\int_0^1 v(x)\,dx$.

$$\int_0^1 v(x)\,dx + \int_1^{10} v(x)\,dx = \int_0^{10} v(x)\,dx$$

$$\int_0^1 v(x)\,dx + 8.8 = 13.6$$

$$\int_0^1 v(x)\,dx = 4.8$$

(c) Given $p(x) = \int_0^{5x} v(2t)\,dt$, find $p'(1)$.

Let $u = 2t$. *Then we have* $du = 2\ dt$ *so that* $dt = \frac{1}{2}\,du$. *Now if* $t = 0$, *we have* $u = 2(0) = 0$. *If* $t = 5x$, *we have* $u = 2(5x) = 10x$. *Then* $p(x) = \dfrac{1}{2}\int_0^{10x} v(u)\,du$. *But then*

$$p'(x) = \frac{d}{dx}\,p(x) = \frac{d}{dx}\left(\frac{1}{2}\int_0^{10x} v(u)\,du\right) = \frac{v(10x)\cdot 10}{2} = 5\,v(10x)$$

$$p'(1) = 5\,v(10) = -15$$