§10.1: 3D Coordinate System

Topics

- Points & Coordinate Axes
- Distances
- Spheres
- 'Basic' Plots

You Should Be Able to....

- Plot points and find distances to axes/planes.
- Find distances between points.
- Find equations of spheres
- Sketch 'basic' regions.

Point in $\mathbb{R}^n$: $P = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, where $a_i \in \mathbb{R}$. Also denoted $P(a_1, \ldots, a_n)$. a_i = coordinates

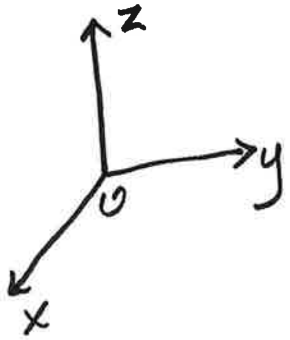
We will usually be working in $\mathbb{R}^3$:

Ex:

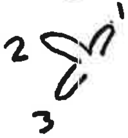
1) (x, y, z)
2) (length, width, height)
3) (height, weight, age)
4) etc.

We need a common language & graphical representation.

Coordinate Axes



- x, y, z axes
- Note positive directions
- Draw with RHR in mind:



1st: Thumb
2nd: Pointer
3rd: Middle

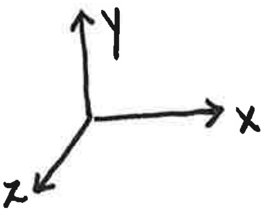
OR LHR and reverse

OR use "curl hand" method

- 8 octants
- 3 planes: xy, xz, yz - planes

Unless otherwise useful, draw these axes consistently

- Animators use



Ex: Sketch $P = (1, 2, 3)$ and find distance to....

a) xy-plane = 3

b) yz-plane = 1

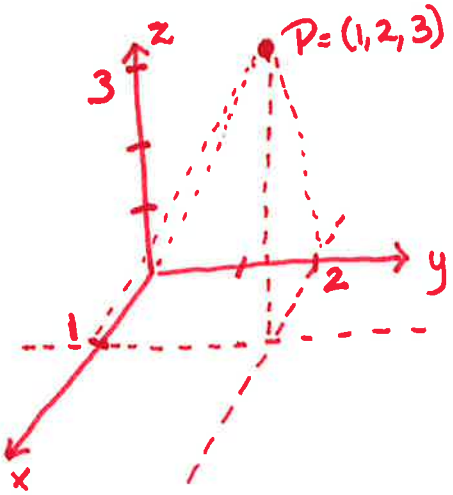
c) xz-plane = 2

d) x-axis = $\sqrt{2^2+3^2} = \sqrt{13}$

e) y-axis = $\sqrt{1^2+3^2} = \sqrt{10}$

f) z-axis = $\sqrt{1^2+2^2} = \sqrt{5}$

g) origin = $\sqrt{1^2+2^2+3^2} = \sqrt{14}$



* We find distances by projecting

* Much of this is based on constructing right Δ's.

Student Ex: Sketch $P = (-2, 1, -1)$ and find dist. to ....

a) xy-plane = 1

b) yz-plane = 2

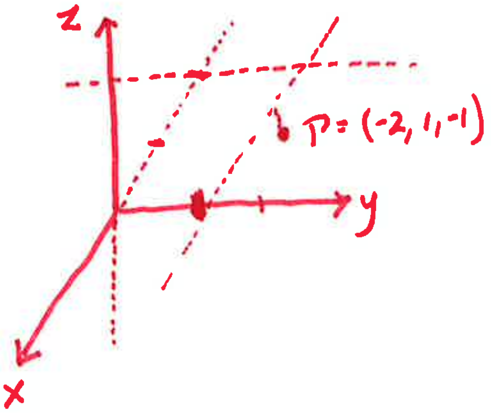
c) xz-plane = 1

d) x-axis = $\sqrt{1^2+(-1)^2} = \sqrt{2}$

e) y-axis = $\sqrt{(-2)^2+(-1)^2} = \sqrt{5}$

f) z-axis = $\sqrt{(-2)^2+1^2} = \sqrt{5}$

g) origin = $\sqrt{(-2)^2+1^2+(-1)^2} = \sqrt{6}$



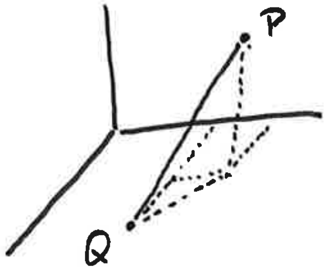
Using this right triangle approach, we find....

Distance P to Q: $P = (x, y, z)$

$Q = (a, b, c)$

$$|PQ| = \sqrt{(x-a)^2 + (y-b)^2 + (c-z)^2}$$

Why? :



Midpoint P, Q : $P = (x, y, z)$

$Q = (a, b, c)$

$$\left(\frac{x+a}{2}, \frac{y+b}{2}, \frac{z+c}{2} \right)$$

Ex: $P = (1, 0, 3)$

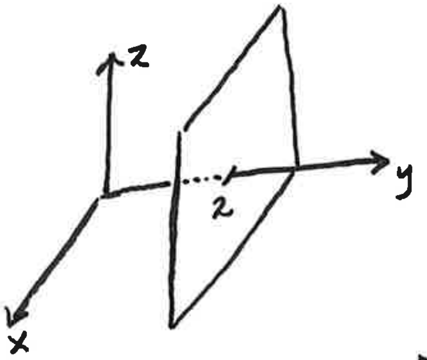
$Q = (-1, 1, 2)$

$$|PQ| = \sqrt{(1 - -1)^2 + (0-1)^2 + (3-2)^2} = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

$$Mpt = \left(\frac{1+-1}{2}, \frac{0+1}{2}, \frac{3+2}{2} \right) = (0, 1/2, 5/2)$$

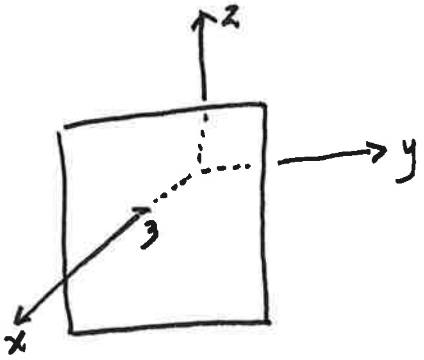
Points behave like in $\mathbb{R}^2$, but equations do not...

Ex: $y=2$



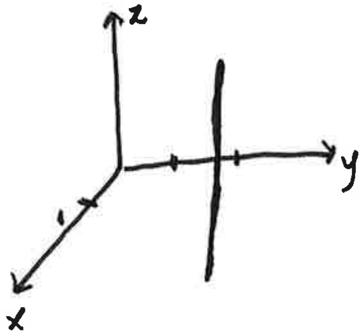
x, z 'free'

Ex: $x=3$



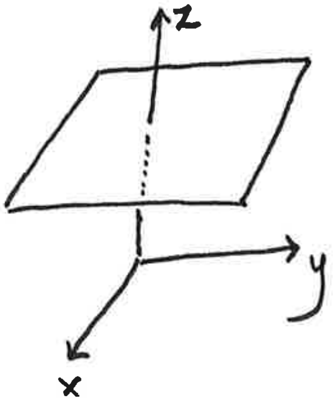
y, z 'free'

Ex: $x=1$, $y=2$



z 'free'

Ex: $z=4$



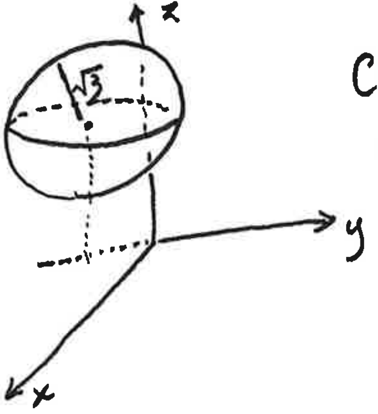
We will see more with lines | planes later.

Spheres:

Sphere with center (a, b, c) and radius r:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Ex: $x^2 + (y+1)^2 + (z-3)^2 = 3$



center: $(0, -1, 3)$

radius: $\sqrt{3}$

To find equation of sphere, need to complete the square ...

Complete the Square

1) 1/2 middle term and square
2) add / subtract
3) write as a square

* Need leading coeff. 1, otherwise factor out.

Ex: $x^2 - 6x - 3$

$$x^2 - \underset{7}{6}x \qquad - 3$$

$$x^2 - 6x + 9 \qquad - 9 - 3$$

$$(x - 3)^2 - 12$$

Ex: $7x^2 - 8x + 3$

$$7\left(x^2 - 8/7\, x + 3/7\right)$$

$$7\left(x^2 - 8/7\, x \qquad + 3/7\right)$$

$$7\left(x^2 - 8/7\, x + 16/49 \qquad - 16/49 + 3/7\right)$$

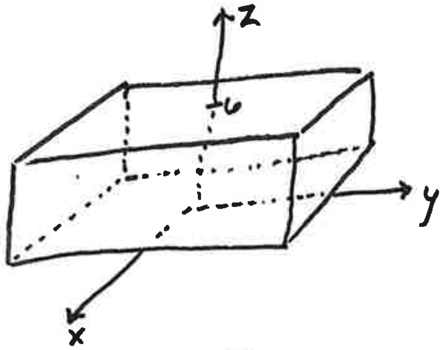
$$7\left((x - 4/7)^2 + 5/49\right)$$

$$7(x - 4/7)^2 + 5/7$$

There also other things one can plot....

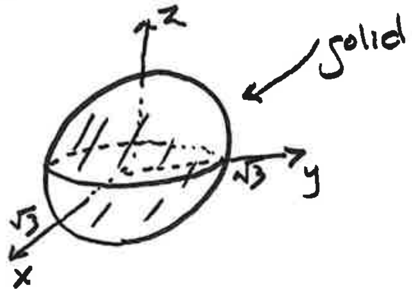
Ex:

$0 \le z \le 6$

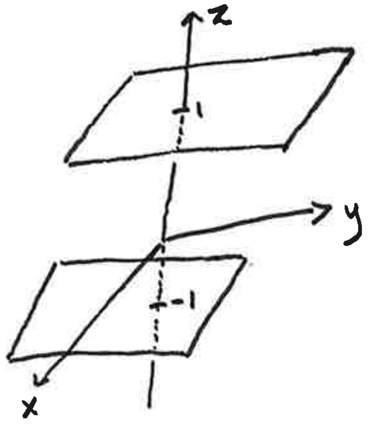


* Goes on in each direction.

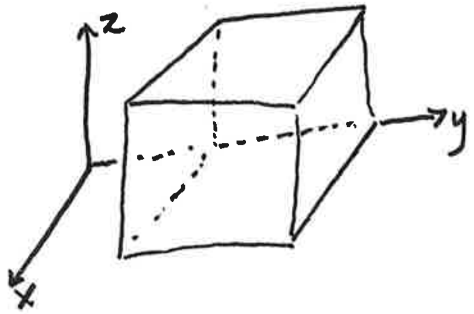
$x^2 + y^2 + z^2 \le 3$



$z^2 = 1$



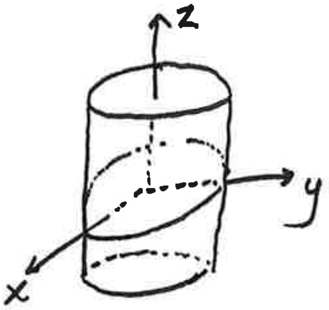
$y > 1$



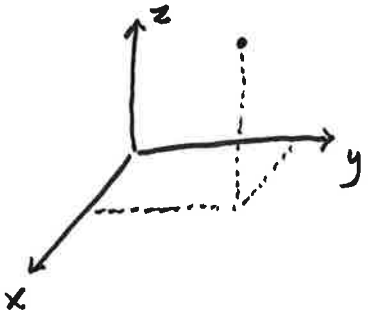
* Goes on in each direction

Ex (cont'd):

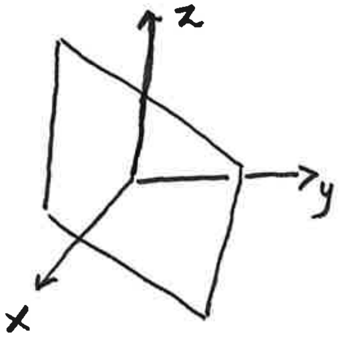
$x^2 + y^2 = 1$



$x = 1,\ y = 2,\ z = 4$



$x = y$



* Goes on in 'each' direction

§10.2 : Vectors

Topics

- Vectors
- Vector Operations
- Lengths
- Vector Applications

You Should Be Able to....

- Sketch vectors.
- Represent vector & vector operations graphically.
- Peform vector operations algebraically.
- Check if vectors are //.
- Create displacement vectors.
- Find vector lengths.
- Create unit vectors.
- Break vectors into components.

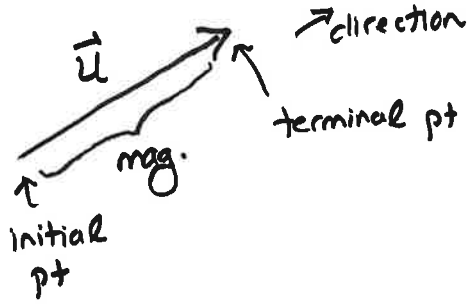
Vector: n-tuple $\vec{v} = \langle \underbrace{v_1, v_2, \dots, v_n}_{\text{components}} \rangle$ with...

1) Magnitude | Length | Norm
2) Direction ← Except zero vector, $\vec{0}$

We need to understand these graphically & algeb.

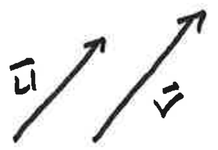
Graphical

Represented by arrow | directed segment

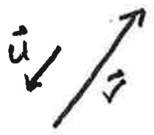


- Parallel: 2 vectors are // if they point in the same (or opposite) direction
- Equal: 2 vectors are =, or equiv., if they point in same direction & have same length.

Ex:



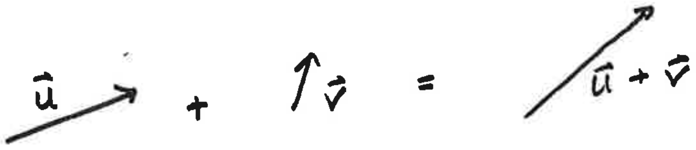
$\vec{u} \parallel \vec{v}$



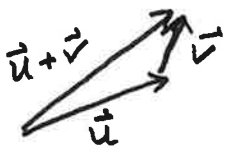
$\vec{u} \parallel \vec{v}$

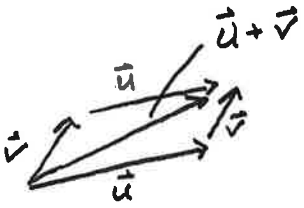
$\vec{u} = \vec{v}$

We can add vectors...



We add using △-Law or ▱-Law....

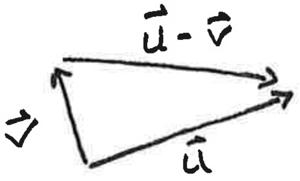




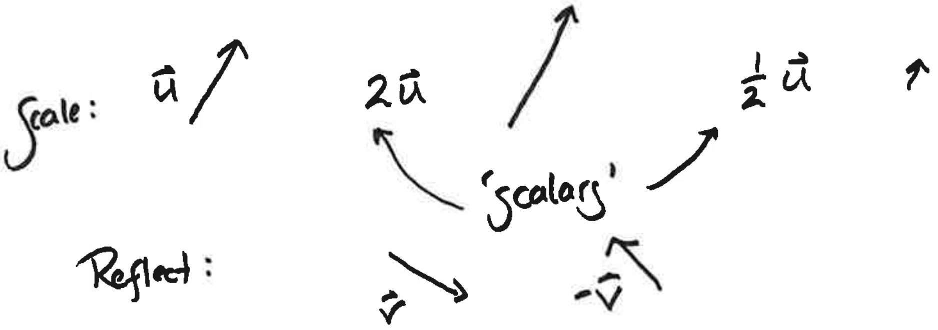
This is also called the tail-to-tip method.

We can subtract vectors....

'From $\vec{v}$ to $\vec{u}$'



We can also scale | reflect vectors....

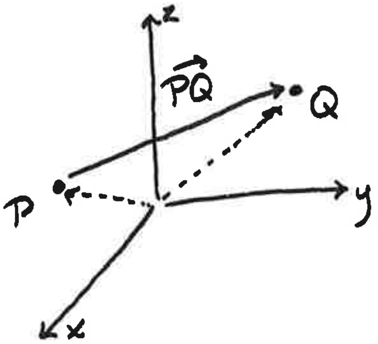


* This is where subtraction comes from:

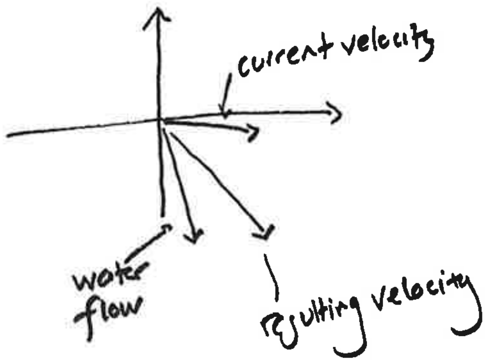
$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

A common type of vector is a displacement vector:

$\vec{PQ}$



* Vectors work in any dimension
* Often represent rate of change, eg velocity



Student Ex :

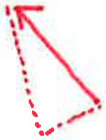
$2\vec{u}$ =

$-\frac{1}{2}\vec{v}$ =

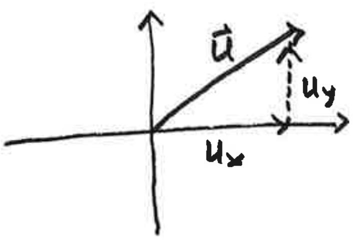
$\vec{u}+\vec{v}$ =

$\vec{u}-\vec{v}$ =

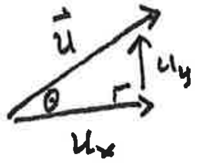
$2\vec{u} - \frac{1}{2}\vec{v}$ =



We can also break a vector into its components



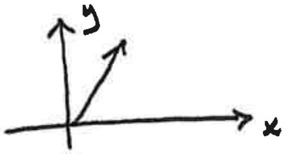
$$\vec{u} = \underbrace{\vec{u}_x}_{\text{Horz}} + \underbrace{\vec{u}_y}_{\text{Vert.}}$$



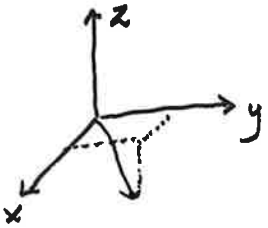
Algebraically

Vector: $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$

Ex: $\vec{v} = \langle 1, 2 \rangle$



$\vec{u} = \langle 1, 2, -3 \rangle$



Parallel: Vectors $\vec{u}$ & $\vec{v}$ are parallel if there is a scalar c so $\vec{u} = c\vec{v}$

Ex: $\vec{u} = \langle 1, 2 \rangle$

$\vec{v} = \langle 2, 4 \rangle$

$\vec{u} \parallel \vec{v}$: $\langle 2, 4 \rangle = 2\langle 1, 2 \rangle = \langle 2, 4 \rangle$

OR

$\langle 1, 2 \rangle = \frac{1}{2}\langle 2, 4 \rangle = \langle 1, 2 \rangle$

$\vec{x} = \langle 3, 1 \rangle$

$\vec{y} = \langle 2, 1 \rangle$

$\vec{x} \nparallel \vec{y}$: $\langle 3, 1 \rangle = c\langle 2, 1 \rangle$

$2c = 3$

$c = 3/2$

But $c \cdot 1 = 3/2 \neq 1$

Equal | Equivalent: Vectors $\vec{u}$ & $\vec{v}$ are equal if all their components are equal.

Algebraic operations work as you'd imagine.

Student Ex: $\vec{u} = \langle 1, -2 \rangle$ $\quad \vec{a} = \langle 1, -2, 3 \rangle$

$\vec{v} = \langle 0, 3 \rangle$ $\quad \vec{b} = \langle 0, 1, -1 \rangle$

$-\vec{u} = \langle -1, 2 \rangle$

$2\vec{v} = \langle 0, 6 \rangle$

$\vec{a} + \vec{b} = \langle 1, -1, 2 \rangle$

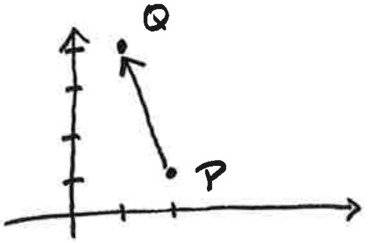
$2(\vec{a} + 3\vec{b}) = 2\langle 1, 1, 0 \rangle = \langle 2, 2, 0 \rangle$

Displacement Vector: $\vec{PQ}$ = Vector pointing from P to Q.

$\vec{PQ} = \langle Q_i - P_i \rangle_i$

Ex: $P = (2, 1)$

$Q = (1, 4)$



$\vec{PQ} = \langle 1 - 2, 4 - 1 \rangle = \langle -1, 3 \rangle$

There are special direction vectors....

$\hat{\imath} = \langle 1,0 \rangle$ or $\langle 1,0,0 \rangle$

$\hat{\jmath} = \langle 0,1 \rangle$ or $\langle 0,1,0 \rangle$

$\hat{k} = \langle 0,0,1 \rangle$

* Sometimes also denoted $\hat{x}, \hat{y}, \hat{z}$

* $\hat{\imath}, \hat{\jmath}, \hat{k}$ come from a special # system called the quaternions, $\mathbb{H}$, invented by Hamilton. These are used heavily in animation because they 'represent' rotations.

This gives us a way of rep. vectors differently.

<u>Ex</u>: $u = \langle 2,1 \rangle = \langle 2,0 \rangle + \langle 0,1 \rangle$

$= 2\langle 1,0 \rangle + 1\langle 0,1 \rangle$

$= 2\hat{\imath} + \hat{\jmath}$

$\vec{v} = \langle -1,4,3 \rangle = -\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$

$\vec{w} = \langle 3,0,2 \rangle = 3\hat{\imath} + 2\hat{k}$

Length | Norm | Mag: $\vec{v} = \langle v_1, \dots, v_n \rangle$

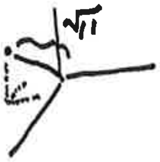
$$|\vec{v}| = \|\vec{v}\| = \sqrt{\sum v_i^2}$$

Ex: $\vec{u} = \langle 1, 2 \rangle$

$$|\vec{u}| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Ex: $\vec{v} = \langle 1, -1, 3 \rangle$

$$\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$$



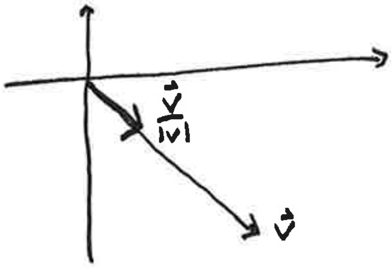
This allows us to do a special thing:

Direction Vectors (unit length): If $\vec{v}$ nonzero vector, then $\frac{\vec{v}}{|\vec{v}|}$ is a unit vector in direction of $\vec{v}$.

Ex: $\vec{v} = \langle 3, -4 \rangle$

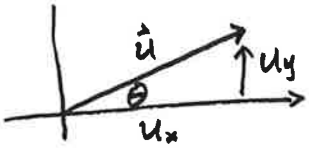
$$|\vec{v}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

$$\frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3, -4 \rangle}{5} = \langle 3/5, -4/5 \rangle$$



Finally, we can decompose a vector into its components. We shall only do this in $\mathbb{R}^2$, but it works in any dimension.

Ex: $u = \langle 3, 2 \rangle$



$|\vec{u}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

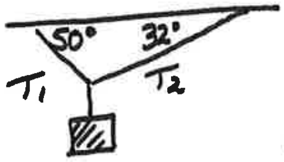
$|u_x|$: $\cos\theta = \frac{|u_x|}{|\vec{u}|} = \frac{3}{\sqrt{13}} \Rightarrow |u_x| = 3 = \sqrt{13}\cos\theta$

$|u_y|$: $\sin\theta = \frac{|u_y|}{|\vec{u}|} = \frac{2}{\sqrt{13}} \Rightarrow |u_y| = 2 = \sqrt{13}\sin\theta$

$$\vec{u} = \sqrt{13}\cos\theta \, \hat{i} + \sqrt{13}\sin\theta \, \hat{j}$$

This is very useful in Physics :

Ex: 100 lb weight hangs from 2 wires. Find T_1 & T_2.



$\vec{T}_1 = -|T_1|\cos 50° \, \hat{i} + |T_1|\sin 50° \, \hat{j}$

$\vec{T}_2 = |T_2|\cos 32° \, \hat{i} + |T_2|\sin 32° \, \hat{j}$

$\vec{T}_1 + \vec{T}_2 = -\vec{w} = 100\hat{j}$

$$\begin{cases} -|T_1|\cos 50° + |T_2|\cos 32 = 0 \\ |T_1|\sin 50° + |T_2|\sin 32 = 100 \end{cases}$$

$|T_1| \approx 85.64 \text{ lb}$
$|T_2| = 64.91 \text{ lb}$
$\rightarrow$
$\vec{T}_1 = -55.05\,\hat{i} + 65.60\,\hat{j}$
$\vec{T}_2 = 55.05\,\hat{i} + 34.40\,\hat{j}$

Student Ex: $\vec{a} = \hat{\imath} - \hat{\jmath}$ $\quad \vec{u} = \langle 1, 2, 3\rangle$

$\vec{b} = -2\hat{\imath} + 3\hat{\jmath}$ $\quad \vec{v} = \langle 0, 1, 3\rangle$

Rewrite $\vec{a}$ = $\langle 1, -1\rangle$

Is $\vec{a} \parallel \vec{b}$? : No

$$\langle 1, -1\rangle$$

$\downarrow \times -2 \quad \downarrow -2 \swarrow$ No

$$\langle -2, 3\rangle$$

Find unit vector $\parallel$ to $\vec{a}$: $|a| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

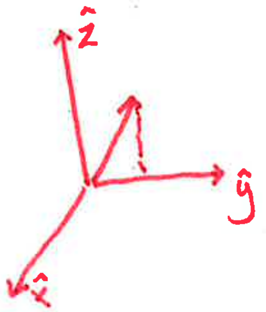
$$\frac{\vec{a}}{|a|} = \frac{\hat{\imath} - \hat{\jmath}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}$$

$\vec{u} - \vec{v}$: $\langle 1, 1, 0\rangle$

$\|\vec{u} - \vec{v}\|$: $\sqrt{2} = \sqrt{1^2 + 1^2 + 0^2}$

Rewrite $\vec{u}$: $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Sketch $\vec{v}$:



§10.3-10.4: Dot Product, Cross Product, Projections

Topics

- Dot Product
- Vector ∡'s
- Cross Products
- Determinants
- Projections

You should Be Able to...

- Compute the dot product.
- Use dot products to find vector lengths.
- Test if vectors are ⊥.
- Find the ∡ between two vectors.
- Compute determinants.
- Compute the cross product.
- Create vectors ⊥ to 2 given vectors.
- Find volumes of parallelepipeds.
- Determine if 3 vectors are coplanar.
- Sketch and compute projections.

We have seen how to multiply a vector by a scalar. There are two more types of vector multiplication.

Dot Product: Given two vectors, $\vec{u}, \vec{v} \in \mathbb{R}^n$, their dot product, denoted $\vec{u} \cdot \vec{v}$, is

$$\vec{u} \cdot \vec{v} = \sum_i u_i v_i$$

Also called scalar / inner product

Ex: $\vec{u} = \langle 1, -2 \rangle$ $\quad$ $\vec{u} \cdot \vec{v} = 1(3) + -2(1)$

$\vec{v} = \langle 3, 1 \rangle$ $\quad$ $= 3 - 2$

$= 1$

Student Ex: $\vec{u} = \langle 1, 0, 2 \rangle$ $\quad$ $\vec{u} \cdot \vec{v} = 1(-1) + 0(2) + 2(0)$

$\vec{v} \cdot \langle -1, 2, 0 \rangle$ $\quad$ $= -1 + 0 + 0$

$= -1$

Recall: $\vec{u} = \langle x, y \rangle \Rightarrow |u| = \sqrt{x^2 + y^2}$

But $\vec{u} \cdot \vec{u} = x^2 + y^2$

We then generally have ...

$$\boxed{\vec{u} \cdot \vec{u} = |\vec{u}|^2}$$

Properties of the Dot Product:

1) $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

2) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

3) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

4) $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$

* Δ-ineq:
$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

* Cauchy-Schwarz
$|a \cdot b| \leq |a||b|$

Note: Dot Product of two vectors always gives a scalar.

In $\mathbb{R}^2$ or $\mathbb{R}^3$, one can prove (which we then just define to be true generally)

$$\boxed{\vec{u} \cdot \vec{v} = |u|\,|v| \cos \theta}$$

So we can find/define the angle, θ, between two vectors

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|u|\,|v|}\right)$$

But $|u|, |v| > 0$ so we then must have....

$$\vec{u} \cdot \vec{v} = 0 \text{ if and only if } \vec{u} \perp \vec{v}$$

Thus, the dot product gives a way of testing if two vectors are $\perp$.

Student Ex: $\vec{u} = \langle 1, 2 \rangle$ $\quad \vec{a} = \hat{i} + \hat{j}$

$\vec{v} = \langle -3, 4 \rangle$ $\quad \vec{b} = \hat{i} - \hat{j} + \hat{k}$

$\vec{u} \cdot \vec{v} = 1(-3) + 2(4) = -3 + 8 = 5$

$u \cdot u = 1^2 + 2^2 = 1 + 4 = 5$

$|u| = \sqrt{u \cdot u} = \sqrt{5}$

Find the $\angle$ between $\vec{u}$ & $\vec{v}$

$\vec{u} \cdot \vec{v} = 5$

$|u| = \sqrt{5}$

$|v| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$

$\vec{u} \cdot \vec{v} = |u| \, |\vec{v}| \cos\theta$

$5 = \sqrt{5} \cdot 5 \cdot \cos\theta$

$\cos\theta = 1/\sqrt{5}$

$\theta = \cos^{-1}(1/\sqrt{5}) \approx 63.43°$

Is $\vec{a} \perp$ to $\vec{b}$? Explain.

$\vec{a} \cdot \vec{b} = 1(1) + 1(-1) + 0(1) = 1 - 1 + 0 = 0$

Yes, $\vec{a} \cdot \vec{b} = 0$

Find a different vector $\perp$ to $\vec{b}$.

$\langle 1, -1, 1 \rangle \cdot \langle x, y, z \rangle = 0$

$x - y + z = 0$

$\langle 1, 1, 0 \rangle \quad \langle 1, 0, -1 \rangle$

$\langle 5, 4, -1 \rangle \quad \langle 5, 10, 5 \rangle$ etc.

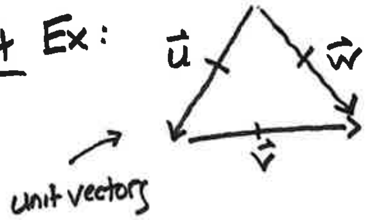
* Note: This will "look like" a plane later.... it is. Why?

Student Ex: $|\vec{u}| = 3$, $|\vec{v}| = \sqrt{6}$, $\theta = \frac{\pi}{4}$

what is $\vec{u} \cdot \vec{v}$?

$$\vec{u} \cdot \vec{v} = |u||v|\cos\theta = 3 \cdot \sqrt{6} \cdot \frac{1}{\sqrt{2}} = 3\sqrt{3}$$

Student Ex:



Find $\vec{u} \cdot \vec{w}$ & $\vec{u} \cdot \vec{v}$.

All angles are 60°.

$\cos 60° = 1/2$

$\vec{u} \cdot \vec{w} = 1/2$; $\vec{u} \cdot \vec{v} = -1/2$

Student Ex: Find acute ∡ between $2x - y = 3$ & $3x + y = 7$

$y = 2x - 3$

$y = -3x + 7$

$\big\Downarrow$

$\vec{u} = \langle 1, 2 \rangle$

$\vec{v} = \langle 1, -3 \rangle$

$|\vec{u}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|\vec{v}| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$\vec{u} \cdot \vec{v} = 1(1) + 2(-3) = -5$

$$\theta = \cos^{-1}\left(\frac{-5}{\sqrt{5} \cdot \sqrt{10}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

Student Ex: Find acute ∡ between curves at their intersection

$y = x^2 - 1$, $y = 1 - x^2$

Intersection $(1, 0)$

$$\begin{cases} y' = 2x \rightsquigarrow 2 \\ y' = -2x \rightsquigarrow -2 \end{cases}$$

$$\begin{cases} \langle 1, 2 \rangle \ \vec{u} \\ \langle 1, -2 \rangle \ \vec{v} \end{cases}$$

$|\vec{u}| = \sqrt{5}$

$|\vec{v}| = \sqrt{5}$

$\vec{u} \cdot \vec{v} = 1(1) + 2(-2) = -3$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{5}\sqrt{5}}\right) = \cos^{-1}\left(\frac{-3}{5}\right) = 53.13°$$

Cross Product: The cross product of two vectors $\vec{u},\vec{v}\in\mathbb{R}^3$, denoted $\vec{u}\times\vec{v}$, is a vector ⊥ to both $\vec{u}$ and $\vec{v}$. Given by determinants.

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex: $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 1(3) - 2(-1) = 3+2=5$

Generally given by cofactor expansion, with signs....

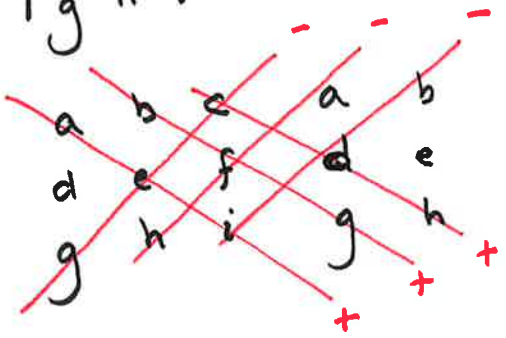
$$\begin{vmatrix} + & - & + & - & \cdots \\ - & & & & \\ + & & \ddots & & \\ - & & & & \\ \vdots & & & & \end{vmatrix}$$

Ex:
$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$
$$= 1(0-4) - 2(6+5) + (-1)(12-0)$$
$$= -4 - 22 - 12$$
$$= -38$$

Any other way works. Easiest if you choose row/col with the most 0's.

Shortcut for 3×3 determinants

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$



aei + bfg + cdh - gec - hfa - idb

Ex:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\begin{array}{ccccc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 1 & 2 & 3 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 \end{array}$$

$$-2i + 0 + \hat{k} - (0 + 3\hat{i} - \hat{j})$$

$$-2\hat{i} + \hat{k} - 3\hat{i} + \hat{j}$$

$$-5\hat{i} + \hat{j} + \hat{k}$$

$$\langle -5, 1, 1 \rangle$$

Ex: $\vec{u} = \langle 1, 3, 4 \rangle$
$\vec{v} = \langle 2, 7, -5 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = (-15-28)\hat{i} - (-5-8)\hat{j} + (7-6)\hat{k}$$
$$= -43\hat{i} + 13\hat{j} + \hat{k}$$
$$= \langle -43, 13, 1 \rangle$$

* For 2D vectors, simply place 0's in their $\hat{k}$ component.

Student Ex: $\vec{u} = \langle 1, 1, -1 \rangle$
$\vec{v} = \langle 1, 0, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = (1-0)\hat{i} - (1--1)\hat{j} + (0-1)\hat{k}$$
$$= \hat{i} - 2\hat{j} - \hat{k}$$
$$= \langle 1, -2, -1 \rangle$$

Again, we can show....

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

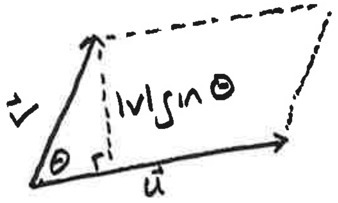
This allows us, again, to compute vector ∡'s. This also shows....

$$\vec{u} \,//\, \vec{v} \text{ if and only if } \vec{u} \times \vec{v} = \vec{0}$$

* Both methods are inefficient ways of doing those.

* Dot products produce scalars
Cross products produce vectors

* $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{w})$



A = bh

Important Fact:

$|\vec{u} \times \vec{v}|$ = Area of ▱ spanned by $\vec{u}$ & $\vec{v}$.

Student Ex: Find vector ⊥ to plane cont. pts
P(1,0,0), Q(0,1,0), R(0,0,1)

(1) $\vec{PQ} = \langle 0,1,0\rangle - \langle 1,0,0\rangle = \langle -1,1,0\rangle$
$\vec{PR} = \langle 0,0,1\rangle - \langle 1,0,0\rangle = \langle -1,0,1\rangle$

(2) $\vec{v} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1,1,1\rangle$

Student Ex: Find area △ with vertices P(1,0,0),
Q(0,1,0), R(0,0,1)

(1) + (2) from above.

$|\vec{v}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

Area $= \frac{1}{2}\cdot\sqrt{3} = \sqrt{3}/2$

Cross products naturally come up in Physics:

Torque: $\vec{\tau} = \vec{r} \times \vec{F}$

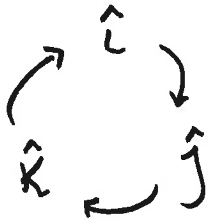
Rigid Body Rotation: $\vec{v} = \vec{\omega} \times \vec{r}$

We can determine the direction of a cross product without components using the RHR.

$\vec{u} \times \vec{v}$

* If awk., use LHR and 'reverse'.

There is a special diagram for the standard unit vectors



There are also special notations based on arrows for into/out of page.

⊙ out of page ⊗ into the page

There is a special relationship between the dot & cross product.

Scalar Triple Product: The volume of the parallelepiped spanned by vectors $\vec{u}, \vec{v}, \vec{w}$ is given by....

$$\boxed{V = |\vec{u} \cdot (\vec{v} \times \vec{w})|}$$

* The arrangements of the vectors does not matter. Why?

* This gives a test if 3 vectors are coplanar. Why?

3 vectors $\vec{u}, \vec{v}, \vec{w}$ are coplanar if and only if $|\vec{u} \cdot (\vec{v} \times \vec{w})| = 0$.

Student Ex: Compute volume of par. spanned by

$\vec{u} = \langle 1,0,1 \rangle$

$\vec{v} = \langle 2,1,0 \rangle$

$\vec{w} = \langle 1,1,1 \rangle$

Are the vectors coplanar?

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \langle -1,2,1 \rangle$$

$$\vec{w} \cdot (\vec{u} \times \vec{v}) = 1(-1) + 1(2) + 1(1)$$
$$= -1 + 2 + 1$$
$$= 2$$

$|2| = 2 \Rightarrow V = 2$

$2 \neq 0$ so vectors are not coplanar.

Summary

- Dot Product gives scalar
- $\vec{u} \perp \vec{v} \rightleftharpoons \vec{u} \cdot \vec{v} = 0$
- Cross product gives a vector.
- Cross product gives vector ⊥ to org. two.
- $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$
- $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta$
- $|\vec{u} \times \vec{v}| =$ Area of ▱ spanned by $\vec{u}, \vec{v}$.
- $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$ volume ⬚ spanned by $\vec{u}, \vec{v}, \vec{w}$
- Scalar triple product tests if 3 vectors are coplanar.

Student Ex: Which of the following 'make sense'.

$\vec{a} \cdot (\vec{b} \times \vec{c})$ ✓

$\vec{a} \times (\vec{b} \times \vec{c})$ ✓

$(\vec{a} \cdot \vec{b}) \times (\vec{c} \cdot \vec{d})$ ✗

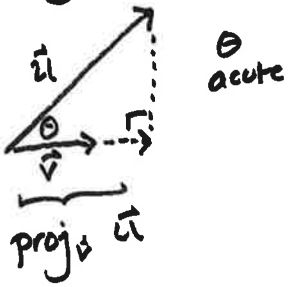
$\vec{a} \times (\vec{b} \cdot \vec{c})$ ✗

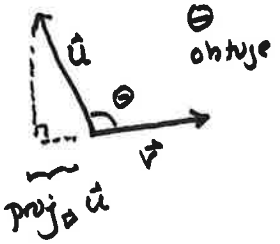
$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ ✓

Projections

'How much' of one vector lies in the direction of another vector.

$\text{proj}_{\vec v}\,\vec u$ = proj. of $\vec u$ onto $\vec v$.





*'Drops' ⊥ from one vector to another.

Using a bit of geometry, we find....

$$\boxed{\text{proj}_{\vec v}\,\vec u = \underbrace{\left(\frac{\vec u\cdot\vec v}{|\vec v|}\right)}_{\text{mag.}}\underbrace{\frac{\vec v}{|\vec v|}}_{\text{direction of }\vec v} = \frac{\vec u\cdot\vec v}{\underset{\overset{\uparrow}{\vec v\cdot\vec v}}{|\vec v|^2}}\,\vec v}$$

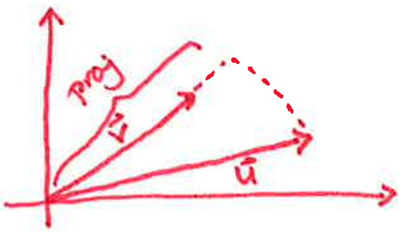
Student Ex: $\vec u = \langle 3, 1\rangle$
$\vec v = \langle 2, 1\rangle$

Find $\text{proj}_{\vec v}\,\vec u$ and sketch.

$\text{proj}_{\vec v}\,\vec u = \frac{7}{5}\langle 2, 1\rangle$

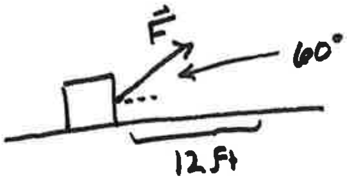
$\vec u\cdot\vec v = 3(2) + 1(1) = 7$

$\vec v\cdot\vec v = 2(2) + 1(1) = 5$

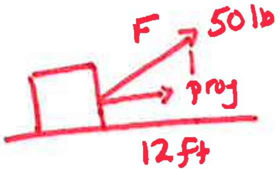


Projections will be useful in finding distances between objects but these also show up in Physics.

Ex: Pull box with constant force of 50 lb at an angle.



What is work done moving box 12 ft?



$W = F \cdot d$

$= |F|\,|d| \cos\theta$

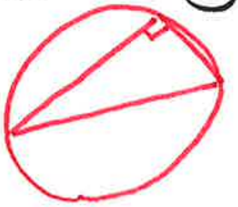
$= 50 \cdot 12 \cdot \cos 60°$

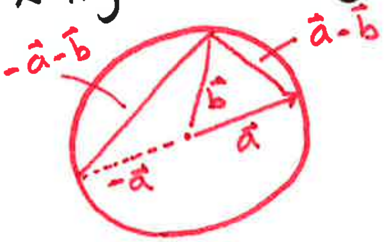
$= 50 \cdot 12 \cdot 1/2$

$= 300$ ft · lb

Vectors & their products are also useful in proofs:

Ex: Every ∡ inscribed in a semicircle is a right ∡.





$(\vec{a}-\vec{b}) \cdot (-\vec{a}-\vec{b})$

$-\vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$-|\vec{a}|^2 + |\vec{b}|^2$

$-r^2 + r^2$

0

§ 10.5: Lines and Planes

Topics

- Lines
- Planes

You Should Be Able to....

- Find the equation of a line.
- Move between different forms of a line.
- Sketch a line.
- Determine if lines are parallel, skew, intersecting, perpendicular, or the same.
- 'Compute' with lines
- Find the equation of a plane.
- Sketch a plane.
- 'Compute' with a plane.

Lines

In 2D, (non-vertical) lines have the form

$$y = \underbrace{m}_{\text{slope}}x + \underbrace{b}_{\text{y-int.}}$$

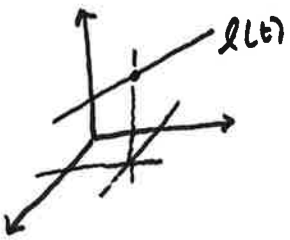
In higher dimensions, lines are...

$$\ell(t) = \underbrace{\vec{m}}_{\substack{\text{slope}\\ \text{vector}}}t + \underbrace{\vec{b}}_{\text{point}}$$

Think of the line as being drawn in time "forward" & "backward."
starting at the point b –

Ex: $b = (1,2,3)$ $\qquad \ell(t) = \langle 4,5,6\rangle t + (1,2,3)$

$m = \langle 4,5,6\rangle$



This is also denoted $r(t) = \underbrace{r_0}_{\substack{\text{initial}\\ \text{position}}} + t\underbrace{v}_{\text{'velocity'}}$

Write $r_0 = b = (x_0, y_0, z_0)$ and $v = m = \langle a,b,c\rangle$

Then...

$$\ell(t) = \langle a, b, c \rangle t + (x_0, y_0, z_0)$$
$$= \langle at, bt, ct \rangle + (x_0, y_0, z_0)$$
$$= (\underbrace{at + x_0}_{x(t)}, \underbrace{bt + y_0}_{y(t)}, \underbrace{ct + z_0}_{z(t)})$$

So we have...

$$\begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

But we can also solve for t in each...

$$(t =) \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

We call these...

Vector Form: $\ell(t) = \vec{m}\, t + \vec{b}$

Parametric Form: $\begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$

Symmetric Form: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

Student Ex: Find all forms of the line

$$\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-4}{5}$$

and find two points on the line.

$t = \frac{x-1}{2}$ $\quad$ $t = \frac{y+2}{-3}$ $\quad$ $t = \frac{z-4}{5}$

$\Downarrow$ $\quad$ $\Downarrow$ $\quad$ $\Downarrow$

$x = 2t+1$ $\quad$ $y = -3t-2$ $\quad$ $z = 5t+4$

$$\begin{cases} x = 2t+1 \\ y = -3t-2 \\ z = 5t+4 \end{cases} \rightsquigarrow \ell(t) = (2t+1, -3t-2, 5t+4)$$

$$\ell(t) = \langle 2,-3,5 \rangle t + (1,-3\ 4)$$

$t=0 \Rightarrow (1,-2,4)$

$t=1 \Rightarrow (3,-5,9)$

* Vector | Parametric form easiest to find points on the line.

* In symmetric form, if 'vertical' in a direction, ie constant, equation has 'missing' piece:

$$\frac{x-1}{1} = \frac{z+2}{6} \; ; \; y=3$$

There are two significant differences from 2D to multi-dim. case:

Lines can intersect, be parallel, or be skew.

Skew: No intersection, not parallel.

Ex: $L(t) = \langle 1,2,3\rangle t + (0,-1,3)$

$M(t) = \langle -1,2,5\rangle t + (0,-1,3)$

L & M intersect at $(0,-1,3)$.

$R(t) = \langle 1,2,3\rangle t + (4,5,6)$

$S(t) = \langle 1,2,3\rangle t + (4,5,7)$

R & S are parallel.

$A(t) = \langle 1,2,3\rangle t + (0,0,0)$

$B(t) = \langle 1,2,-3\rangle t + (1,0,0)$

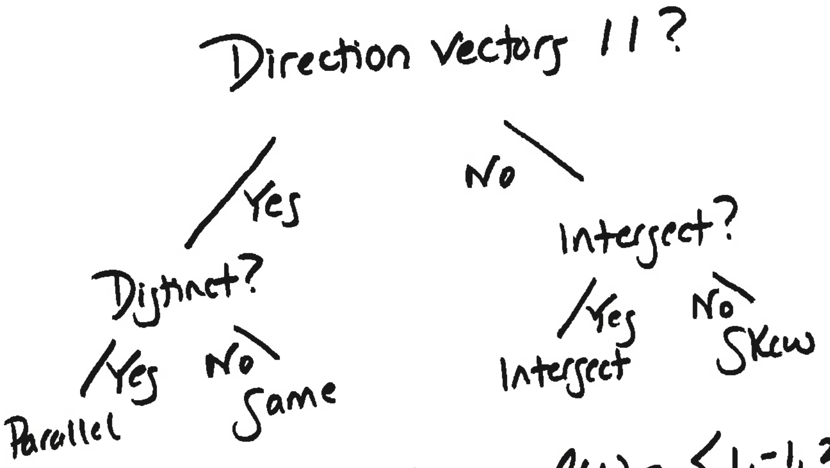
A & B are skew.

* Lines (if distinct) are parallel if and only if their direction vectors are parallel. $\perp$ if and only if direction vectors $\perp$.

This leads to the next issue:

* Lines are not 'unique'. It is non-trivial to tell if two given lines are the same.

Note: To be the same, must have // direction vectors.



<u>Student</u> Ex: Are the lines $\ell(t) = \langle 1,-1,2\rangle t + (0,2,-1)$ and $m(t) = \langle 2,-2,4\rangle t + (3,-1,5)$ parallel, skew, intersecting, or distinct?

$\langle 2,-2,4\rangle = 2\langle 1,-1,2\rangle$ so cannot be skew or intersecting. So parallel or the same.

$\langle 1,-1,2\rangle t + (0,2,-1) = (3,-1,5)$

$(t,-t,2t) = (3,-3,6)$

$t=3$ works

So the lines are the same.

* If no single t worked in prev. example, then lines are parallel.

* If in prev. example, lines were not 'parallel', then set up similar equations to see if single t works, if so they intersect. If not, skew.

Perpendicular: Lines ⊥ if their direction vectors are ⊥ <u>and</u> they intersect. So just an extra step to the above procedures.

One special form of the line we will use much later in the course:

Line through two points $\vec{x}_0$, $\vec{y}_0$:

$$\ell(t) = (1-t)x_0 + t\,y_0$$

We often restrict to line segment connecting them: $0 \le t \le 1$. [Hint: Consider cases of $t=0$ and $t=1$.]

Student Ex: Where does line

$$\ell(t) = \langle 1,2,3\rangle t + (0,1,-1)$$

intersect... $= (t, 2t+1, 3t-1)$

xy-plane: $z=0 \to 3t-1=0$

$t = 1/3$

$(1/3, 5/3, 0)$

yz-plane:

$x=0 \to t=0 \quad (0,1,-1)$

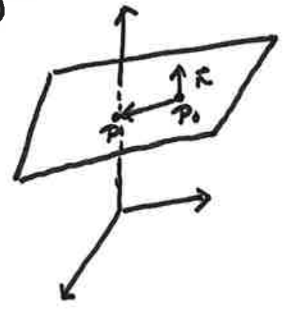
xz-plane:

$y=0 \to 2t+1=0 \quad \to \quad (-1/2, 0, -5/2)$

$t=-1/2$

Planes

Planes are determined by a normal vector (and a point). See previous example from dot product section.



$$\underbrace{\vec{n}}_{\text{normal}} \cdot (\underbrace{\vec{r} - P_0}_{\text{vector to point}}) = 0$$

Writing: $\vec{n} = \langle a, b, c \rangle$

$\vec{r} = \langle x, y, z \rangle$

$P_0 = (x_0, y_0, z_0)$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

Which we clean up as....

$$\boxed{ax + by + cz = d}$$

Not only way of expressing the plane, π (can scale $\vec{n}$). But in some ways 'more unique' than with lines.

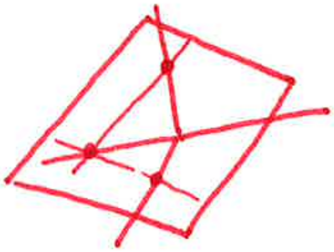
Student Ex: Find normal vector to and 3 points on plane $2x - y + 3z = 5$. Sketch the plane.

normal vector : $\langle 2, -1, 3 \rangle$

$x = y = 0: \; z = 5/3 \Rightarrow (0, 0, 5/3)$

$x = z = 0: \; y = -5 \Rightarrow (0, -5, 0)$

$y = z = 0: \; x = 5/2 \Rightarrow (5/2, 0, 0)$



Student Ex: Find equation of plane $\perp$ to line $\ell(t) = \langle 2, -1, 4 \rangle t + (0, 0, 1)$ containing $(3, 2, 1)$.

Need: pt, $\vec{n}$

$n = \langle 2, -1, 4 \rangle$

$\langle 2, -1, 4 \rangle \cdot (x-3, y-2, z-1) = 0$

$2(x-3) - (y-2) + 4(z-1) = 0$

$2x - y + 4z = 8$

* If it were $n = \langle 4, -2, 8 \rangle$, use $\langle 2, -1, 4 \rangle$ (still $\parallel$, so still $\perp$) but 'nicer' #'s.

General note : $\parallel$ planes $\rightleftarrows$ $\parallel$ normal vectors (with distinct d's)

Student Ex: Find equation of the plane containing...
$(1,2,0)$, $(3,1,2)$, $(0,1,1)$

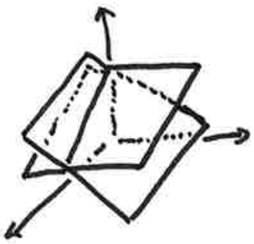
We create vectors:

Need: pt, $\vec{n}$

$$\vec{a} = (3,1,2) - (1,2,0) = \langle 2,-1,2\rangle$$
$$\vec{b} = (0,1,1) - (1,2,0) = \langle -1,-1,1\rangle$$
$$\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ -1 & -1 & 1 \end{vmatrix} = \langle 1,-4,-3\rangle$$
$$\langle 1,-4,-3\rangle \cdot \langle x-1, y-2, z-0\rangle = 0$$
$$1(x-1) - 4(y-2) - 3(z-0) = 0$$
$$x - 4y - 3z = -7$$

Student Ex: Find equation of line formed by intersection of $x-2y+z=4$ & $2x+y+3z=-7$

Need: pt, $\hat{n}$



$$\begin{cases} x-2y+z=4 \\ 2x+y+3z=-7 \end{cases}$$

$z=0$ (hopefully will work)

$$\begin{cases} x-2y=4 \\ 2x+y=-7 \end{cases} \rightsquigarrow x=-2,\ y=-3 \rightsquigarrow (-2,-3,0)$$

Have normals $\vec{n}_1 = \langle 1,-2,1\rangle$, $\vec{n}_2 = \langle 2,1,3\rangle$. Note $\vec{n}_1 \times \vec{n}_2 = \langle -7,-1,5\rangle$. So...

$$\ell(t) = \langle -7,-1,5\rangle t + (-2,-3,0).$$

§ ? : Distances and Miscellaneous

Topics

- Review of lines, planes, & their equations
- Distances for lines.
- Distances for planes.

You Should Be Able to...

- Find equations of lines | planes given varying data.
- Check if lines lie in a given plane.
- Find the $\measuredangle$ between planes.
- Find the line of intersection between 2 planes.
- Determine if planes are $\parallel$, $\perp$, intersecting, or the same.
- Find distance from point to a line.
- Find distance between $\parallel$ planes.
- Find distance between skew & parallel lines.

Misc.

Student Ex: Find equation of line through $(1,0,1)$ and parallel to line $\frac{x-1}{2} = y+1 = \frac{z-3}{-4}$.

We need direction vector $\langle 2,1,-4\rangle$ and point $(1,0,1)$.

$$\ell(t) = \langle 2,1,-4\rangle t + (1,0,1)$$

Student Ex: Find the equation of a line perpendicular to $\ell(t) = (1-2t, t, 3t+1)$.

Point: $t=0 \rightsquigarrow (1,0,1)$

Direction vector: $\langle -2,1,3\rangle$

$$0 = \langle -2,1,3\rangle \cdot \langle x,y,z\rangle = -2x+y+3z$$

So $\langle 1,2,0\rangle$ works.

$\ell(t) = \langle 1,2,0\rangle t + (1,0,1)$ is such a line.

Student Ex: Find equation of plane containing $(1,1,1)$ and $(1,0,1)$ and $\perp$ to plane $z = 2x - y$.

$\vec{v} = (1,1,1) - (1,0,1) = (0,1,0)$; $\quad 0 = 2x - y - z$, $\vec{n} = \langle 2,-1,-1\rangle$

$$\vec{v} \times \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 2 & -1 & -1 \end{vmatrix} = \langle -1,0,-2\rangle \rightsquigarrow \text{use } \langle 1,0,2\rangle$$

$$\langle 1,0,2\rangle \cdot \langle x-1, y-0, z-1\rangle = 0$$

$$1(x-1) + 0(y-0) + 2(z-1) = 0$$

$$x + 2z = 3$$

Student Ex: Find equation of plane containing $(3,-1,4)$ and $\parallel$ to $x+3y=z$.

$x+3y-z=0 \rightarrow \vec{n}=\langle 1,3,-1\rangle$, point $(3,-1,4)$

$$\langle 1,3,-1\rangle \cdot \langle x-3, y+1, z-4\rangle = 0$$
$$1(x-3)+3(y+1)-(z-4)=0$$
$$x+3y-z=-4$$

Student Ex: Find equation of plane cont. $\ell(t)=(t+1, 2-t, 4-3t)$ and $\parallel$ to $5x+2y+z=1$.

$\vec{n}=\langle 5,2,1\rangle$. Point: $t=0 \rightarrow (1,2,4)$

$$\langle 5,2,1\rangle \cdot \langle x-1, y-2, z-4\rangle = 0$$
$$5(x-1)+2(y-2)+(z-4)=0$$
$$5x+2y+z=13$$

Student Ex: Does the plane $x+2y-z=5$ contain the line $\ell(t)=\langle 1,2,3\rangle t+(1,3,-1)$?

$x=t+1$
$y=2t+3$
$z=3t-1$

$$x+2y-z=5$$
$$(t+1)+2(2t+3)-(3t-1)=5$$
$$t+1+4t+6-3t+1=5$$
$$2t+8=5$$

No. $2t+8 \neq 5$ for all t.

Student Ex: Are the planes $x+y+z=1$ and $x-y+z=1$ $\parallel$ or $\perp$?

$\langle 1,1,1\rangle \nparallel \langle 1,-1,1\rangle$

$\langle 1,1,1\rangle \cdot \langle 1,-1,1\rangle = 1-1+1 = 1 \neq 0$

So neither $\parallel$ nor $\perp$.

Student Ex: Are the planes $2x+y-z=4$ and $x+3y+5z=0$ $\perp$?

$\langle 2,1,-1\rangle \cdot \langle 1,3,5\rangle$

$2+3-5=0$

So planes are $\perp$

* Make sure all on same side so proper $\vec{n}$'s.

Student Ex: Are the planes $x-y+z=3$ and $2x-2y+2z=3$ $\parallel$, $\perp$, or the same?

$\langle 1,-1,1\rangle \parallel \langle 2,-2,2\rangle = 2\langle 1,-1,1\rangle$ so $\parallel$ or the same. First contains pt $(3,0,0)$ but on second $2(3)-0+0=6\neq 3$. So $\parallel$. If it was $2x-2y+2z=6$, then would be same.

Student Ex: Show planes $x+y+z=1$ & $x+2y+z=3$ intersect and find $\measuredangle$ of intersection.

$\langle 1,1,1\rangle \nparallel \langle 1,2,1\rangle$ so must intersect.

* If $\perp$, then obv. $90° = \frac{\pi}{2}$

$|\langle 1,1,1\rangle| = \sqrt{3}$

$|\langle 1,2,1\rangle| = \sqrt{6}$

$\langle 1,1,1\rangle \cdot \langle 1,2,1\rangle = 1+2+1 = 4$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{3}\sqrt{6}}\right) = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

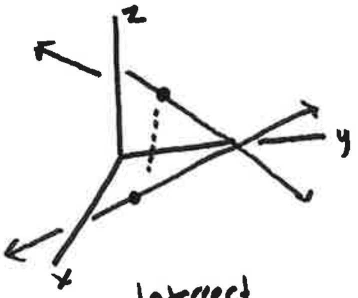
$\approx 19.47°$

Distances

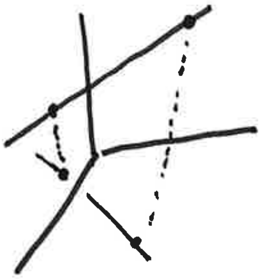
We have already seen distances between points. We have also seen distance from lines | planes to other 'special' points | lines | planes. But we want to do this generally. But first...

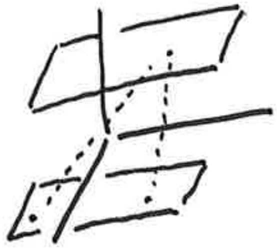
What do we mean by distance, because this varies from point to point.

Ex:



Intersect
Distance = 0 ??
(varies, see ..... line)





Obviously, if objects intersect, we want distance to be 0. So what is general definition?

Distance : Min distance between objects

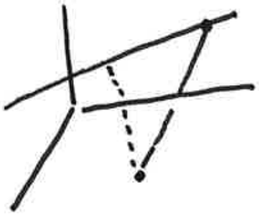
→ sets

So distance is 0 when intersect.

* Technically, we want sup.... but who cares.

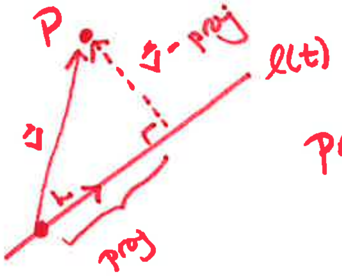
* This definition works generally.

Distance from a point to a line:



Ex: $P = (2,1,3)$

$\ell(t) = \langle -1,1,-2\rangle t + (2,3,-2)$



$\vec{v} = (2,1,3) - (2,3,-2) = \langle 0,-2,5\rangle$

$$\text{proj}_h \vec{v} = \left(\frac{h\cdot v}{h\cdot h}\right) h = \frac{\langle -1,1,-2\rangle\cdot\langle 0,-2,5\rangle}{\langle -1,1,-2\rangle\langle -1,1,-2\rangle}\langle -1,1,-2\rangle$$

$$= \langle 2,-2,4\rangle$$

$$|v - \text{proj}| = |\langle 0,-2,5\rangle - \langle 2,-2,4\rangle| = |\langle -2,0,1\rangle| = \sqrt{5}$$

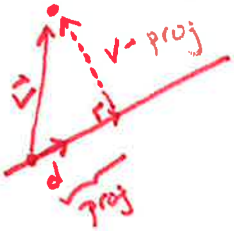
*(and call distance D and use right △ trig. $\sin\theta = \frac{D}{|\vec{v}|}$

$D = \sqrt{5}$

$$D = \frac{|v|\sin\theta\cdot|h|}{|h|} = \frac{|h\times v|}{|h|}$$

Student Ex: $P = (1,0,1)$

$\ell(t) = \langle 2,1,0\rangle t + (0,0,1)$



$\vec{v} = (1,0,1) - (0,0,1) = \langle 1,0,0\rangle$

$$\text{proj}_d v = \left(\frac{d\cdot v}{d\cdot d}\right) d = \frac{\langle 2,1,0\rangle\cdot\langle 1,0,0\rangle}{\langle 2,1,0\rangle\cdot\langle 2,1,0\rangle}\langle 2,1,0\rangle$$

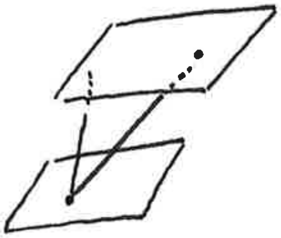
$$= \frac{2}{5}\langle 2,1,0\rangle$$

$$= \langle 4/5, 2/5, 0\rangle$$

$$|v - \text{proj}| = |\langle 1/5, -2/5, 0\rangle| = \left|\frac{1}{5}\langle 1,-2,0\rangle\right| = \frac{1}{5}\cdot\sqrt{5} = \frac{1}{\sqrt{5}}$$

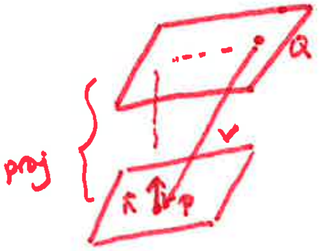
$$D = \frac{1}{\sqrt{5}}$$

Distance between parallel planes:



Ex: $\Pi_1 : 2x-2y+z=5$ * Not 15!

$\Pi_2 : 2x-2y+z=20$



$P=(0,0,5)$ $\rightarrow$ $\vec{v}=(0,0,20)-(0,0,5)=\langle 0,0,15\rangle$

$Q=(0,0,20)$

$\hat{n}=\langle 2,-2,1\rangle$

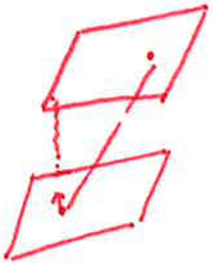
$$\text{proj}_{\hat{n}}\vec{v} = \frac{\langle 2,-2,1\rangle\cdot\langle 0,0,15\rangle}{\langle 2,-2,1\rangle\cdot\langle 2,-2,1\rangle}\langle 2,-2,1\rangle = -\frac{5}{3}\langle 2,-2,1\rangle$$

$$|\text{proj}_{\hat{n}}\vec{v}| = \left|-\frac{5}{3}\langle 2,-2,1\rangle\right| = \frac{5}{3}\cdot\sqrt{9} = 5$$

$D=5$

Student Ex: $\Pi_1 : x-2y+z=1$

$\Pi_2 : x-2y+z=3$



$\hat{n}=\langle 1,-2,1\rangle$

$P=(1,0,0)$ $\rightarrow$ $\vec{v}=(3,0,0)-(1,0,0)=\langle 2,0,0\rangle$

$Q=(3,0,0)$

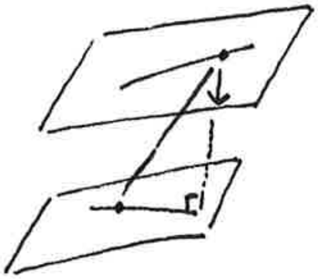
$$\text{proj}_{\hat{n}}\vec{v} = \frac{\langle 1,-2,1\rangle\cdot\langle 2,0,0\rangle}{\langle 1,-2,1\rangle\cdot\langle 1,-2,1\rangle}\langle 1,-2,1\rangle = \frac{2}{6}\langle 2,0,0\rangle$$

$$= 1/3\,\langle 2,0,0\rangle$$

$$|\text{proj}_{\hat{n}}\vec{v}| = \left|\frac{1}{3}\langle 2,0,0\rangle\right| = 2/3$$

$D=\frac{2}{3}$

Distance between skew (≠ parallel) lines:



Ex: $\ell_1(t) = \langle 2,1,3\rangle t + (0,5,-1)$

$\ell_2(t) = \langle 1,-1,0\rangle t + (-1,2,0)$

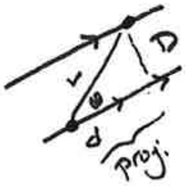
$\vec{v} = (-1,2,0) - (0,5,-1) = (-1,-3,1)$

$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix} = \langle 3,3,-3\rangle \rightsquigarrow 3\underbrace{\langle 1,1,-1\rangle}_{\vec{n}}$

$\text{proj}_{\vec{n}}\vec{v} = \dfrac{\langle 1,1,-1\rangle\cdot\langle -1,-3,1\rangle}{\langle 1,1,-1\rangle\cdot\langle 1,1,-1\rangle}\langle 1,1,-1\rangle = \dfrac{-5}{3}\langle 1,1,-1\rangle$

$|\text{proj}_n\vec{v}| = |{-5/3}\langle 1,1,-1\rangle| = \frac{5}{3}\cdot\sqrt{3} = 5/\sqrt{3}$

*If lines //, cross product $= \vec{0}$.



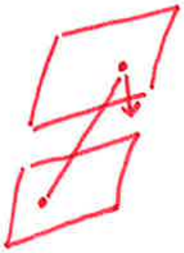
$D = |\text{proj}|\sin\theta$

$= |v|\dfrac{|\text{proj}|}{|v|}\sin\theta$

$= \dfrac{|v \times \text{proj}|}{|v|}$

Student Ex: $\ell_1(t) = \langle 1,0,2\rangle t + (1,0,1)$

$\ell_2(t) = \langle 0,1,0\rangle t + (1,1,1)$



$\vec{v} = (1,1,1) - (1,0,1) = \langle 0,1,0\rangle$

$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = \langle -2,0,1\rangle$

* $\vec{v} \perp \vec{n}$ so $\text{proj} = \vec{0}$

$\text{proj}_{\hat{n}}\vec{v} = \dfrac{\langle -2,0,1\rangle\cdot\langle 0,1,0\rangle}{\langle -2,0,1\rangle\cdot\langle -2,0,1\rangle}\langle -2,0,1\rangle = \vec{0}$

$D = 0 \rightsquigarrow$ intersect?

$\left.\begin{aligned} t+1&=1 \\ 0&=s+1 \end{aligned}\right\}$ gives $t=0$ and $s=-1$

$2t+1=1$

Works in last so intersect (at (1,0,1)).

§10.6: Cylinders, Quadratic Surfaces, & Level Curves

Topics

- (Parabolic) Cylinders
- Quadratic Surfaces
- Level Curves

You Should Be Able to...

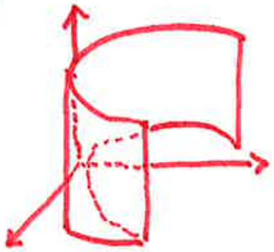
- Identify/plot (parabolic) cylinders
- Identify and sketch quadratic surfaces
- Plot spheres
- Sketch level curves of surfaces.

(Parabolic) Cylinders & the 'like':

Student Ex: Plot $y = x^2$ (in 3-space)

If $z=0$:

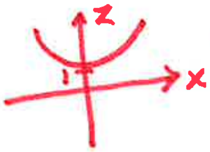
. But indep. of z. So same for any value of z. Then...



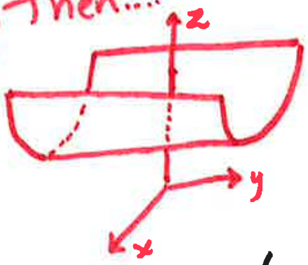
← continues indefinitely

Student Ex: Plot $z = x^2 + 1$

In the xz-plane (ie $y=0$):



. But indep. of y. So same for every value of y. Then....

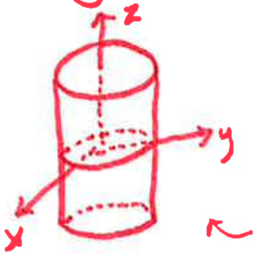


← cont. indef.

Student Ex: Plot $x^2 + y^2 = 1$ (in 3-space)

If $z=0$ (ie xy-plane):

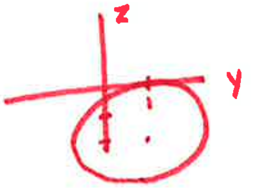
. But this is the same for any value of z. Then....



← cont. indef.

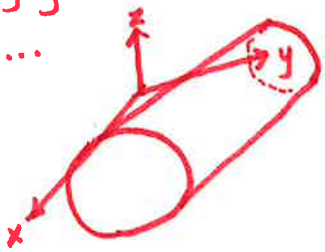
Student Ex: Plot $(y-1)^2 + (z+2)^2 = 4$

In the yz-plane (ie $x=0$):



Center: $(1,-2)$
radius: 2

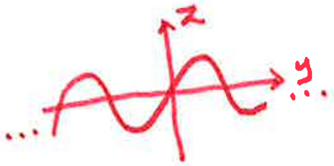
But this is indep. of x. So we have...



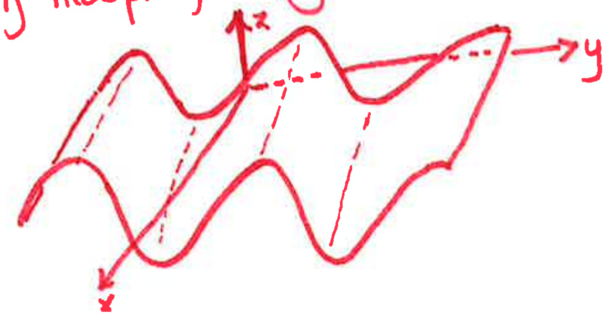
Notice we plot the surface by seeing what a 'piece' of it looks like and 'stretch' this through space. This works more generally:

Student Ex: Plot $z = \sin y$

In the yz-plane, we have



But this is indep. of x. So...



These are 'easy' example of a general concept, called level curves, where we find out what 'sections' of the surface look like and piece together to discern the bigger picture.

Quadratic Surfaces:

Quadratic for degree two. In 3 variables, this is....

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

After a series of general translations and rotations, 'equivalent' to

$$\boxed{Ax^2 + By^2 + Cz^2 + J = 0}$$

You will need to memorize which combinations correspond to which quadratic surfaces.

Show Quad. Surface Sheet

*Be careful of permutations of variables. Corresponds to orientation of surface but not which is which.

Ex: Identify $x^2 + \frac{y^2}{3} + 4z^2 = 6$

Ellipsoid

Ex: Identify $x - 4z^2 = \frac{y^2}{3}$

(Elliptic) Paraboloid

Ex: Identify $2y^2 - x^2 = 4 + 3z^2$

Hyperboloid of two sheets

Ex: Identify $z^2 = 2x^2 + 3y^2$

Cone

Ex: Identify $y = z^2 - x^2$

Hyperbolic Paraboloid

Ex: Identify $x^2 + z^2 = 3 + y^2$

Hyperboloid of one sheet

but the traces in the xz- and yz-planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \qquad y = 0 \qquad \text{and} \qquad y^2 - \frac{z^2}{4} = 1 \qquad x = 0$$

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9. ■

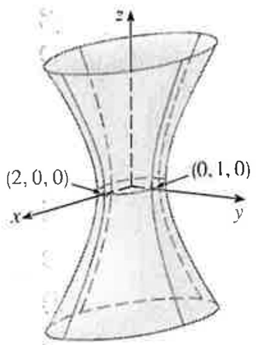
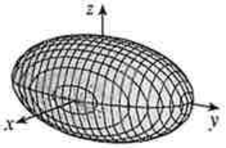
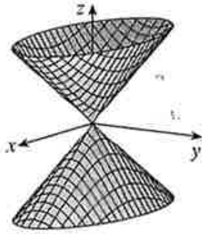
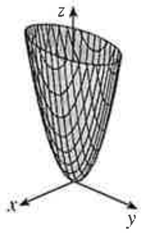
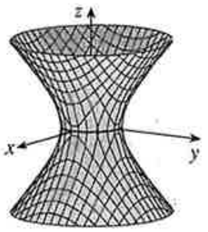
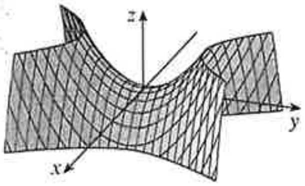
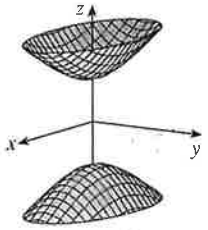


FIGURE 9

The idea of using traces to draw a surface is employed in three-dimensional graphing software for computers. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k, and parts of the graph are eliminated using hidden line removal. Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

TABLE 1 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid  	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone  	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid  	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet  	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid  	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets  	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

But generally, how does one determine which surface is which and plot them?

Level Curves: Fix one of x, y, z and see what curve(s) arise, then stitch these curves together.

Also called traces.

* Before, all the level curves in a particular direction were all <u>exactly</u> the same. Not generally the case.

Student Ex: What is $z = 4 - x^2 - y^2$? Plot the surface.

Fix z:

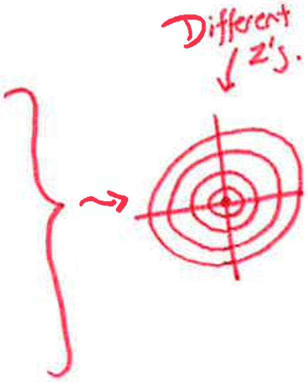
$z = 3$: $x^2 + y^2 = 1$

$z = 0$: $x^2 + y^2 = 4$

↓z, bigger circle

$z = 4$: $x^2 + y^2 = 0 \rightarrow .$

$z > 4$: $x^2 + y^2 < 0 \Rightarrow\Leftarrow$

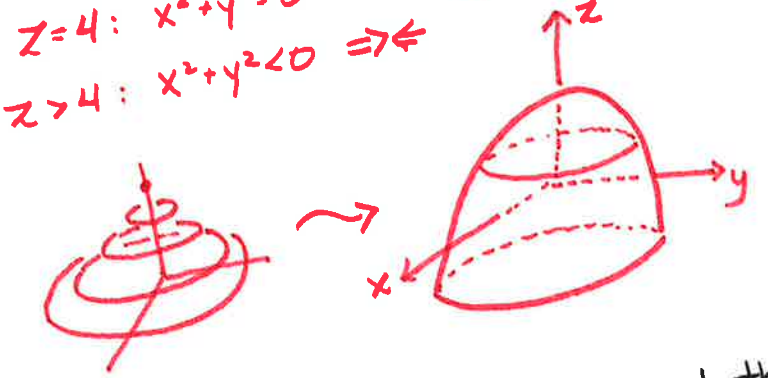


Fix x|y:

$y = 0$: $z = 4 - x^2$

$x = 0$: $z = 4 - y^2$

Always parabolas, just different 'start' height



This is how one would arrive at the fact that $z = 4 - x^2 - y^2$ is a paraboloid. Note the level curve diagrams should remind you of something: topographical maps.

Student Ex: What is $4x^2 - 3y^2 + 12z^2 + 12 = 0$? Plot the surface.

First, put the surface in standard form:

$$4x^2 - 3y^2 + 12z^2 + 12 = 0$$

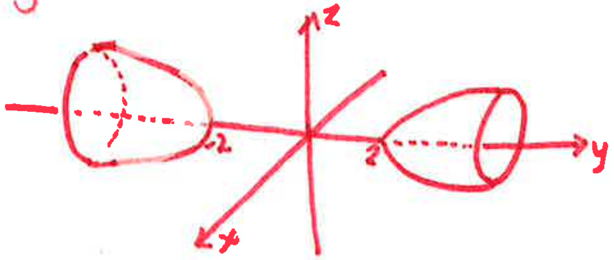
$$\frac{x^2}{-3} + \frac{y^2}{4} - z^2 - 1 = 0$$

$$\frac{y^2}{4} - \frac{x^2}{3} - \frac{z^2}{1} = 1$$

xy-plane: $\frac{y^2}{4} - \frac{x^2}{3} = 1$ Hyperbola

xz-plane: $\frac{x^2}{3} + \frac{z^2}{1} = -1$ Nothing

yz-plane: $\frac{y^2}{4} - \frac{z^2}{1} = 1$ Hyperbola

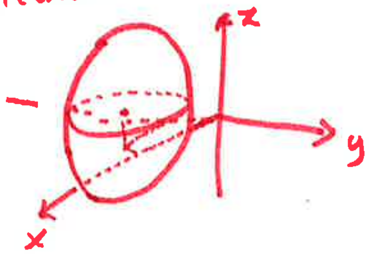


Student Ex: Classify & sketch $x^2 + 2y^2 + z^2 - 4x + 4y - 2z + 3 = 0$

Complete the square in each variable....

$$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{2} + \frac{(z-1)^2}{4} = 1$$

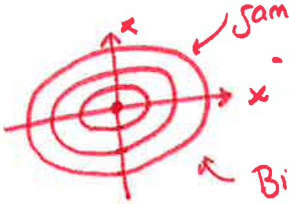
'Clearly', ellipsoid centered at $(2,-1,1)$. 'smallest' in y-direction.



Student Ex: Classify and sketch $2x^2+3z^2-y^2=0$.

Write $y^2=2x^2+3z^2$

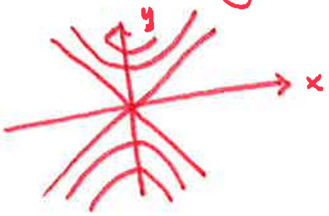
Fix y:



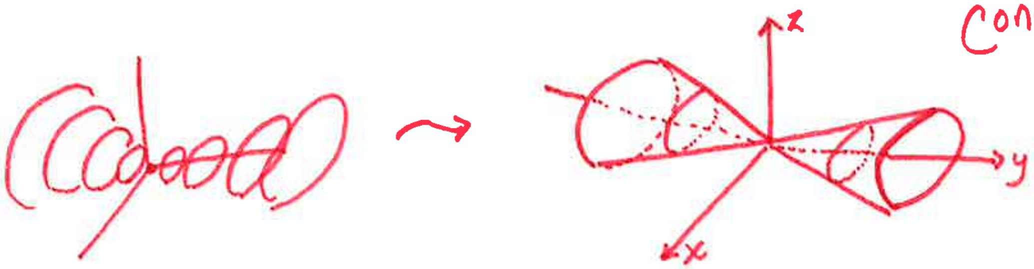
same for y as $-y$. Ellipses

Bigger in x.

Fix x or z: (say z)



lines / hyperbolas



Cone

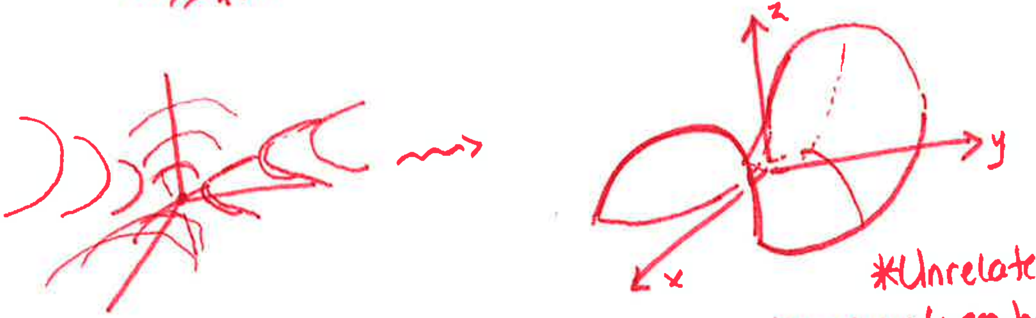
Student Ex: Classify and sketch $z=x^2-y^2$

Fix z:

lines / hyperbola

Fix x or y: (say y)

parabolas



Hyperbolic paraboloid

*Unrelated note: 1-sheet hyperboloids can be made with string rings:

twist

Some 'fun' examples:

Saddle: $z=x^2-y^2$

Monkey Saddle: $z=x^3-3xy^2$

Dog Saddle: $z=4x^3y-4xy^3$

$z=(x^2+3y^2)e^{1-(x^2+y^2)}$

$(x-2)^2(x+2)^2+(y-2)^2(y+2)^2+(z-2)^2(z+2)^2$
$+3(x^2y^2+x^2z^2+y^2z^2)+6xyz$
$-10(x^2+y^2+z^2)+22=0$

As a 'fun' final note, we can create formulas for the surfaces in 3-space arising from revolving curves.

Suppose $y = f(x)$ is revolved about the x-axis. If (x_0, y_0) is on the curve, the point (x_0, y, z) has distance y_0 from $(x_0, 0, 0)$. Squaring...

$$(x_0 - x_0)^2 + y^2 + z^2 = (y_0)^2 = (f(x_0))^2$$

But then...

$$y^2 + z^2 = (f(x_0))^2$$

So we generally have...

$$y^2 + z^2 = (f(x))^2$$

A similar argument works for other rotational lines.

§10.7-10.9: Vector Functions & Curves - Evaluation, Diff., Integration, Velocity | Accel., Arclength, Curvature, Moving Frames

Topics

- Vector functions and their algebra | vector operations
- Graphing & creating vector functions
- Diff | Int. vector valued functions
- Applications of vector functions
- Graphical inter. of vector functions
- Position, speed, vel., accel.
- Arclength
- T, N, B vectors & plane
- Curvature

You Should Be Able to...

- Perform basic vector valued function operations.
- Parametrize line segments, circles | ellipses, helixes, and plane curves
- Sketch | identify vector valued functions
- Take limits of & diff | int. vector valued functions
- Find where vector valued functions are smooth
- Compute position, velocity, acceleration, and speed
- Graphically interpret pos., vel., acc.
- Compute position from acceleration
- Compute arclength of a vector valued function.
- Compute & plot $\vec{T}$, $\vec{N}$, and $\vec{B}$ for a vector valued function.
- Compute the curvature of a vector valued function.
- Graphically interpret $\vec{T}$, $\vec{N}$, $\vec{B}$, and κ.

There are a few special types of functions we focus on in this course:

1) Vector functions | curves | paths:

$$f : \mathbb{R} \to \mathbb{R}^n$$

2) Multivariable functions:

$$f : \mathbb{R}^n \to \mathbb{R}$$

3) Vector Fields:

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

For now, we focus on the first type.

Vector Valued Function | Space Curve | Path:

$$\vec{r}(t) = \text{`}(f(t), g(t))\text{'} = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath}$$

OR

$$\vec{r}(t) = \text{`}(f(t), g(t), h(t))\text{'} = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{k}$$

OR

$$\vdots$$

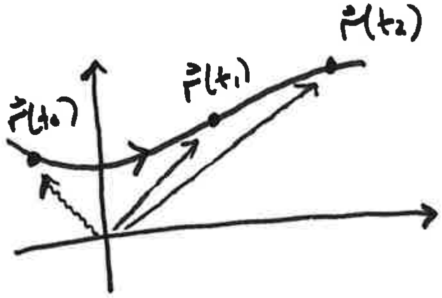
We also use the notation $\vec{x}(t)$ or simple boldface.

* The input 't' is the parameter. The functions $f, g, h, \ldots$ are the components.

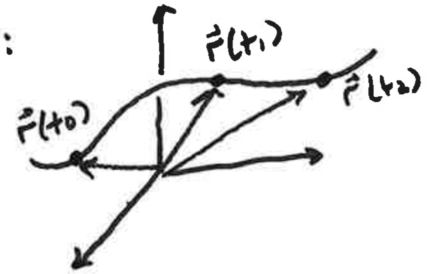
Strictly Speaking:

Path in $\mathbb{R}^n$: Is a continuous function $\vec{x}: I \to \mathbb{R}^n$. If $I = [a, b]$, then $\vec{x}(a)$ and $\vec{x}(b)$ are called the endpoints of $\vec{x}$. If $\vec{x}(a) = \vec{x}(b)$, we call $\vec{x}(t)$ a closed path | loop.

Ex: Curve in the plane:



Ex: Curve in space:



Ex: 'Line': $\vec{x}(t) = \langle 2t+1, 1-t, t+2 \rangle$

Strictly speaking, this is vectors pointing to points on the line. The plot of the path is a line.

* Difference between a curve and vector function is subtle, and mostly unimportant

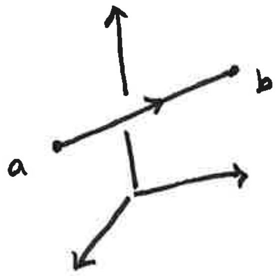
* Different curves can have the same graph, eg. $\vec{x}(t) = \langle \sin t, \cos t \rangle$ and $\vec{x}(t) = \langle \cos t, \sin t \rangle$ have unit circle as their graph but have different values at different t's, ie trace curve out differently.

* We use arrows on the curve to denote the orientation of the curve.

Important examples you will need to know

<u>Line</u> (segment): Line (segment) through 2 points $\vec{a}$, $\vec{b}$ (from $\vec{a}$ to $\vec{b}$).

$$\vec{r}(t) = (1-t)\vec{a} + t\vec{b} \; ; \; 0 \le t \le 1$$

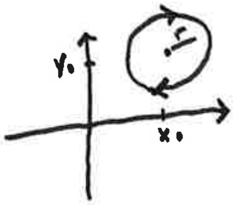


Eg, $a = \langle 1, 0, 2 \rangle$
$b = \langle 1, 1, 3 \rangle$

$$\vec{r}(t) = (1-t)\langle 1, 0, 2\rangle + t\langle 1, 1, 3\rangle$$
$$= \langle 1-t, 0, 2-2t\rangle + \langle t, t, 3t\rangle$$
$$= \langle 1, t, t+2\rangle$$

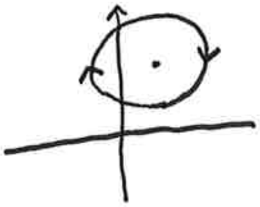
<u>Circle</u> | <u>Ellipse</u>: Radius r, center (x_0, y_0), counterclockwise

$$\vec{x}(t) = \langle r\cos t + x_0, r\sin t + y_0 \rangle$$



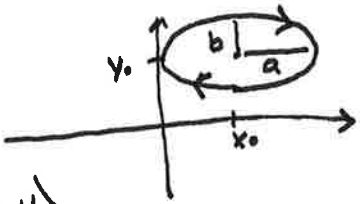
Eg. $r = 2$, $(1,3)$

$$\vec{x}(t) = \langle 2\cos t + 1, 2\sin t + 3\rangle$$



For an ellipse, simply have different radii

$$\vec{x}(t) = \langle a\cos t + x_0, b\sin t + y_0 \rangle$$

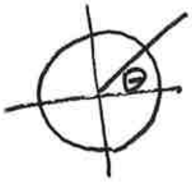


And can do the same in 3 space:

$$\vec{x}(t) = \langle 2\cos t + 1, 3, 2\sin t + 4\rangle$$

* This resolves that a circle is a function, just not in x or y

$$\vec{r}(\theta) = \langle \cos\theta, \sin\theta \rangle$$



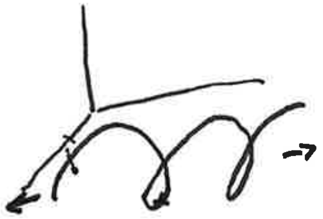
* Think of curves as being traced in time.

Helix: radius r, 'center' (x_0, y_0, z_0), counterclockwise

$$\vec{x}(t) = \langle r\cos t + x_0,\ r\sin t + y_0,\ Ct + z_0 \rangle$$

Note: C just determines rate of up/down movement. Written this way, helix around 'z-axis' (z_0). But can permute to get curving around other axes. The linear component is the axis of revolution.

Ex: $\vec{r}(t) = \langle 2\cos t + 1,\ 3t,\ 2\sin t - 1 \rangle$

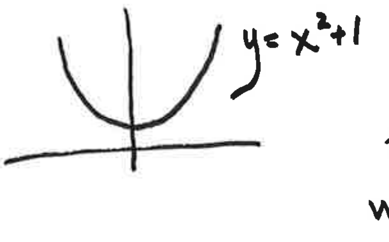


* Note as in circle case, can vary 'r's' to get an elliptical helix.

Plane Curves: Given plane curve $y = f(x)$ or $x = f(y)$, then $\vec{r}(t) = \langle t, f(t) \rangle$ or $\vec{r}(t) = \langle f(t), t \rangle$ work as parametrization.

* Simple substitution of variables really - now thinking of curve being sketched in time.

Ex:



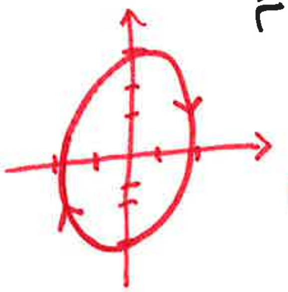
$\vec{r}(t) = \langle t,\ t^2 + 1 \rangle$

* Note: Any linear sub. for x/y will work - simply changes rate/direction of trace.

Cole

Ex: Sketch curve given by

$\vec{r}(t) = 2\cos t\,\hat{\imath} - 3\sin t\,\hat{\jmath}$; $0 \le t \le 2\pi$



Ellipse w/ axes 2, 3 centered at (0,0).

Note $x = 2\cos t$ at all times.
$y = -3\sin t$

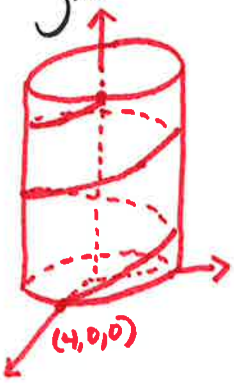
$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = \frac{(2\cos t)^2}{2^2} + \frac{(-3\sin t)^2}{3^2} = \cos^2 t + \sin^2 t = 1$$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ so indeed an ellipse.

Ex: Sketch curve given by...

$\vec{x}(t) = \langle 4\cos t, 4\sin t, t \rangle$; $0 \le t \le 4\pi$

Show the curve lies along a cylinder.



Helix (circular) with radius 4. Notice $0 \le t \le 4\pi$ so it makes 2 loops around. Now note:

$$x^2 + y^2 = (4\cos t)^2 + (4\sin t)^2 = 16\cos^2 t + 16\sin^2 t$$
$$= 16$$

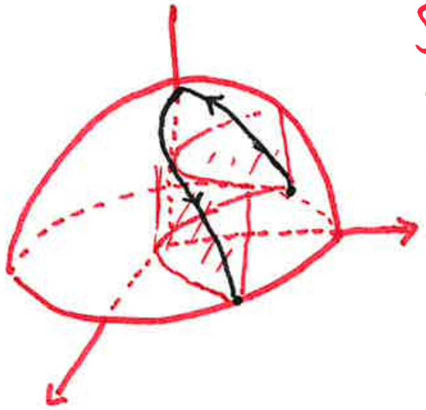
So curve lies along $x^2+y^2=16$, which in 3-space is a cylinder.

These show we can go from a space curve to its graph/surface, assuming we are clever enough to find a relation. But we can also, in some sense, go the other way.

Ex: Find and sketch space curve formed by the intersection of the semiellipsoid

$$\frac{x^2}{12}+\frac{y^2}{24}+\frac{z^2}{4}=1, \quad z\geq 0$$

and the parabolic cylinder $y=x^2$.



Notice first in 3 variables. But points on both surfaces. If we know x, we know y. But then can solve for z in terms of x, y, so when we know x, we know x & y, hence z.

$$x=t$$

Then $y=x^2=t^2$

Then $\frac{x^2}{12}+\frac{y^2}{24}+\frac{z^2}{4}=1$

$$\frac{z^2}{4}=1-\frac{x^2}{12}+\frac{y^2}{24}$$

$$\frac{z^2}{4}=\frac{24-2t^2-t^4}{24}=\frac{(t^2+6)(t^2-4)}{24}$$

$$\underbrace{z=\pm\sqrt{\frac{(t^2+6)(t^2-4)}{6}}}_{\text{Take + root as } z\geq 0}$$

Then $\vec{r}(t)=\left\langle t, t^2, \sqrt{\frac{(t^2+6)(t^2-4)}{6}}\right\rangle$

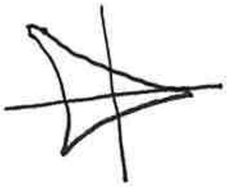
is a possible parametrization. Note we want $z\geq 0$. This is when $t=\pm 2$, so $-2\leq t\leq 2$

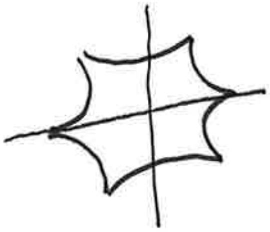
* We can do something similar but with surfaces instead. This will come later in the course. (Paramet. Surfaces)

As a final aside, these vector valued functions are exactly how one creates equations for more complicated, and beautiful, curves.

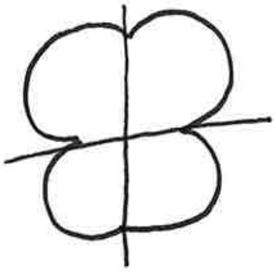
Spirographs®

Hypocycloids

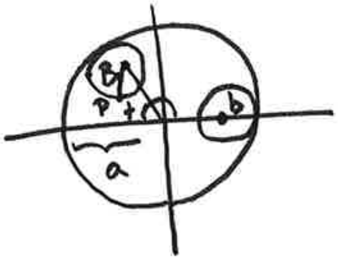


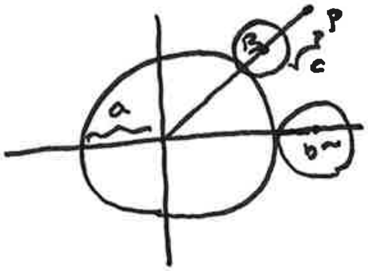


Epicycloid



Hypotrochoids, epitrochoids, etc.





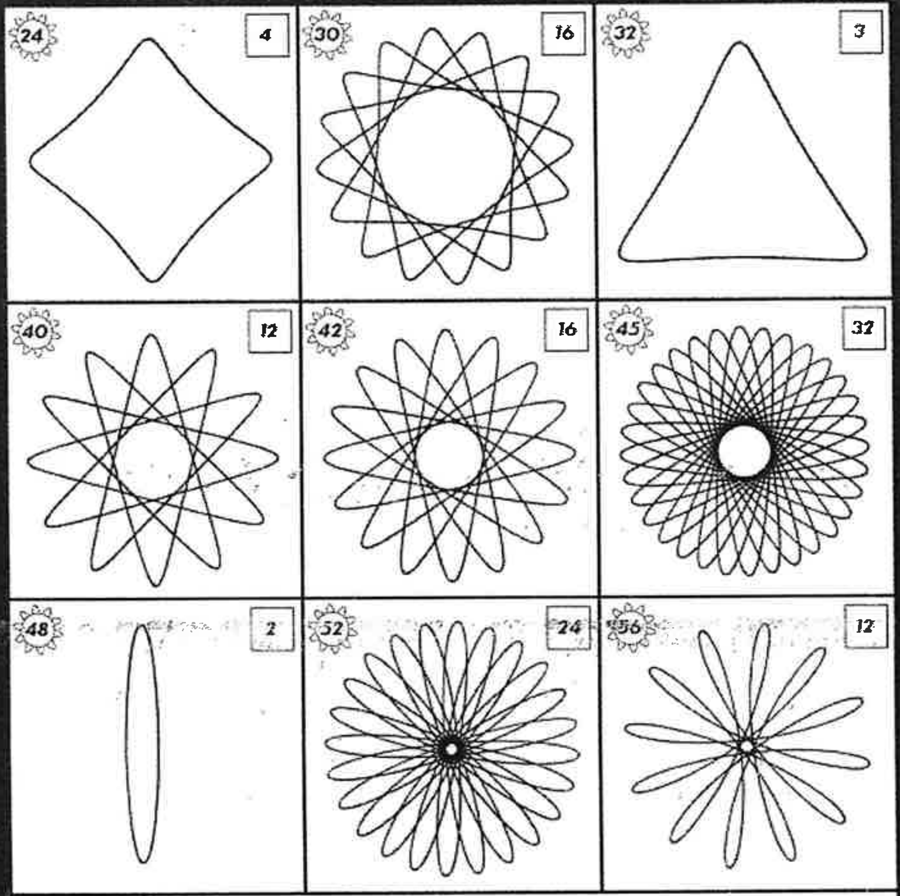
Spirograph® Basic Pattern Guide

The patterns below were drawn with the pen in Hole No. 1 of each WHEEL.

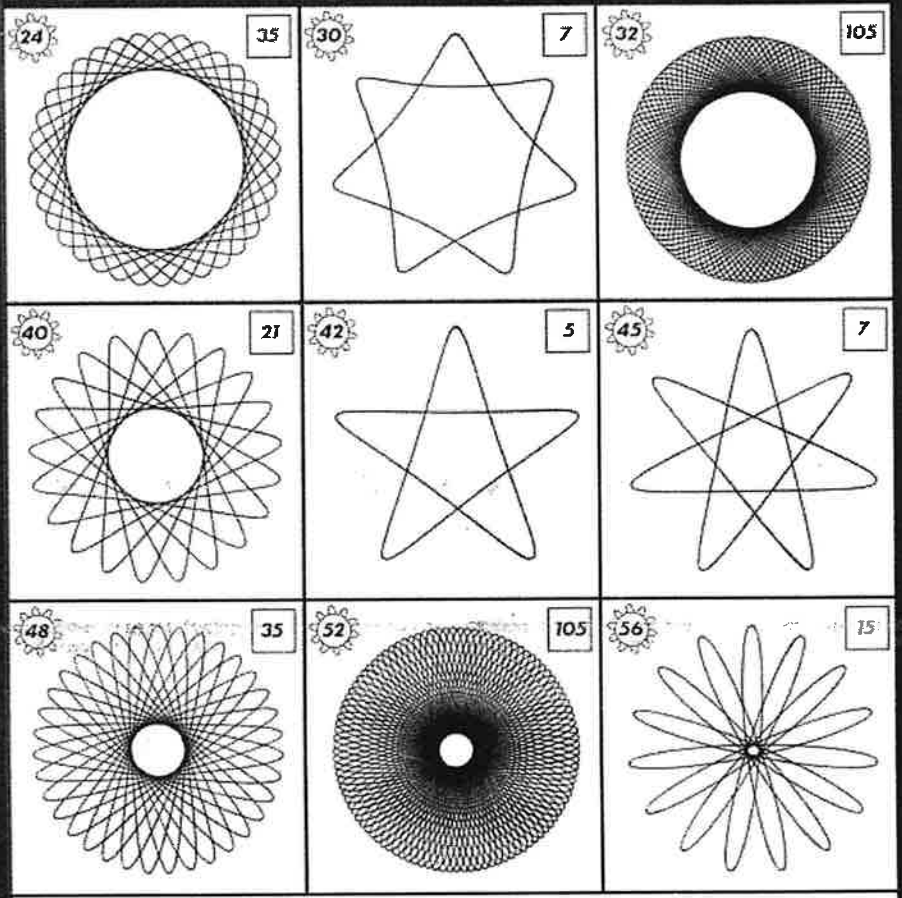
⚙ = WHEEL used to create pattern

☐ = Number of points in pattern

Patterns created using RING No. $\frac{144}{96}$



Patterns created using RING No. $\frac{150}{105}$



Domains, Evaluations, Limits, Continuity, Diff., Int.:

* The domain of a vector valued function is the common domain of its components

Ex: $\vec{r}(t) = \langle \sin t, 1/t, \sqrt{t}, t+1 \rangle$ 4D

$\sin t : \mathbb{R}$ $\sqrt{t} : t \geq 0$ So domain is $t > 0$.

$1/t : t \neq 0$ $t+1 : \mathbb{R}$

Now vector valued functions are evaluated in the 'obvious' way:

Ex: $\vec{x}(t) = \langle t^2+1, t\sin t, e^t \rangle$

$\vec{x}(0) = \langle 0+1, 0\sin 0, e^0 \rangle = \langle 1, 0, 1 \rangle = \hat{i} + \hat{k}$

Notice also the different components do not interact. But then one can immediately prove:

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

* Note the limit here (if it exists) is a vector, ie what vector do we 'get close to' as $t \to a$. If the limit is $\vec{L}$, then $\lim_{t \to a} \| \vec{r}(t) - \vec{L} \| = 0$

* Again, as in Calculus I, $\vec{r}(a)$ need not be defined.

* Addition | Subtraction | Scaling | Dot & Cross products are defined also in the 'obvious' way. and behave well with limits.

Continuity is defined just as in Calculus I :

$\vec{r}(t)$ is continuous at $t=a$ if the left/right limits at 'a' exist, are equal, and $\vec{r}(a)$ is defined.

Derivatives are defined, again, as in Calculus I :

Newton notation $\longrightarrow$
$$\vec{r}\,'(t) := \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$\dot{\vec{r}}, \ddot{\vec{r}}$, etc.

Again, because the components do not interact, we have:

If $\vec{x}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\vec{x}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Ex: Let $\vec{r}(t) = t\,\hat{i} + (t^2+1)\,\hat{j} + \hat{k}$

Find $\vec{r}(0)$, $\vec{r}\,'(t)$, $\vec{r}\,'(1)$, $\vec{r}\,''(t)$, $\vec{r}\,'(t)\cdot\vec{r}\,''(t)$

$$\vec{r}(0) = 0\hat{i} + 1\hat{j} + \hat{k} = \hat{j} + \hat{k} = \langle 0,1,1 \rangle$$

$$\vec{r}\,'(t) = \frac{d}{dt}\langle t, t^2+1, 1 \rangle = \langle 1, 2t, 0 \rangle$$

$$\vec{r}\,'(1) = \langle 1, 2t, 0 \rangle \Big|_{t=1} = \langle 1, 2, 0 \rangle$$

$$\vec{r}\,''(t) = \langle 0, 2, 0 \rangle$$

$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = \langle 1, 2t, 0 \rangle \cdot \langle 0, 2, 0 \rangle = 0 + 4t + 0 = 4t$$

Again, because the components are indepent, integration works similarly.

$$\int \vec{x}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$$

* Do not forget $+\vec{C}$ (vector valued here)!

Ex: Let $\vec{r}(t) = t\hat{i} + (t^2+1)\hat{j} + k$. Find $\int \vec{r}(t)\,dt$.

$$\int \vec{r}(t)\,dt = \left\langle \int t\,dt, \int (t^2+1)\,dt, \int dt \right\rangle$$
$$= \left\langle t^2/2 + C_1, t^3/3 + t + C_2, t + C_3 \right\rangle$$
$$= \left\langle t^2/2, t^3/3 + t, t \right\rangle + \left\langle C_1, C_2, C_3 \right\rangle$$
$$= \left\langle t^2/2, t^3/3 + t, t \right\rangle + \vec{C}$$

A few special properties of the derivative:

If $\vec{r}, \vec{s}$ are vector valued functions and f is a function of t (real-valued), then...

1) $D_t(w(t)\vec{r}(t)) = w(t)\vec{r}(t) + w'(t)\vec{r}'(t)$

2) $D_t(\vec{r}(t)\cdot\vec{s}(t)) = \vec{r}(t)\cdot\vec{s}'(t) + \vec{r}'(t)\cdot\vec{s}(t)$

3) $D_t(\vec{r}(t)\times\vec{s}(t)) = \vec{r}'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}'(t)$

4) $D_t(\vec{r}(w(t))) = \vec{r}'(w(t))\,w'(t)$

5) If $|\vec{r}(t)|$ is constant, then $\vec{r}(t)\cdot\vec{r}'(t) = 0$.

Before moving on, because it will come up later in the course, we say $\vec{r}(t)$ is smooth on an interval (open) I if $\vec{r}'(t)$ exists on I and $\vec{r}'(t) \neq \vec{0}$ on I. Means no 'pinch' points.

Ex: $\vec{x}(t) = \langle t^2+1, t, \cos t \rangle$. Where is $\vec{x}(t)$ smooth?

$\vec{x}'(t) = \langle 2t, 1, -\sin t \rangle$

$2t = 0$
$t = 0$

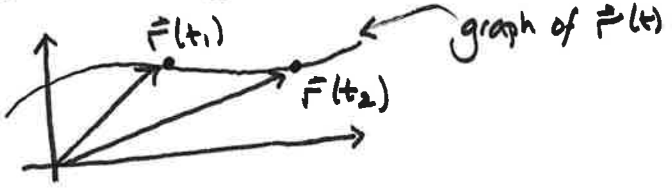
$1 = 0$
Never

$-\sin t = 0$
$t = \pm k\pi$

No value which makes all components vanish (to form $\vec{0}$). So $\vec{x}'(t)$ is smooth for all $t \in \mathbb{R}$.

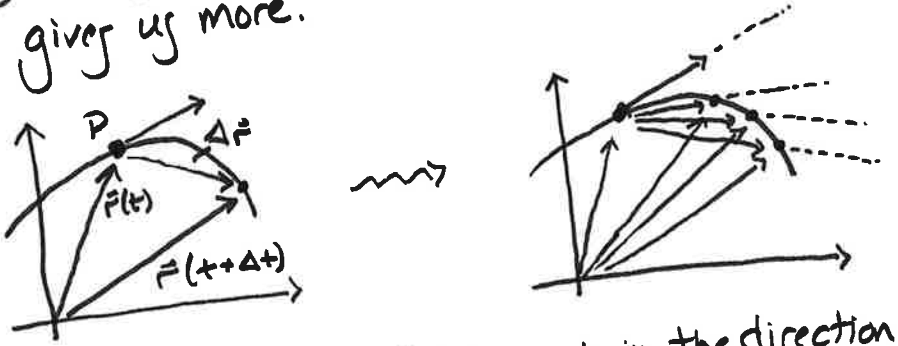
Differentiation | Integration of vector valued functions have many uses | interpretations in Physics.

Position: $\vec{r}(t)$ or $\vec{x}(t)$ denote position of an object at time t. (Really vector pointing to the current position). This traces a curve out in space.



Also gives parametrization of tangent lines.

Velocity: $\vec{r}\,'(t)$ or $\vec{x}\,'(t)$ gives the rate of change of the components as a vector, ie the velocity vector. But this gives us more.



As $\Delta t \to 0$

$\frac{\Delta \vec{r}}{\Delta t} \to \vec{v}(t)$

So the velocity vector, $\vec{v}(t)$ points in the direction that the object is moving. This also give the current speed.

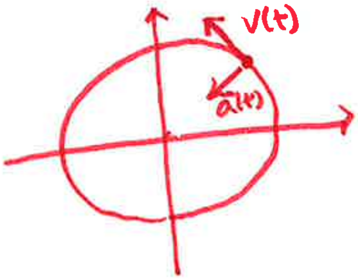
$$\boxed{\text{Speed} = |\vec{v}(t)| = |\vec{r}\,'(t)| = |\vec{x}\,'(t)|}$$

✱ As they emphasize in Physics, velocity $\neq$ speed. Velocity is a <u>vector</u> and has direction, whereas speed is a scalar.

Acceleration: Applying the above logic to $\vec{v}(t)$, $\vec{a}(t) := \vec{v}\,'(t)$, ie $\vec{a}(t) = \vec{r}\,''(t) = \vec{x}\,''(t)$ is the acceleration vector, and tells the direction of acceleration, ie direction of velocity shift.

Ex: Sketch $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$. Find $\vec{r}(\frac{\pi}{4})$. Find $\vec{v}(t)$, $\vec{a}(t)$, as well as $\vec{v}(\frac{\pi}{4})$, $\vec{a}(\frac{\pi}{4})$, and the speed at $t = \frac{\pi}{4}$. Sketch these all together.

$$\vec{r}(t) = \langle \cos t, \sin t \rangle \Big|_{t=\frac{\pi}{4}} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$$



$$v(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle \Big|_{t=\frac{\pi}{4}} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$$

$$a(t) = \vec{r}\,''(t) = \langle -\cos t, -\sin t \rangle \Big|_{t=\frac{\pi}{4}} = \langle -1/\sqrt{2}, -1/\sqrt{2} \rangle$$

$$\text{Speed} = |\langle -1/\sqrt{2}, 1/\sqrt{2} \rangle| = \left|\tfrac{1}{\sqrt{2}} \langle -1, 1 \rangle\right| = \frac{1}{\sqrt{2}} \cdot \sqrt{2} = 1$$

Generally, $\text{speed}(t) = |\langle -\sin t, \cos t \rangle| = \sin^2 t + \cos^2 t = 1$

This example should remind you of planetary motion in Physics, esp. the 'center seeking' force.

Often, we can only find the forces, ie accelerations, on a body. We work 'backwards' to find the resulting path of motion.

Ex: An object starts at rest at $P(1,2,0)$, with the only force of $\vec{a}(t) = \hat{j} + 2\hat{k}$ acting on it. Find the position after 2 seconds.

Know:
$\vec{r}(0) = \langle 1,2,0 \rangle$
$\vec{v}(0) = \langle 0,0,0 \rangle$
$\vec{a}(0) = \langle 0,1,2 \rangle$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0,1,2 \rangle\,dt = \langle 0, t, 2t \rangle + \langle C_1, C_2, C_3 \rangle$$

$$\vec{v}(0) = \langle 0,0,0 \rangle = \langle 0,0,0 \rangle + \langle C_1, C_2, C_3 \rangle = \langle C_1, C_2, C_3 \rangle$$

$$\vec{v}(t) = \langle 0, t, 2t \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 0, t, 2t \rangle\,dt = \langle 0, t^2/2, t^2 \rangle + \langle D_1, D_2, D_3 \rangle$$

$$\vec{r}(0) = \langle 1,2,0 \rangle = \langle 0,0,0 \rangle + \langle D_1, D_2, D_3 \rangle = \langle D_1, D_2, D_3 \rangle$$

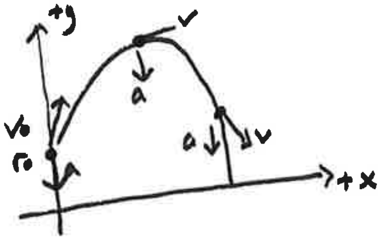
$$\vec{r}(t) = \langle 1, t^2/2 + 2, t^2 \rangle = \hat{i} + (t^2/2 + 2)\hat{j} + t^2\hat{k}$$

$$\vec{r}(2) = \hat{i} + 4\hat{j} + 4\hat{k} = \langle 1, 4, 4 \rangle$$

*This is the problem in Physics & starting idea for Diff Eq., PDE, Num. Analysis

Projectile Motion

Consider a particle (small so no spin and without air resistance) launched from a starting position $\vec{r}_0$ at velocity $|\vec{v}_0|$, acting only under the influence of gravity.



The only force is gravity: $\vec{F} = -mg\hat{j}$

But $\vec{F} = \frac{dp}{dt} = m\vec{a}$ $\quad m\vec{a} = -mg\hat{j}$

$\vec{a} = -g\hat{j}$

So note acceleration independent of mass (ie all fall equally fast in a vacuum).

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int -g\hat{j}\,dt = \langle 0, -gt\rangle + \langle C_1, C_2\rangle$$

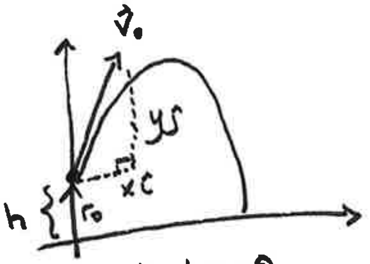
$\vec{v}(0) = \vec{v}_0$ so we have $\vec{v}(0) = \vec{v}_0 = \langle 0, 0\rangle + \langle C_1, C_2\rangle$

$$\vec{v}(t) = \langle 0, -gt\rangle + \vec{v}_0$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 0, -gt\rangle + \vec{v}_0\,dt = \langle 0, -\tfrac{1}{2}gt^2\rangle + \vec{v}_0 t + \vec{D}$$

$$\vec{r}(0) = \vec{r}_0 = \langle 0, 0\rangle + \vec{0} + \vec{D}$$

$$\begin{cases} \vec{a}(t) = -g\hat{j} \\ \vec{v}(t) = -gt\hat{j} + v_0 \\ \vec{r}(t) = -\frac{1}{2}gt^2\hat{j} + t\vec{v}_0 + \vec{r}_0 \end{cases}$$



$x = |v_0| \cos\theta$

$y = |v_0| \sin\theta$

Break $\vec{v}_0$ into components; do the same for r

$$\vec{v}_0 = v_0 \cos\theta\,\hat{i} + v_0 \sin\theta\,\hat{j} = \langle v_0 \cos\theta, v_0 \sin\theta\rangle$$

*Check $|\vec{v}_0| = v_0$

Plugging in for $\vec{r}(t)$:

$$\boxed{\vec{r}(t) = (v_0 \cos\theta)t\,\hat{i} + \left(h + (v_0 \sin\theta)t - \tfrac{1}{2}gt^2\right)\hat{j}}$$

ie, $\begin{cases} x = (v_0 \cos\theta)t \\ y = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \end{cases}$

One can check that $x(t)$, $y(t)$ satisfy the equation of some parabola, ie the motion is parabolic.

When does the projectile hit the ground?

This is when $y=0$. Write

$$0 = \left(-\tfrac{1}{2}g\right)t^2 + (v_0 \sin\theta)\,t + h$$

One can then use the quadratic equation to find this time (take $t>0$ root). This is simpler if $h=0$, ie start from ground.

$$0 = \left(\tfrac{-1}{2}g\right)t^2 + (v_0 \sin\theta)\,t$$

$$0 = t\left(\tfrac{-1}{2}g\,t + v_0 \sin\theta\right)$$

$$\underbrace{t=0}_{\text{Naturally}} \quad \text{or} \quad t = \frac{2v_0 \sin\theta}{g}$$

How far does the projectile travel (horz.) in this time?

General case: Plug solution into $\vec{x}(t)$ (assuming $\vec{x}(0)=0$).

For $h=0$, this is $x = (v_0 \cos\theta)\left(\frac{2v_0 \sin\theta}{g}\right) = \frac{v_0^2 \sin 2\theta}{g}$

What initial ∡ maximizes the distance? [Expect 45°?]

In case $h=0$: $x = \frac{v_0^2 \sin 2\theta}{g}$. This is greatest when $\sin 2\theta$ greatest, ie $\sin\sim = 1$, ie $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

* Note: To find total distance traveled (not horz. dist.) we need arc length from (in case where $h=0$) $t=0$ to $t = \frac{2v_0 \sin\theta}{g}$.

* The applications are endless ! See derivation of Kepler's laws.

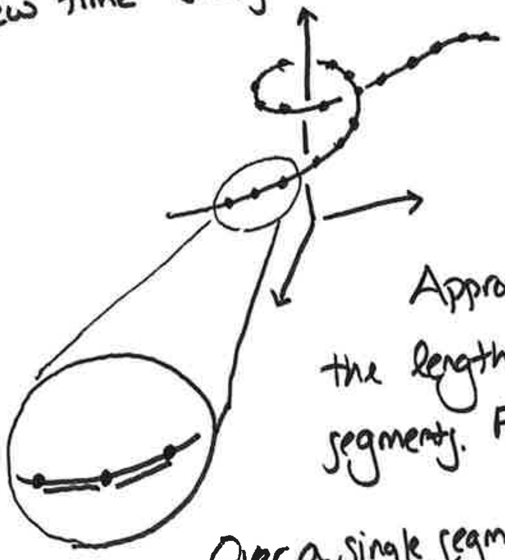
Arc length

Recall from Calculus II:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

This came from $\sqrt{(dx)^2 + (dy)^2}$. We show this idea works for more generally:

Break a curve up into small pieces by looking where object is located every few time 'clicks'



Approximate the total length by the length of all these small straight line segments. For simplicity, take same Δt.

Over a single segment, the distance traveled Δs_i, is...

distance moved:
velocity × time
$x' \times \Delta t$ →

$$
\begin{aligned}
s_i &= \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \\
&\approx \sqrt{(x'\Delta t)^2 + (y'\Delta t)^2 + (z'\Delta t)^2} \\
&= \sqrt{(x')^2 + (y')^2 + (z')^2}\quad \Delta t
\end{aligned}
$$

So $L \approx \sum_i s_i = \sum_i \sqrt{(x')^2 + (y')^2 + (z')^2}\,\Delta t$. Taking limit as $\Delta t \to 0$, we have...

$$L = \int_{t_0}^{t_1} |\vec{r}\,'(t)|\ dt = \int_{t_0}^{t_1} |\vec{x}\,'(t)|\ dt$$

Ex: Find the arclength of the helix:

$\vec{x}(t) = \langle r\cos t + x_0, r\sin t + y_0, Ct + z_0 \rangle$; $0 \le t \le 2\pi$

$\vec{x}'(t) = \langle -r\sin t, r\cos t, C \rangle$

$|\vec{x}'(t)| = \sqrt{(-r\sin t)^2 + (r\cos t)^2 + C^2} = \sqrt{r^2 + C^2}$

$L = \int_0^{2\pi} |x'(t)|\,dt = \int_0^{2\pi} \sqrt{r^2 + C^2}\,dt = 2\pi\sqrt{r^2 + C^2}$

Ex: Find the arclength of...

$\vec{r}(t) = t^2\,\hat{i} + \frac{2}{3}(2t+1)^{3/2}\,\hat{j}$; $0 \le t \le 4$

$\vec{r}'(t) = \langle 2t, (2t+1)^{1/2} \cdot 2 \rangle$

$$
\begin{aligned}
|\vec{r}'(t)| &= \sqrt{(2t)^2 + (2(2t+1)^{1/2})^2} \\
&= \sqrt{4t^2 + 4(2t+1)} \\
&= \sqrt{4t^2 + 8t + 4} \\
&= \sqrt{4(t^2 + 2t + 1)} \\
&= 2\sqrt{(t+1)^2} \\
&= 2(t+1)
\end{aligned}
$$

$$
\begin{aligned}
L = \int_0^4 |\vec{r}'(t)|\,dt &= \int_0^4 2(t+1)\,dt \\
&= 2\int_0^4 (t+1)\,dt \\
&= 2\left(\frac{t^2}{2} + t\right)\Big|_0^4 \\
&= 2\left[\left(\frac{4^2}{2} + 4\right) - 0\right] \\
&= 24
\end{aligned}
$$

Differential Geometry: Moving Frames & Curvature

For simplicity, throughout this section, assume $\vec{x}(t)$, $\vec{F}(t)$ are smooth.

Normalize the velocity vector (which points in direction of motion)

Unit Tangent Vector: $\boxed{\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{x}'(t)}{|\vec{x}'(t)|}}$

Two special facts:

1) $\frac{d\vec{T}}{dt}$ is $\perp$ $\vec{T}$ for all t.
2) $\left|\frac{d\vec{T}}{dt}\right|$ is the angular rate of change in the direction of $\vec{T}$.

Using this, we define...

Principal Unit Normal Vector: $\boxed{\vec{N}(t) = \frac{T'(t)}{|T'(t)|}}$

*Note: $\vec{N}(t) \neq \frac{x''(t)}{|x''(t)|}$!

* Using fact (5) from earlier and $|\vec{T}| = \vec{T}\cdot\vec{T} = 1$, we know $\vec{T}\cdot\vec{T}' = 0$. But then $\vec{T} \perp \vec{N}$ (hence the name).

* In 2D, $\vec{N}(t)$ points toward the concave side of the curve

* $\vec{N}(t)$ is the direction a curve is turning.

Ex: Find $\vec{T}(t)$ and $\vec{N}(t)$ for the helix...

$$\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + t\,\hat{k}$$

Sketch the results.

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 1\rangle$

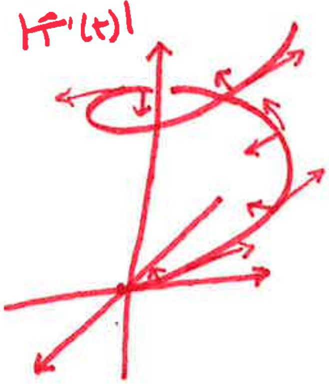
$|\vec{r}\,'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2} = \sqrt{5} \leftarrow$ Generally, can be a function of t.

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{1}{\sqrt{5}}\langle -2\sin t, 2\cos t, 1\rangle$

$\vec{T}'(t) = \dfrac{1}{\sqrt{5}}\langle -2\cos t, -2\sin t, 0\rangle$

$|\vec{T}'(t)| = \dfrac{1}{\sqrt{5}}\sqrt{(-2\cos t)^2 + (-2\sin t)^2 + 0^2} = 2/\sqrt{5}$

$\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos t, -\sin t, 0\rangle$



- * Generally, $|\vec{r}\,'(t)|$ and $|\vec{T}'(t)|$ will be functions of t.
- * These problems will tend to be time consuming (even worse when we soon add $\vec{B}$)
- * The geometry of these scenarios are important.

Finally, we can define...

Binormal vector : $\boxed{\vec{B} = \vec{T} \times \vec{N}}$

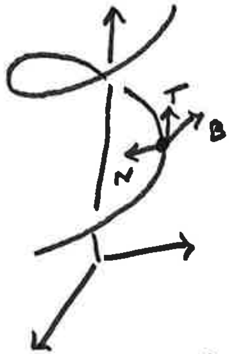
By construction, $\vec{B} \perp$ to $\vec{T}$ and $\vec{N}$. Moreover,

$$|\vec{B}| = |\vec{T}|\,|\vec{N}| \sin\tfrac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1$$

so that it is a unit vector, as are $\vec{T}$ and $\vec{N}$. Then

$$(\vec{T}, \vec{N}, \vec{B})$$

forms a 'right-handed' coordinate frame at the current position.



This is very useful in many real life applications and in animation - to have a frame of reference (coord. system) that moves with an object. But difficult to compute by hand.

Ex: Compute $\vec{B}$ for $\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$, ie the prev. helix.

$$\vec{T}(t) = \langle -2/\sqrt{5} \sin t, 2/\sqrt{3} \cos t, 1 \rangle$$

$$\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

$$\vec{B} = \vec{T} \times \vec{N} = \left\langle t \sin t, -\cos t, \frac{2\cos^2 t + 2 + \sin^2 t}{\sqrt{5}} \right\rangle$$

The plane spanned by $\vec{T}$ and $\vec{N}$ is called the osculating plane and 'instantaneously' contains the path at a point P.

* osculating from Latin osculare, ie "to kiss".

The binormal vector's derivative is // to $\vec{N}$, ie

$$\frac{d\vec{B}}{ds} = c\vec{N}$$

for some scalar c, we write...

$$\frac{d\vec{B}}{ds} = -\tau \vec{N}$$

where τ is the 'torsion', ie the amount the path 'twists' out of the osculating plane. More shortly.

* If $\vec{r}(t)$ position vector and $\vec{N}$ exists, then $\vec{a}(t)$ lies in plane spanned by $\vec{T}$ & $\vec{N}$.

Thus, we can decompose $\vec{a}$ in terms of these components:

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

where,

$$a_T = D_t|\vec{v}| = \vec{a}\cdot\vec{T} = \frac{\vec{v}\cdot\vec{a}}{|\vec{v}|}$$

$$a_N = |\vec{v}||T'| = \vec{a}\cdot\vec{N} = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|} = \sqrt{|a|^2 - a_T^2}$$

This is easier to see if we write $\vec{v} = \dot{s}\vec{T}$, where $s(t) = \int_a^t |r'(t)|\,dt$.

Note $\dot{s} = \frac{d}{dt}\int - dt = |r'| =$ speed. Then...

$$\vec{a} = \vec{v}'(t) = \frac{d}{dt}(\dot{s}\vec{T}) = \ddot{s}\vec{T} + \dot{s}\boxed{\frac{d\vec{T}}{dt}} \rightarrow = \dot{s}\frac{d\vec{T}}{ds} \cdot \dot{s}\kappa\vec{N}$$

$$= \ddot{s}\vec{T} + \kappa\dot{s}^2\vec{N}$$

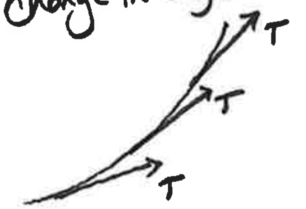
But we need to define κ, which is useful on its own.

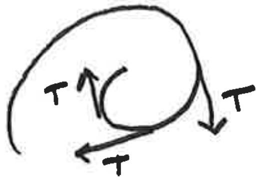
Curvature:

We want to measure how much a curve 'curves'.

$$\boxed{\kappa(t) := \frac{\left|\frac{d\vec{T}}{dt}\right|}{ds/dt} = \left|\frac{d\vec{T}}{ds}\right|}$$

where $s(t) = \int_a^t |x'(\tau)|\,d\tau$. The curvature of a path (in $\mathbb{R}^3$) is the angular rate of change of direction of $\vec{T}$ per unit change in distance along the path.





The $|dT/dt|$ part measures the 'amount of turning'. But we don't care about this alone – it's about 'how long' you took to turn, measured by ds/dt.

Ex: Find the curvature $\kappa(t)$ for the circle $\vec{x}(t) = \langle r\cos t, r\sin t\rangle, 0 \le t \le 2\pi$

$\vec{x}'(t) = \langle -r\sin t, r\cos t\rangle$

$|\vec{x}'(t)| = \sqrt{(-r\sin t)^2 + (r\cos t)^2} = r = \frac{ds}{dt} \leftarrow$ speed

$\vec{T}(t) = \frac{x'(t)}{|x'(t)|} = \langle -\sin t, \cos t\rangle$

$\vec{T}'(t) = \langle -\cos t, -\sin t\rangle$

$\kappa(t) = \frac{|dT/dt|}{ds/dt} = \frac{|\langle -\cos t, \sin t\rangle|}{r} = \frac{1}{r}$

The curvature of a fixed circle is constant, and smaller the larger the circle, which should make sense.

Ex: If $\vec{m}, \vec{b}$ are constant vectors in $\mathbb{R}^3$ (with $\vec{m} \neq \vec{0}$) then...

$$\vec{x}(t) = \vec{m}t + \vec{b}$$

is a line. Then $\vec{x}'(t) = \vec{m}$, $\frac{ds}{dt} = |\vec{m}|$ so that $\vec{T}(t) = \frac{\vec{m}}{|\vec{m}|}$, which is constant. Then $\vec{T}'(t) \equiv \vec{0}$ so that $\kappa = 0$, which matches our intuition that a line does not curve.

Using $\vec{v} = \dot{s}\vec{T}$ and $\vec{a} = \ddot{s}\vec{T} + \kappa \dot{s}^2 \vec{N}$, we have...

$$\boxed{\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}}$$

In the special case of $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$, we have...

$$\boxed{\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{|\dot{x}^2 + \dot{y}^2|^{3/2}}}$$

One can also show $\tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}$, $r' \times r'' = \kappa (s')^3 \vec{B}$, and $r''' = (s''' - \kappa^2 (s')^3)\vec{T} + (3\kappa s' s'' + \kappa'(s')^2)\vec{N} + \kappa\tau(s')^3 \vec{B}$

Finally, with a lot more computation, one can derive...

Frenet-Serret Formulas

$$\begin{cases} \vec{T}'(s) = \kappa \vec{N} \\ \vec{N}'(s) = -\kappa\vec{T} + \vec{B} \\ \vec{B}'(s) = -\tau\vec{N} \end{cases} \qquad \begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}$$

§11.1: Multivariable Functions

Topics

- Multivariable functions
- Level curves
- Domains

You Should Be Able to...

- Perform computations with multivariable functions
- Sketch level curves for functions
- Sketch multivariable functions
- Graphically interpret contour plots
- Estimate values from contour plots
- Find and sketch domains for multivariable functions
- Find functions with specified domains.

The next class of functions we will see are multivariable functions:

$$f : \mathbb{R}^n \to \mathbb{R}$$

That is, $f(x_1, x_2, \dots, x_n)$ takes a point $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ and outputs a real number.

Ex: $f(l, w, h) = lwh$ ⇝ Volume of box

Ex: $f(\text{hours}, \text{pay}, \text{taxes}, \text{spent}) = \text{hours}(\text{pay}) - \text{taxes} - \text{spent}$ is your weekly net income

$$\vdots$$

Shows that these are the 'real life' functions (even if we cannot write them down explicitly).

Ex: $z = f(x,y) = \dots$: Many surfaces can take this form.

$f(x,y) = x+y$ ⇝ write $z = f(x,y)$, ie $z(x,y)$, then $z = x+y$ so $x+y-z=0$ is a plane.

Ex: $f(x,y) = \sqrt{4-x^2-y^2}$ ⇝ $z = \sqrt{4-x^2-y^2}$ ⇝ $x^2+y^2+z^2=4$ is a sphere

Multivariable functions are operated on in the obvious way:

Ex: $f(x,y,z) = x+y-z$

$g(x,y,z) = x+y^2-\sin z$

$f(3,2,1) = 3+2-1=4$

$f(x,y,z) - g(x,y,z) = (x+y-z) - (x+y^2-\sin z) = y - y^2 - z + \sin z$

$$\frac{f(x,y,z)}{g(x,y,z)} = \frac{x+y-z}{x+y^2-\sin z}$$

Two types of such functions we will refer to:

Polynomial: Sums / differences of products of nonnegative powers of the variables.

Ex: $f(x) = x^2 + x - 1$

$f(x,y) = x^3 + y^3 + 4$

$f(x,y) = y^3 + xy^2 - yx^2 + x + y - 3$

$f(x,y,z) = x^3 z^2 - xy^3 + xyz + 5$

$f(x,y,z,w) = (x+w+z)^3 - (y+z+w)^4$

Rational Functions: Quotients of polynomial functions

Ex: $f(x) = \frac{x+1}{x-1}$

$f(x,y) = \frac{x+y-1}{x+y+3}$

$f(x,y,z) = \frac{x^2 + xy - y^2}{x + y - 5}$

We can draw level curves with these functions:

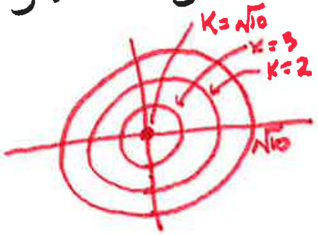
Sketches also called contour plots

Level Curve: Curves with the equation $f(x,y) = K$ or $f(x,y,z) = K$ for some fixed K. (in the range of f).

Student Ex: Sketch level curves for $f(x,y) = \sqrt{10 - x^2 - y^2}$. Sketch $f(x,y)$.

$K = \sqrt{10 - x^2 - y^2}$

$x^2 + y^2 = 10 - K^2$

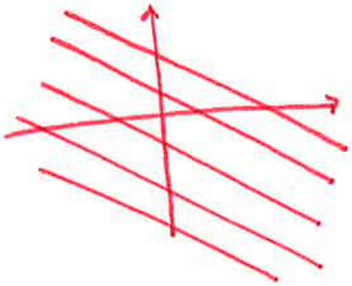


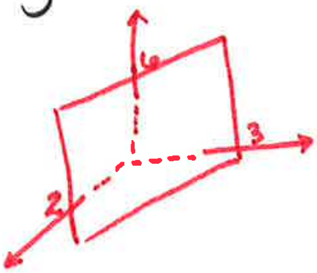
Student Ex: Sketch level curves for $f(x,y) = 6-3x-2y$. Sketch $f(x,y)$

$K = 6-3x-2y$

$2y = -3x + (6-K)$

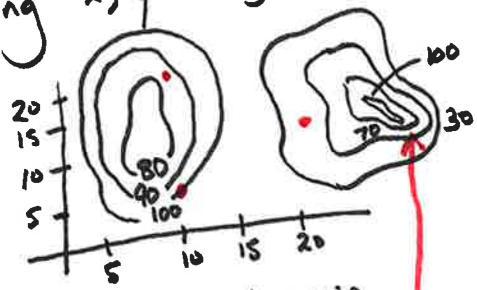
$y = -\frac{3}{2}x + \frac{6-K}{2}$





Level curves should remind one of temperature/pressure maps or topographic maps. The lines are called isotherms/isobars or equipotential lines. Lines are closest where the gradient ('steepness') is greatest.

Student Ex: Below is a level curve map for $f(x,y)$, the income for producing x, y units of certain products. Approximate the following:



Where is the change in income most extreme?

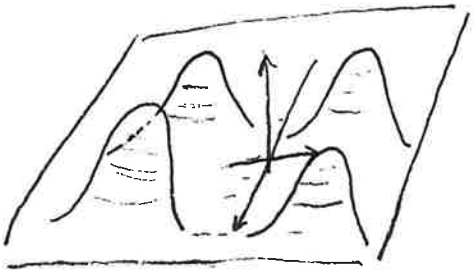
$f(10,20) \approx 85$

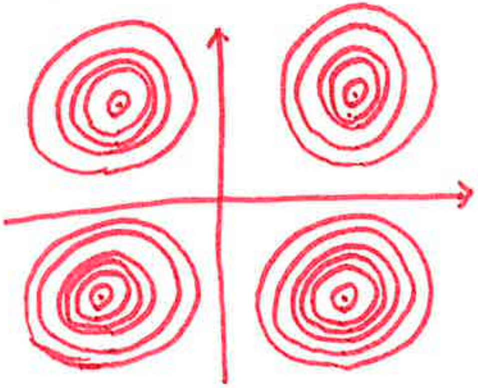
$f(10,5) \approx 100$

What happens if you double production to 20, 10?

$f(20,10) \approx 50$. Cuts in half.

Student Ex: Sketch level curves for the following surface.





For now, we will be most interested in the domains of these functions. Recall the domains of the following.

Polynomials: Any value

'Rational' Functions: Anywhere except where the den. is zero.

$\ln(x)$: $x > 0$

$\sqrt{x}$: $x \geq 0$

$\sin x, \cos x$: Any value

$\arcsin x, \arccos x$: $-1 \leq x \leq 1$

$\vdots$

Student Ex: Find the domain for $f(x,y) = x^2 + xy + y - 4$. Sketch the domain.

Any $x,y \rightsquigarrow \mathbb{R}^2$

Student Ex: Find the domain for $f(x,y) = \frac{\sin(xy)}{x^2+y^2}$. Sketch the domain.

$\sin(xy)$: Fine for all x,y

$x^2+y^2 \neq 0$ so x and y cannot both be 0, ie $(x,y) \neq (0,0)$

$\mathbb{R} \setminus \{(0,0)\}$

Student Ex: Find the domain for $f(x,y,z) = \sqrt{1-x^2-y^2-z^2}$. Sketch the domain.

Need $1-x^2-y^2-z^2 \geq 0$

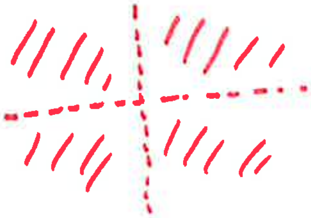
$x^2+y^2+z^2 \leq 1$

Solid sphere at $(0,0,0)$ with radius 1.

Student Ex: Find and sketch the domain for $f(x,y) = \frac{x+y}{xy}$

$x+y$: Any x,y

xy: $xy \neq 0$
so $x \neq 0$ or $y \neq 0$

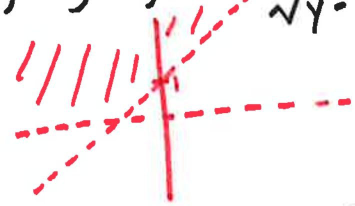


plane without axes.

Student Ex: Find and sketch the domain for $f(x,y) = \frac{x + \ln y}{\sqrt{y-x-1}}$

$x + \ln y$: 'Fine' if ln – is 'fine' so $y>0$

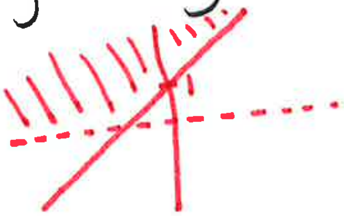
$\sqrt{y-x-1}$: $\sqrt{y-x-1} \neq 0$ and
$y-x-1 \geq 0 \rightarrow y-x-1 > 0$
$y > x+1$



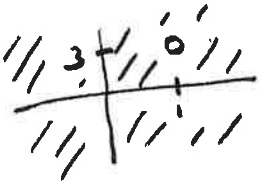
Student Ex: Find and sketch the domain for $f(x,y) = x + \ln y + \sqrt{y-x-1}$

$x + \ln y$: 'Fine' if ln – 'fine' so $y>0$

$\sqrt{y-x-1}$: $y-x-1 \geq 0$
$y \geq x+1$

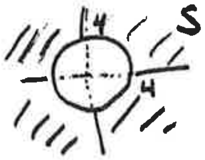


Student Ex: Construct a function with the following region for a domain



Infinitely many answers. Here is one:

$$f(x,y) = \frac{1}{(x-1)^2 + (y-3)^2}$$

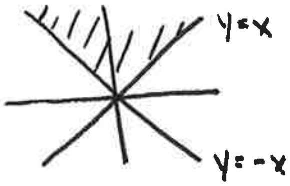


Student Ex: Construct a function with S as its domain

Want to exclude things inside $x^2+y^2=16$. Infinitely many answers. One possible answer.

$$f(x,y) = \sqrt{x^2+y^2-16}$$

Student Ex: Construct a function with W as its domain



Want $y \geq x$ and $y \geq -x$

Infinitely many answers, one such is...

$$f(x,y) = \sqrt{y-x} + \sqrt{y+x}$$

§ 11.2 : Limits and Continuity

Topics

- Limit definition
- Limit methods
- Continuity

You Should Be Able to...

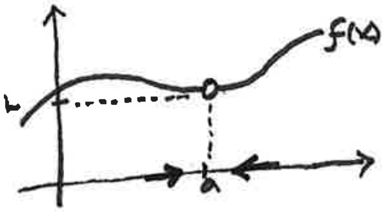
- Intuitively define a limit and describe it graphically
- Evaluate limits by 'plug'n chug'
- Evaluate limits by algebraic 'tricks'
- Show limits do not exist using paths.
- Evaluate limits using polar coordinates
- Show a limit exists by using the Squeeze Theorem
- Explain what a continuous function is.
- Find where a function is continuous
- Extend (when possible) a discontinuous function to a continuous one.

Recall from Calculus I:

$$\lim_{x \to a} f(x) = L$$

if $f(x)$ gets 'close' to L as x gets 'close' to a.

* $f(x)$ need not be defined at $x = a$



$\lim_{x \to a} f(x) = L$

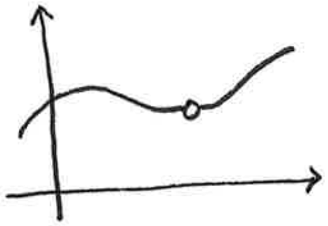
Recall also that if $\lim_{x \to a} f(x)$ exists, then...

- $\lim_{x \to a^+} f(x)$ exists
- $\lim_{x \to a^-} f(x)$ exists
- $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

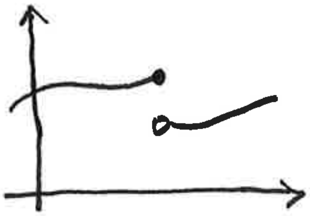
We said that $f(x)$ is continuous at $x = a$ if...

- $f(a)$ is defined
- $\lim_{x \to a} f(x) = f(a)$ ← so all the above hold

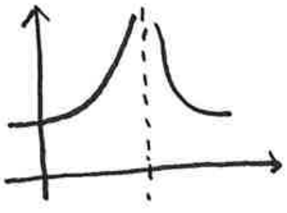
If not, we say that $f(x)$ is discontinuous at $x = a$.



Removable discont.



Jump discontinuity



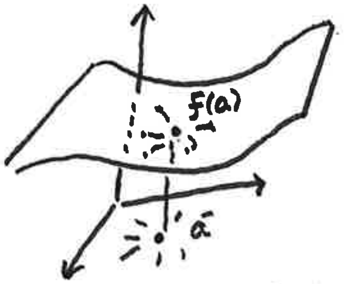
Infinite discontinuity

We need a more general definition.

Limit: Let $\vec{f}: X \subseteq \mathbb{R}^n \to \mathbb{R}$. We write $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = L$, if for all $\epsilon > 0$, there exists $\delta > 0$ (possibly depending on ϵ) so that for all $|\vec{x} - \vec{a}| < \delta$, we have $|f(\vec{x}) - L| < \epsilon$

Okay..... what?

This just means the following: $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ exists if for any point 'near' $\vec{a}$, $\vec{f}(\vec{x})$ is 'close to' L.



$$\lim_{(x,y)\to a} f(x,y) = f(a)$$

Notice no matter how $\vec{x} \to \vec{a}$ in the plane, the values $f(\vec{x})$ 'slide toward' $f(\vec{a})$.

The situation is far more complicated in $\mathbb{R}^n$ than in $\mathbb{R}^1$ because there are far more ways to approach $\vec{x} = \vec{a}$. Whereas before, there were only 2 (left & right). The general methods stay the same:

1) Plug 'n Chug
2) Algebra prestidigitation
3) Paths
4) Squeeze Theorem

$\downarrow$ order to try in

* Note: All traditional limit operations, eg $\lim(f+g) = \lim f + \lim g$, hold under the same conditions as in Calculus I.

Plug 'n Chug

If you can evaluate the function at $\vec{x}=\vec{a}$, then this is the limit. [Not generally true, but will mostly be for us as our functions will mostly have, at most, a removable discontinuity and are otherwise continuous.]

Ex: $\lim\limits_{(x,y)\to(1,3)} (x+2y-3) = 1+2(3)-3 = 4$

Ex: $\lim\limits_{(x,y,z)\to(1,-1,0)} \dfrac{x+y+\sin(xz)}{x\cos(yz)} = \dfrac{1-1+\sin 0}{\cos(0)} = \dfrac{1-1+0}{1} = 0$

Student Ex: $\lim\limits_{(x,y)\to(\pi,0)} \dfrac{\sin x-\cos(xy)}{x^2+xy-y} = \dfrac{\sin\pi-\cos 0}{\pi^2+0-0} = \dfrac{0-1}{\pi^2} = -1/\pi^2$

Algebra Prestidigitation

Ex: $\lim\limits_{(x,y)\to(0,0)} \underbrace{\dfrac{x^2-y^2}{x+y}}_{0/0 \text{ undef.}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)\cancel{(x+y)}}{\cancel{x+y}} = \lim\limits_{(x,y)\to(0,0)} (x-y) = 0$

Ex: $\lim\limits_{(x,y)\to(1,2)} \dfrac{xy-y}{x-1} = \lim\limits_{(x,y)\to(1,2)} \dfrac{y\cancel{(x-1)}}{\cancel{x-1}} = \lim\limits_{(x,y)\to(1,2)} y = 2$

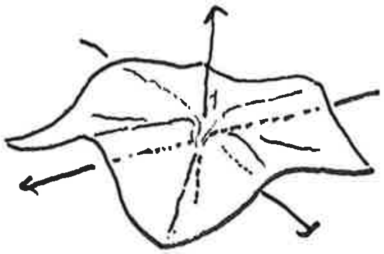
Student Ex: $\lim\limits_{(x,y)\to(2,1)} \dfrac{x^3-2x^2y+xy^2-2y^3}{x-2y} = \lim\limits_{(x,y)\to(2,1)} \dfrac{x^2(x-2y)+y^2(x-2y)}{x-2y}$

$= \lim\limits_{(x,y)\to(2,1)} \dfrac{(x^2+y^2)(x-2y)}{x-2y}$

$= \lim\limits_{(x,y)\to(2,1)} (x^2+y^2) = 2^2+1^2 = 5$

Paths

Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ This looks like a 'factor-cancel' method will work here but it does not. We plot the function



There looks to be a 'drop off'. Some values are 'high' and others 'low' near (0,0). But for limit to exist, must be same no matter how you get 'near' (0,0). So perhaps there is no limit? Perhaps we can show this by trying different approaches to (0,0).

Along $y=x$ toward (0,0):

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{'='} \lim_{x\to 0} \frac{x^2-x^2}{x^2+x^2} = 0$$

Along x-axis: $(y=0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{'='} \lim_{x\to 0} \frac{x^2-0}{x^2+0} = 1$$

Along y-axis: $(x=0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{'='} \lim_{y\to 0} \frac{0-y^2}{0+y^2} = -1$$

The limit depends on the approach; therefore, the limit does not exist.

* '=' Because this is a test, <u>if</u> limit exists, must have = (because same for all directions)

* Be sure your path actually ends at the point you are wanting to approach!!

Student Ex: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Along x-axis $(y=0)$: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1$

Along y-axis $(x=0)$: $\lim\limits_{(x,y)\to(0,0)} \frac{0}{0+y^2} = 0$

Along $y=x$: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}$

Therefore, the limit does not exist.

* As soon as you get <u>any</u> 2 mismatching values, the limit DNE. (Just be sure they are correct).

Student Ex: $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{2x^2+3y^2+z^2}{x^2+y^2+z^2}$

Along x: $\lim\limits_{(x,y,z)\to 0} \dfrac{2x^2}{x^2} = 2$

Therefore, the limit does not exist.

Along y: $\lim\limits_{(x,y,z)\to 0} \dfrac{3y^2}{y^2} = 3$

Along z: $\lim\limits_{(x,y,z)\to 0} \dfrac{z^2}{z^2} = 1$

You may need to experiment a bit with paths as examples can be pathological. For example, there are functions whose limit does not exist, even when it exists along every line to the point.

Ex: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

Every line in the plane (including the x-axis + y-axis) can be written in the form $(Mt+x_0,\ Nt+y_0)$ for some M, N, x_0, y_0

For example, $y=2x+1 \rightsquigarrow (t, 2t+1)$, $x=3 \rightsquigarrow (3,t)$, etc.

Then every line through the origin can be written (Mt, Nt) (so $t=0$ gives $(0,0)$).

Along line (Mt, Nt): $\lim\limits_{t\to 0} \dfrac{(Mt)^2(Nt)}{(Mt)^4+(Nt)^2} = \lim\limits_{t\to 0} \dfrac{M^2Nt^3}{M^4t^4+N^2t^2} =$

$$\lim_{t\to 0} \frac{M^2Nt^3}{t^2(M^4t^2+N^2)} = \lim_{t\to 0} \frac{M^2Nt}{M^4t^2+N^2} = 0$$

But along the parabola $y=x^2$: (t, t^2) & $t\to 0$

$$\lim_{t\to 0} \frac{(t)^2\cdot(t^2)}{(t)^4+(t^2)^2} = \lim_{t\to 0} \frac{t^4}{t^4+t^4} = 1/2$$

So that the limit does not exist.

Remember, it's about ending up at the final point. It does not (or at least should not) matter how you get there or how you think of it. So no need to think 'Cartesian-ally'.

Ex: Use polar coordinates to evaluate...

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

$x = r\cos\theta$
$y = r\sin\theta$

$r^2 = \underbrace{x^2+y^2}_{(r\cos\theta)^2+(r\sin\theta)^2 = r^2}$

$$\lim_{r\to0} \frac{(r\cos\theta)^2(r\sin\theta)}{r^2} = \lim_{r\to0} \frac{r^3\sin\theta\cos\theta}{\underbrace{r^2}} = 0$$

independent of what θ is or how it varies.

Squeeze Theorem

The squeeze theorem holds just as in Calculus I: if $\lim_{\vec{x}\to\vec{a}} f = \lim_{\vec{x}\to\vec{a}} h = L$ and $f \le g \le h$, then $\lim_{\vec{x}\to\vec{a}} g = L$

Ex: $$\lim_{(x,y)\to(0,0)} \frac{y^3\cos^2x}{x^2+y^2}$$

$|\text{anything}| \ge 0$

$|\cos(\text{anything})| \le 1$

$x^2+y^2 \ge x^2$ & $x^2+y^2 \ge y^2$, so

$\frac{1}{x^2+y^2} \le \frac{1}{x^2}$ & $\frac{1}{x^2+y^2} \le \frac{1}{y^2}$

Then we must have

$$\underbrace{0}_{f} \le \left|\frac{y^3\cos^2x}{x^2+y^2}\right| \le \frac{y^3\cdot 1}{y^2} = \underbrace{y}_{h}$$

$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} y = 0$. Therefore, $\lim_{(x,y)\to(0,0)} \frac{y^3\cos^2x}{x^2+y^2} = 0$

* Trying many paths will all result in 0, which hints, but does not prove, that the limit is actually 0.

Student Ex: Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$.

We have seen this using polar coordinates. But we use Squeeze Theorem.

$$x^2+y^2 \geq x^2$$

$$\frac{x^2+y^2}{x^2+y^2} \geq \frac{x^2}{x^2+y^2}$$

$$\frac{x^2}{x^2+y^2} \leq 1$$

$$0 \leq \left|\frac{x^2y}{x^2+y^2}\right| = |y|\left|\frac{x^2}{x^2+y^2}\right| \leq |y|\cdot 1 = |y|$$

$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} |y| = 0$. Therefore by the Squeeze Theorem

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$$

Continuity

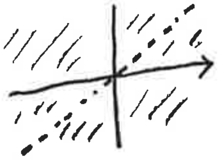
Continuity is defined just as in Calculus I: $f(\vec{x})$ is cont. at $\vec{x}=\vec{a}$ if $f(\vec{a}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x})$.

Ex: Where is $f(x,y,z) = x+y^2-z+3$ cont?

$f(x,y,z)$ is defined on all of $\mathbb{R}^3$ and $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$ for all $\vec{a}\in\mathbb{R}^3$. So $f(x,y,z)$ is everywhere continuous.

Ex: Where is $f(x,y) = \frac{x+y}{x-y}$ cont?

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ for all $x\neq y$. Along $x=y$, the function is not defined. So f is cont. on $\mathbb{R}^2 \setminus \{(x,y): x=y\}$



We can also extend discontinuous functions to continuous ones just as in Calculus I.

Ex: Let $f(x,y) = \frac{x^2 y}{x^2+y^2}$. Find a function $g(x,y)$ so that $g(x,y) = f(x,y)$ whenever $f(x,y)$ is defined and so that $g(x,y)$ is everywhere continuous.

We know $f(x,y)$ is defined (and continuous) for all (x,y) with $(x,y) \neq 0$. We have also seen that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Then...

$$g(x,y) = \begin{cases} f(x,y) = \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is such a function

All the traditional Calculus I continuity theorems hold:

Suppose $f(\vec{x}), g(\vec{x})$ are continuous on $\mathbb{R}^n$, then...

1) $f(\vec{x}) \pm g(\vec{x})$ is continuous
2) $a f(\vec{x})$ is continuous for all $a \in \mathbb{R}$
3) $f(\vec{x}) g(\vec{x})$ is continuous
4) $\frac{f(\vec{x})}{g(\vec{x})}$ is continuous for all $\vec{x} \in \mathbb{R}^n$ with $g(\vec{x}) \neq 0$
5) $f(g(\vec{x}))$ is continuous.

§ 11.3: Partial Derivatives

Topics

- Partial Derivatives
- Higher partials
- Differentiability theorems
- The derivative
- Tangent plane
- Implicit Differentiation

You Should Be Able to...

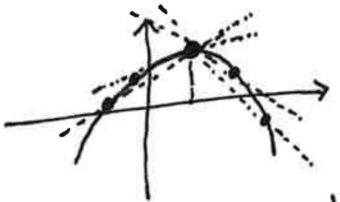
- Use the definition to compute a partial derivative
- Compute partial derivatives using the 'shortcuts'
- Compute higher order partials.
- Explain what smooth means and 'see' it graphically
- Explain what f_x, f_y, f_z are and their graphical interpretation.
- Explain what f_x, f_y, and f_z mean in context.
- Approximate f_x, f_y, and f_z from a picture.
- Differentiate implicitly.
- Explain the general notion of the derivative.
- Identify the equation of the tangent plane.
- Know the basic counterexamples for differentiability.

As in Calculus I, we want a notion of rate of change for a function. Recall that before we had...

$$f'(x) = \frac{df}{dx} := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\leftarrow (x+h) - x = h$

The limit of the slopes of 'converging' secant lines



But this depended on only the change in one direction - x-direction. As always, things are more complicated because we have many possible directions. In fact, there will be 3 notions of rate of change for us:

1) (Total) Derivative
2) Partial Derivative
3) Directional Derivative

These all interact and are different in subtle ways. We will discuss, but not worry ourselves, over these nuances. We will first discuss the partial derivative.

Partial Deriv. (with respect to x_i): Denoted $\frac{\partial f}{\partial x_i}$, $D_{x_i} f(x_1, \dots, x_n)$ or $f_{x_i}(x_1, \dots, x_n)$, is

$$\frac{\partial f}{\partial x_i} := \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

provided the limit exists.

For simplicity, Consider the case in $\mathbb{R}^2$:

$$\frac{\partial f}{\partial x} := \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} := \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Ex: Let $f(x,y)= x^2+y+3$. Find (using the definition) $f_x(1,2)$ and $f_y(1,2)$.

First, note that $f(1,2)=1^2+2+3=6$

$$f_x(1,2)=\lim_{h\to 0}\frac{f(1+h,2)-f(1,2)}{h}=\lim_{h\to 0}\frac{((1+h)^2+2+3)-6}{h}=\lim_{h\to 0}\frac{h^2+2h}{h}=2$$

$$f_y(1,2)=\lim_{h\to 0}\frac{f(1,2+h)-f(1,2)}{h}=\lim_{h\to 0}\frac{(1^2+(2+h)+3)-6}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

OR

$$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\lim_{h\to 0}\frac{((x+h)^2+y+3)-(x^2+y+3)}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2+y+3-x^2-y-3}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}=2x$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)}=\left.2x\right|_{(1,2)}=2$$

$$\frac{\partial f}{\partial y}=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}=\lim_{h\to 0}\frac{(x^2+(y+h)+3)-(x^2+y+3)}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)}=\left.1\right|_{(1,2)}=1$$

But life is too short to always compute partial derivatives using this limit definition. Luckily, there is a shortcut. In fact, it is the same shortcut (holding the other variables constant) as in Calculus I: derivative tricks !

Ex: $f(x,y) = x^2 + y^2$

$f_x = 2x$; $f_y = 2y$

f_x you 'see' $x^2 + \square$

f_y you 'see' $\square + y^2$

Ex: $f(x,y) = x^3 + xy^2 + 2$

$f_x = 3x^2 + y^2$

$f_y = 2xy$

f_x you 'see' $x^3 + x\square + \square$

f_y you 'see' $\square + \square y^2 + \square$

Ex: $f(x,y,z) = xz + xy + yz$

$f_x = z + y$

$f_y = x + z$

$f_z = x + y$

f_x you 'see' $x\square + x\square + \square$

f_y you 'see' $\square + \square y + y\square$

f_z you 'see' $\square z + \square + \square z$

Student Ex: $f(x,y) = x\sin(xy)$

$\frac{\partial f}{\partial x} = \sin(xy) + xy\cos(xy)$

$\frac{\partial f}{\partial y} = x^2\cos(xy)$

f_x you 'see' $x\sin(x\square)$

f_y you 'see' $\square\sin(\square y)$

* This shows f_x, f_y, f_z will not always involve the same derivative rules.

Student Ex: $f(x,y) = x e^{x^2 y}$

$$f_x = e^{x^2y} + x e^{x^2y}\cdot 2xy = e^{x^2y} + 2x^2y e^{x^2y} = e^{x^2y}(1+2x^2y)$$

$$f_y = x^3 e^{x^2y}$$

Student Ex: $f(x,y) = \frac{x+y}{x+1}$

$$f_x = \frac{(x+1)\cdot 1 - 1(x+y)}{(x+1)^2} = \frac{x+1-x-y}{(x+1)^2} = \frac{1-y}{(x+1)^2}$$

$$f_y = \frac{1}{x+1}$$

Student Ex: $f(x,y,z) = xy + z^2 - \cos(xz) + \ln(xz)$

$$\frac{\partial f}{\partial x} = y + z\sin(xz) + \frac{1}{xz}\cdot z = y + z\sin(xz) + \frac{1}{x}$$

$$\frac{\partial f}{\partial y} = x$$

$$\frac{\partial f}{\partial z} = 2z + x\sin(xz) + \frac{1}{xz}\cdot x = 2z + x\sin(xz) + \frac{1}{z}$$

Student Ex: $f(x,y,z) = xz\sqrt{\ln(xy)} + e^z\sin y$

$$f_x = z\sqrt{\ln(xy)} + xz\cdot\frac{1}{2}(\ln(xy))^{-1/2}\cdot\frac{1}{xy}\cdot y = z\sqrt{\ln(xy)} + \frac{z}{2\sqrt{\ln(xy)}} = \frac{z+2z\ln(xy)}{2\sqrt{\ln(xy)}}$$

$$f_y = xz\cdot\frac{1}{2}(\ln(xy))^{-1/2}\cdot\frac{1}{xy}\cdot x + e^z\cos y = \frac{xz}{2y\sqrt{\ln(xy)}} + e^z\cos y$$

$$f_z = x\sqrt{\ln(xy)} + e^z\sin y$$

As in Calculus I, there are higher order derivatives. However, we have many more combinations due to the number of variables.

$$f_{xx} \rightsquigarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{xy} \rightsquigarrow \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yzx} \rightsquigarrow \frac{\partial^3 f}{\partial x\, \partial z\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right)$$

$$\vdots$$

∗ Note the difference in the order in the two notations.

$\dfrac{\partial^n f}{\underbrace{\partial\, \partial\partial\cdots}_{\text{'reverse' order}}}$ due to the "$\underbrace{\text{mult}}_{\text{not really}}$" in den. $\underbrace{f_{\cdots}}_{\text{inorder}}$

Student Ex: Let $f(x,y,z) = ye^x + x\ln z$

$f_x = ye^x + \ln z$

$f_y = e^x$

$f_z = \frac{x}{z}$

$f_{xy} = e^x$

$f_{yx} = e^x$

$f_{xz} = \frac{1}{z}$

$f_{zx} = \frac{1}{z}$

$f_{yz} = 0$

$f_{zy} = 0$

$f_{xzz} = \frac{-1}{z^2}$

$f_{zxz} = \frac{-1}{z^2}$

$f_{zzx} = \frac{-1}{z^2}$

Notice in this previous example,

$$f_{xx} = f_{zx} \; ; \; f_{xy} = f_{yx} \; ; \; f_{yz} = f_{zy} \; ; \; f_{xzz} = f_{xzx} = f_{zxx}$$

This is not a coincidence.

Clairaut's Theorem: Suppose $f(x,y)$ is defined on a disk D containing (a,b), then if f_{xy} and f_{yx} are continuous on D, we have...

$$f_{xy}(a,b) = f_{yx}(a,b)$$

This generalizes to more variables and higher derivatives.

So you can compute higher derivatives in 'any' order

that meet this theorem.

* By any, any

* Be sure you compute the correct derivative.

Ex: To compute f_{xyxy}, compute f_{xxyy} or f_{xyyx} or f_{yyxx} or .... (assuming f meets the theorem req.) but you could not use f_{xxxx} or f_{xxxy} or f_{xyyy} or...

* Use this technique to compute derivatives in the simplest way possible.

Ex: $f(x,y) = x \dfrac{y^3 \sin y}{e^y + \sin(\ln y)}$. Compute f_{yyxx}.

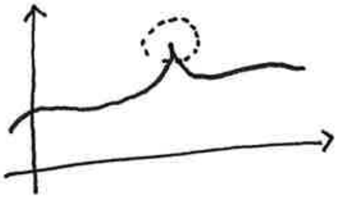
In the given order, you have nightmarish quotient rules. But we can compute f_{xxyy} and this is clearly 0. So....

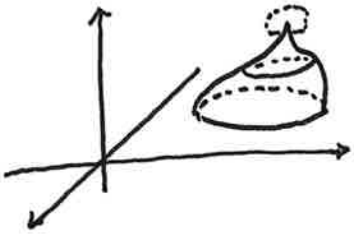
$$f_{yyxx} = f_{xxyy} = 0$$

The following definition will be important later in the course.

Class C^k | Smooth: A function $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be of class C^k if all the partials of order $\leq K$ exist and are continuous. We say f is smooth, or of class C^∞, if all partials of all orders exist and are continuous. A function $f: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k if and only if its components are C^k.

Essentially, this prevents 'pinch' points on curves | surfaces.

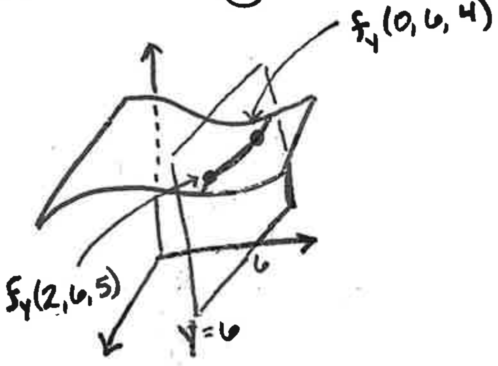




What do f_x, f_y, and f_z mean?

The partials f_x, f_y, and f_z are the slopes | rates of change of the function $f(x,y,z)$ in the x, y, and z, ie $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, directions.

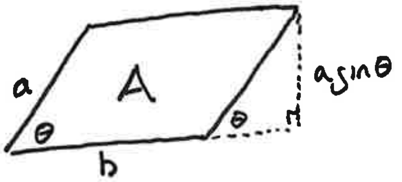
So if we intersect $f(x,y,z)$ with the appropriate plane, f_x, f_y, f_z are the slopes of the resulting curve in the x, y, z direction, respectively.



We will learn how to find rates of change in directions that are not $\hat{x}, \hat{y}, \hat{z}$ later when we see directional derivatives.

There are also other interpretations for partials, depending on the context.

Ex:



The area of this parallelogram is

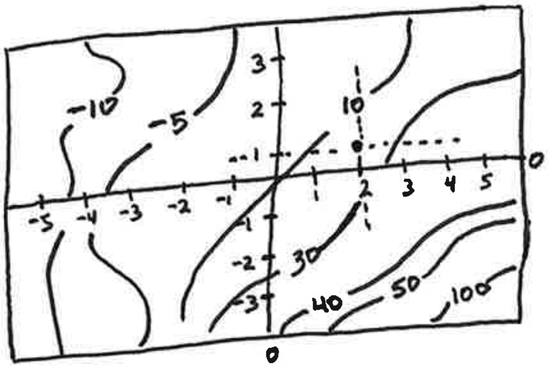
$$A = ab\sin\theta$$

Then...

$$\left.\begin{aligned}\frac{\partial A}{\partial a} &= b\sin\theta\\ \frac{\partial A}{\partial b} &= a\sin\theta\\ \frac{\partial A}{\partial \theta} &= ab\cos\theta\end{aligned}\right\}$$

The rate of change of the area if we increase/decrease a, b, θ, respectively.

Ex:



Estimate...

$$f_x(2,1) \approx \frac{10-30}{1-3} = 10$$

$$f_y(2,1) \approx \frac{10-30}{2--1} = -20/3$$

The partials f_{xx}, f_{yy}, f_{zz} have similar graphical interpretations in the $\hat{x}, \hat{y}; \hat{z}$ directions, respectively. This also generalizes to any number of variables. Higher partials, such as f_{xxx}, f_{xyx}, f_{xy}, etc are harder to interpret but all have meaning, ie

f_{xy}: "rate of change of rate of change in x if y changes"

Recall implicit differentiation from Calculus I:

$$\frac{d}{dx}\left(x^2 + xy + y^2\right) = 2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx}$$

This allowed us to find slopes for things we did not have explicit equations for, eg $xy^3 + xy - \frac{1}{y} = 4$ is some curve, but we have no $f(x) = -$ but we can still use implicit diff to find slopes, tangent lines, and approximations. We can do the same in many variables.

Ex: $e^x = x^2 + y^2$:

$$\frac{\partial}{\partial x}(e^x) = \frac{\partial}{\partial x}(x^2 + y^2)$$

$$e^x = 2x + 2y\frac{\partial y}{\partial x}$$

$$\frac{\partial y}{\partial x} = \frac{e^x - 2x}{2y}$$

Ex: $z = xz + y^2 \sin z$:

$$\frac{\partial}{\partial x}(z) = \frac{\partial}{\partial x}\left(xz + y^2 \sin z\right)$$

$$\frac{\partial z}{\partial x} = \left(z + x\frac{\partial z}{\partial x}\right) + y^2 \cos z \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{z}{1 - x - y^2 \cos z}$$

$$\frac{\partial}{\partial y}(z) = \frac{\partial}{\partial y}\left(xz + y^2 \sin z\right)$$

$$\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial y} + \left(2y \sin z + y^2 \cos z \frac{\partial z}{\partial y}\right) \Rightarrow \frac{\partial z}{\partial y} = \frac{2y \sin z}{1 - x - y^2 \cos z}$$

The Derivative

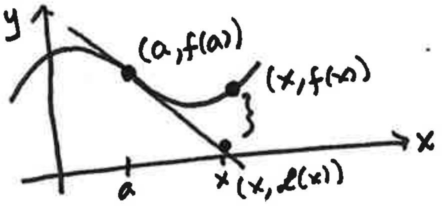
What about the general derivative? In one variable, the existence of the derivative depends on how well a line (the tangent line) approximates the function at a point.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

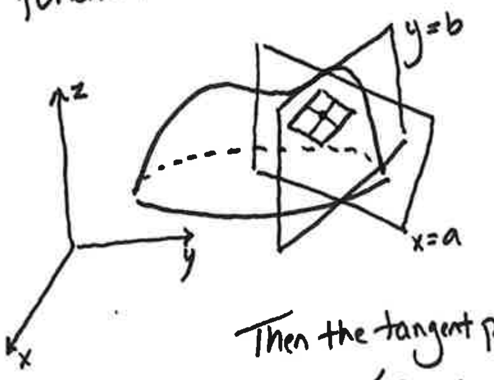
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a) = 0$$

$$\lim_{x \to a} \frac{f(x) - \overbrace{\left(f(a) + f'(a)(x-a)\right)}^{\text{tangent line, } \mathcal{L}(x)}}{x - a} = 0$$

We need the vertical distance (the difference between f(x) and the tangent line) to approach zero faster than the horizontal distance between 'x' and 'a' does.



In the case of two variables, we need not just a line but a plane to approximate f(x,y) 'well', in the sense above. In one variable, the tangent line matched the function value at a point and the slope in the x-direction. The tangent plane should match the function value and the slopes in the x & y directions.



Point: $(a, b, f(a,b))$

Vector // x tangent: $\vec{u} = \hat{\imath} + f_x(a,b)\,\hat{k}$

Vector // y tangent: $\vec{v} = \hat{\jmath} + f_y(a,b)\,\hat{k}$

normal to tangent plane: $\vec{n} = \vec{u} \times \vec{v} = \langle -f_x(a,b), -f_y(a,b), 1 \rangle$

Then the tangent plane is...

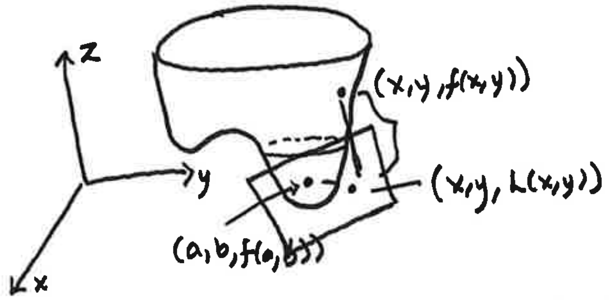
$$\langle -f_x(a,b), -f_y(a,b), 1 \rangle \cdot \langle x-a, y-b, z-f(a,b) \rangle = 0$$

So $\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$

Call this tangent plane $\mathcal{L}(x,y)$. Check for yourself that it has the desired properties:

1) $\mathcal{L}(a,b) = f(a,b)$
2) $\mathcal{L}_x(a,b) = f_x(a,b)$
3) $\mathcal{L}_y(a,b) = f_y(a,b)$

We again want the vertical distance between $f(x,y)$ and $\mathcal{L}(x,y)$ to approach zero faster than the distance between (x,y) and (a,b) does.



Differentiable: Let X be open in $\mathbb{R}^2$ and $f: X \subseteq \mathbb{R}^2 \to \mathbb{R}$. We say f is differentiable at $(a,b) \in X$ if $f_x(a,b)$, $f_y(a,b)$ exist and if $\mathcal{L}(x,y)$ is a 'good' approximation to f 'near' (a,b), ie

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-\mathcal{L}(x,y)}{\|(x,y)-(a,b)\|} = 0$$

If f is differentiable at (a,b), then $z = \mathcal{L}(x,y)$ is the tangent plane to f.

The notion 'easily' generalizes mutatis mutandis to $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ by considering the ~~tangent~~ hyperplane. To generalize to $f: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, we need ~~the~~ matrix of mixed partials to measure slopes in each direction of each component:

$$D\vec{f}(x_1,\dots,x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \vdots \\ \vdots & & \ddots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Differentiable: Let $X \subseteq \mathbb{R}^n$ be open and $\vec{f}: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{a} \in X$. We say $\vec{f}$ is diff. at $\vec{a}$ if $D\vec{f}(\vec{a})$ exists and $\vec{\mathcal{L}}: \mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{\mathcal{L}}(\vec{x}) = \vec{f}(\vec{a}) + D\vec{f}(\vec{a})(\vec{x}-\vec{a})$ approx. $\vec{f}$ 'well', ie

$$\lim_{\vec{x}\to\vec{a}} \frac{\|\vec{f}(\vec{x})-\vec{\mathcal{L}}(\vec{x})\|}{\|\vec{x}-\vec{a}\|} = 0$$

You need only walk away with being able to know and explain the general notion of differentiability - not necessarily be able to use it. You should also know the following:

- * If $\vec{f}$ is differentiable at $\vec{x} = \hat{a}$, then $\vec{f}$ is continuous at $\vec{x} = \vec{a}$.
- * If $\vec{f}$ is such that all its partials exist $\underline{\underline{\text{and}}}$ are continuous at $\vec{x} = \vec{a}$, then $\vec{f}$ is differentiable at $\vec{x} = \vec{a}$.
- * A function $\vec{f}$ is differentiable | continuous at $\vec{x} = \vec{a}$ if and only if each component of $\vec{f}$ is differentiable | continuous.

It is possible, as in one variable, for a function to be continuous but not differentiable.

Ex: One can show that $h(x,y) = (xy)^{1/3}$ is continuous at $(0,0)$, and that $h_x(0,0)$, $h_y(0,0)$ exist, but that $h(x,y)$ is not differentiable at $(0,0)$.

It is also possible for a function to have partial derivatives but not be differentiable.

Ex:
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

One can show that $f_x(0,0)$ and $f_y(0,0)$ exist but that f is not differentiable at $(0,0)$.

§11.5: The Chain Rule

Topics

- Chain Rule
- Implicit partial differentiation

You Should Be Able to....

- Compute derivatives with the chain rule.
- Explain the 'total change' interpretation of the chain rule.
- Use the chain rule in applied scenarios.
- Find slopes using implicit partial differentiation.

Recall the chain rule from Calculus I:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Ex: $f(x) = \sin x$
$x(t) = t^3 + t$

$$f(x(t)) = \sin(t^3 + t)$$

$$\frac{d}{dt} f(x(t)) = \cos(t^3 + t) \cdot (3t^2 + 1)$$

$$= \cos(x(t)) \cdot x'(t)$$

$$= f'(x(t)) \cdot x'(t)$$

We want to think about this differently.

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

$$= \cos x \cdot (3t^2 + 1)$$

$$= \cos(t^3 + t) \cdot (3t^2 + 1)$$

Let's think of this as follows: $\frac{df}{dt}$ is the amount f changes if t changes. We simply break this into its pieces:

$$\underbrace{\frac{df}{dt} = \underbrace{\frac{df}{dx}}_{\text{amount } f \text{ changes if } x \text{ changes}} \cdot \underbrace{\frac{dx}{dt}}_{\text{amount } x \text{ changes if } t \text{ changes}}}_{\text{amount } f \text{ changes if } t \text{ changes}}$$

We want to generalize this to more variables. This will save us messy substitutions and then even messier derivatives if we do not need the function written explicitly.

For instance in two variables with $z=f(x,y)$, where $x=x(t)$ and $y=y(t)$, we have...

$$\frac{dz}{dt} = \underbrace{\underbrace{\underbrace{\frac{\partial z}{\partial x}}_{\text{amt z change in x}} \underbrace{\frac{dx}{dt}}_{\text{amt x change in t}}}_{\text{amt z changes due to x change}} + \underbrace{\underbrace{\frac{\partial z}{\partial y}}_{\text{amt z change in y}} \underbrace{\frac{dy}{dt}}_{\text{amt y change in t}}}_{\text{amt z changes due to y change}}}_{\text{total change in z due to change in t}}$$

Ex: $z(x,y) = x^2y - y^2$, $x = \sin t$, $y = e^t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy)(\cos t) + (x^2 - 2y)e^t$$

If need be, we can substitute to find $\frac{dz}{dt}$ in terms of t alone:

$$\frac{dz}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$$

Student Ex: $z = x^2y + xy^2$; $x(t) = t^2+2$; $y(t) = \cos(2t)$

Find $\frac{dz}{dt}$ when $t=0$. If $t=0$, then $x(0)=2$ and $y(0) = \cos 0 = 1$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy+y^2)(2t) + (x^2+2xy)(-2\sin(2t)) \Big|_{t=0}$$

$$= (2\cdot2\cdot1 + 1^2)(2\cdot0) + (2^2 + 2\cdot2\cdot1)(-2\sin(0))$$

$$= 0 + 0$$

$$= 0$$

This easily extends to a function of many variables, all a function of a single variable:

$$Z = f(x_1, x_2, \dots, x_n)$$

$$x_i = f_i(t)$$

$$\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}$$

We can also do this for functions which are a composite of many variables, each a composite in their own way. For example, If $f(x,y,z)$ is a function and $x = x(s,t)$, $y = y(s,t)$, and $z = z(s,t)$, then...

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

The same logic of 'net change' from before applies mutatis mutandis.

Student Ex: $f(x,y,z) = xy + yz + xz$; $\vec{r}(s,t) = (s\cos t, s\sin t, t)$.
Find $\frac{\partial f}{\partial s}(1, 2\pi)$ and $\frac{\partial f}{\partial t}(1, 2\pi)$. If $s=1$, $t=2\pi$, $\vec{r}(s,t) = (1, 0, 2\pi)$,
ie $x=1$, $y=0$, $z=2\pi$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = (y+z)(\cos t) + (x+z)(\sin t) + (y+x)(0)$$

At $(s,t) = (1, 2\pi)$

$$= (0+2\pi)1 + (1+2\pi)0 + 0 = 2\pi$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = (y+z)(-s\sin t) + (x+z)(s\cos t) + (y+x)(1)$$

At $(s,t) = (1, 2\pi)$

$$= (0+2\pi)(0) + (1+2\pi)(1) + (0+1)(1)$$

$$= 2 + 2\pi$$

Student Ex: $z = e^x \sin y$; $x(s,t) = s^2 + t$; $y(s,t) = ts^2$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ in terms of s and t.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (e^x \sin y)(2s) + (e^x \cos y)(2ts)$$

$$= 2s\, e^{s^2+t} \sin(ts^2) + 2st\, e^{s^2+t} \cos(ts^2)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (e^x \sin y)(1) + (e^x \cos y)(s^2)$$

$$= e^{s^2+t} \sin(ts^2) + s^2 e^{s^2+t} \cos(ts^2)$$

These chain rules have more practical applications.

Student Ex: The equation $PV = nRT$ relates physical characteristics of ideal gases. P the pressure, V the volume, n the number of moles, R the ideal gas constant, T the temperature. $R \approx 8.314\ J\ mol^{-1} K^{-1}$. Find the rate of pressure change when the temp. is 300K and increasing at 0.1 K/s and the volume is 100L and dec. at 0.2 L/s. [One mole of material.]

$$P = 8.314 \frac{T}{V} \ ; \qquad \frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt}$$

$$= \frac{8.314}{V}\frac{dT}{dt} - \frac{8.314T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.314}{100}(0.1) - \frac{8.314(300)}{100^2}(-0.2)$$

$$= 0.0088 \text{ KPa/s}$$

Implicit Partial Differentiation:

We can use these chain rule techniques to find slopes for functions which are defined implicitly.

If $F(x,y)=0$ and $y=f(x)$, then differentiating...

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we have...

$$\boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}}$$

Student Ex: Use the above to find dy/dx for $x^3 + y^2 = 4xy$

$F(x,y) = x^3 + y^2 - 4xy$; $F_x = 3x^2 - 4y$

$F_y = 2y - 4x$

$$\frac{dy}{dx} = -\frac{3x^2-4y}{2y-4x} = \frac{3x^2-4y}{4x-2y}$$

Similarly, if $F(x,y,z)=0$, where $z=f(x,y)$: $\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \ ; \ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$

Student Ex: Find $\partial z/\partial x$ and $\partial z/\partial y$ if $3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$

$F(x,y,z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5$

$F_x = 6xz - 2xy^2$

$F_y = -2x^2y + 3z$

$F_z = 3x^2 + 6z^2 + 3y$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{2xy^2 - 6xz}{3x^2+6z^2+3y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{2x^2y - 3z}{3x^2+6z^2+3y}$$

§11.4: Tangent Planes, Differentials, Linear Approx.

Topics

- Tangent plane
- Linearization
- Linear approximation
- Gradient
- Differentials
- Applications

You Should Be Able to...

- Find the tangent plane to surfaces.
- Compute the gradient of functions.
- Explain the idea of linearization
- Compute the linearization of a function
- Approximate values using linearizations
- Compute the differential.
- Approximate values using the differential
- Use linearizations & differentials in applications.

Recall that for a function $z = f(x,y)$, the tangent plane to $f(x,y)$ at (x_0, y_0) was....

$$z = \underbrace{f(x_0, y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}_{L(x,y)}$$

Again, confirm this by checking...

1) $L(x_0,y_0) = f(x_0,y_0)$
2) $L_x(x_0,y_0) = f_x(x_0,y_0)$
3) $L_y(x_0,y_0) = f_y(x_0,y_0)$

Review the derivation of this from earlier in §11.3.

Student Ex: Find the tangent plane to $x^2y - xy + z = 0$ at $(2,-1,2)$.

$-z = x^2y - xy \Rightarrow z = xy - x^2y =: f(x,y)$

$f_x = y - 2xy \rightsquigarrow f_x(2,-1) = -1 - 2(2)(-1) = 3$

$f_y = x - x^2 \longrightarrow f_y(2,-1) = 2 - 2^2 = -2$

$f(2,-1) = 2$; $\quad z = 2 + 3(x-2) - 2(y+1)$

$z = 2 + 3x - 6 - 2y - 2$

$3x - 2y - z = 6$

* Always be sure that the point is actually on the surface.

We want a more systematic way of computing these tangent planes, especially for cases where we cannot explicitly solve for z, eg a sphere $x^2 + y^2 + z^2 = r^2$.

Gradient:

The gradient of a function $f: \mathbb{R}^3 \to \mathbb{R}$, denoted ∇f or grad f, is.....

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Essentially, ∇f keeps track of the rate of change in the 'standard' directions. It generalizes to functions $f: \mathbb{R}^n \to \mathbb{R}$ in the 'obvious' way. Note that ∇f is a vector. The symbol ∇ is called the 'del operator', and we say 'del f' for ∇f.

Student Ex: $f(x,y) = x\cos(xy)$. Find ∇f and $\nabla f(1,0)$.

$f_x = \cos(xy) - xy\sin(xy)$

$f_y = -x^2\sin(xy)$

Then $\nabla f = \langle \cos(xy) - xy\sin(xy), -x^2\sin(xy) \rangle$

$\nabla f(1,0) = \langle 1-0, 0 \rangle = \langle 1,0 \rangle = \hat{\imath}$

Student Ex: $f(x,y,z) = xz + y^2 - 5$. Find ∇f.

$f_x = z$

$f_y = 2y$

$f_z = x$

$\nabla f = \langle z, 2y, x \rangle$

We can now formulize the tangent plane generally so that we can compute it systematically.

Given a surface given by $F(x,y,z)=0$, the tangent plane to F at (x_0,y_0,z_0) is...

$$\boxed{\nabla F(x_0,y_0,z_0)\cdot\langle x-x_0,\, y-y_0,\, z-z_0\rangle = 0}$$

* Note: This generalizes the case from before. If $z=f(x,y)$, we have $\underbrace{z-f(x,y)}_{F(x,y,z)}=0$

$$\nabla F(x_0,y_0,z_0)\cdot\langle x-x_0,\, y-y_0,\, z-z_0\rangle = 0$$

$$\langle -f_x, -f_y, 1\rangle \cdot \langle x-x_0,\, y-y_0,\, z-z_0\rangle = 0$$

$$-f_x(x-x_0) - f_y(y-y_0) + z-z_0 = 0$$

$$z = z_0 + f_x(x-x_0) + f_y(y-y_0)$$

* From this definition, it is simple to see that this is a plane.

* As always, be sure (x_0,y_0,z_0) is on the surface, ie $F(x_0,y_0,z_0)=0$.

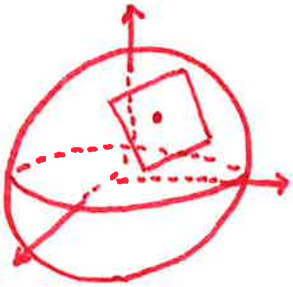
Student Ex: Find the tangent plane to the sphere $x^2+y^2+z^2=3$ at $(1,1,1)$.

$$F(x,y,z) = x^2+y^2+z^2-3$$

$$F_x = 2x \rightsquigarrow F_x(1,1,1) = 2$$

$$F_y = 2y \rightsquigarrow F_y(1,1,1) = 2$$

$$F_z = 2z \rightsquigarrow F_z(1,1,1) = 2$$



$$\langle 2,2,2\rangle \cdot \langle x-1,\, y-1,\, z-1\rangle = 0$$

$$2(x-1)+2(y-1)+2(z-1) = 0$$

$$x+y+z = 3$$

The tangent plane matches the function exactly at one specific point (possibly more). But one of the whole purposes of the tangent plane is that it should approximate the function 'near' the point of tangency. We can also reason this as follows in the case of tangent planes to $z = f(x,y)$.

$$z = \underbrace{z_0}_{\text{initial value}} + \underbrace{\underbrace{\underbrace{f_x}_{\substack{\text{amount z}\\\text{changes if}\\\text{x changes}}} \underbrace{(x-x_0)}_{\substack{\text{change}\\\text{in x}}}}_{\substack{\text{change in z}\\\text{due to change in x}}} + \underbrace{\underbrace{f_y}_{\substack{\text{amount z}\\\text{changes if}\\\text{y changes}}} \underbrace{(y-y_0)}_{\substack{\text{amount}\\\text{y changes}}}}_{\substack{\text{change in z}\\\text{due to change in y}}}}_{\text{total change in z}}$$

So the tangent plane matches the value of the function at the point of tangency and tries to keep track of the amount of the change near this point. We sum this up as...

$$\boxed{F(x,y,z) \approx \mathcal{L}(x,y,z)}$$

for (x,y,z) 'near' (x_0,y_0,z_0), where $\mathcal{L}(x,y,z)$ is the tangent plane to F at (x_0,y_0,z_0).

Ex: Approximate $(1.1)^3 e^{-0.1} \cos(0.2)$ Let $f(x,y,z) = x^3 e^y \cos z - 1$

Now $f(1,0,0) = 1$. $f_x = 3x^2 e^y \cos z \rightarrow f_x(1,0,0) = 3$. $f_y = x^3 e^y \cos z$

$f_y(1,0,0) = 1$. $f_z = -x^3 e^y \sin z \rightarrow f_z(1,0,0) = 0$

$\langle 3,1,0 \rangle \cdot \langle x-1, y-0, z-0 \rangle = 0$

$3(x-1) + 1(y-0) + 0(z-0) = 0$

$3x - 3 + y$

$\mathcal{L}(x,y,z) = 3x - 3 + y$

$f(1.1,-0.1,0.2) \approx \mathcal{L}(1.1,-0.1,0.2) + 1$

$= 3.3 - 3 + (-0.1) + 1$

$= 1.2$

The actual value is 1.18033..., an error of 1%!

The tangent plane is also called the linearization of f (at $\vec{x}$). Hence, this process is called linear approx. There are higher dimensional generalizations.

Student Ex: Approximate $(2.01)^7(9.02) - (2.01)(9.02)^2$

$$f(x,y) = x^7 y - x y^2$$

$$f(2,9) = 2^7 \cdot 9 - 2 \cdot 9^2 = 128(9) - 2(81) = 1152 - 162 = 990$$

$$f_x = 7x^6 y - y^2 \rightsquigarrow f_x(2,9) = 7(64)9 - 81 = 4032 - 81 = 3951$$

$$f_y = x^7 - 2xy \rightsquigarrow f_y(2,9) = 128 - 2(2)9 = 128 - 36 = 92$$

$$\mathcal{L}(x,y) = 990 + 3951(x-2) + 92(y-9)$$

$$f(2.01, 9.02) \simeq \mathcal{L}(2.01, 9.02) = 990 + 3951(2.01-2) + 92(9.02-9)$$

$$= 990 + 39.51 + 1.84$$

$$= 1031.35$$

The actual value is
1032.0464...
for an error of 0.067%.

Essentially, what we are doing is saying new value = old value + change. This can be expressed in terms of the total differential: $z = f(x,y)$

$$\boxed{dz = f_x\, dx + f_y\, dy}$$

The same logic applies to a function of more variables. Note this is understandable using the logic of tracking change from earlier.

Student Ex: Find the total differential for $w = x^2 + y^2 + z^2$

$$dw = 2x\,dx + 2y\,dy + 2z\,dz$$

Student Ex: Suppose $f(1,2,-1)=5$, $f_x(1,2,-1)=2$, $f_y(1,2,-1)=3$, and $f_z(1,2,-1)=-1$. Use total differentials to approximate $f(0.8, 2.1, -0.9)$.

$$
\begin{aligned}
df &= f_x\,dx + f_y\,dy + f_z\,dz \\
&= 2(-0.2) + 3(0.1) + (-1)0.1 \\
&= -0.4 + 0.3 - 0.1 \\
&= -0.2
\end{aligned}
$$

Then $f(0.8,2.1,-0.9) \approx f(1,2,-1) + df = 5 - 0.2 = 4.8$

Student Ex: A rectangular box is measured as being 50cm × 20cm × 15cm. Due to the method of measurement, each side is only measured with an accuracy of ± 0.01 cm. Estimate the error in measuring the boxes volume.

$V = lwh$; $V(50,20,15) = 15{,}000\text{cm}^3$

$$
\begin{aligned}
dV &= V_l\,dl + V_w\,dw + V_h\,dh \\
&= wh\,dl + lh\,dw + lw\,dh \\
&= 20(15)(\pm 0.01) + 50(15)(\pm 0.01) + 50(20)(\pm 0.01) \\
&= (20\cdot 15 + 50\cdot 15 + 50\cdot 20)(\pm 0.01) \\
&= (300 + 750 + 1000)(\pm 0.01) \\
&= 2050\,(\pm 0.01) \\
&= \pm 20.50\text{ cm}^3
\end{aligned}
$$

Which is a relative error of $\frac{\Delta V}{V} = \frac{20.5}{15000} \approx 0.14\%$

§11.6: Directional Derivatives

Topics

- Directional Derivatives
- Tangent planes to surfaces
- Normal Lines
- Inverse/Implicit Function Theorems

You Should Be Able to...

- Compute a directional derivative using the definition.
- Compute the directional derivative using the 'shortcut'.
- Find the direction of steepest ascent/descent.
- Find the max rate inc./dec. for a function in any direction.
- Compute the tangent plane to a surface.
- Compute the normal line to a surface.

We have seen how to compute f_x, f_y, f_z, ... , and more generally ∇f. But what if we wanted to compute the rate of change of f, not in the direction of coordinate axes like x, y, z, but in any direction. To figure out what this should mean, let's reexamine f_x, f_y:

$$\frac{\partial f}{\partial x}(a,b) := \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{f((a,b) + (h,0)) - f(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{f((a,b) + h(1,0)) - f(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{f(\vec{p} + h\hat{\imath}) - f(a,b)}{h} \quad ; \quad \vec{p} = (a,b)$$

Similarly, $\frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(\vec{p} + h\hat{\jmath}) - f(a,b)}{h}$. So if we want to find the rate of change of f in the direction of $\vec{v}$, we should define:

Directional Deriv: The directional derivative of f at $\vec{p}$ in the direction of $\vec{v}$, denoted $D_{\vec{v}} f(\vec{p})$, is

$$D_{\vec{v}} f(\vec{p}) = \lim_{h \to 0} \frac{f(\vec{p} + h\frac{\vec{v}}{|\vec{v}|}) - f(\vec{a})}{h}$$

* We want in the direction of $\vec{v}$, but not $\vec{v}$ itself, merely its direction. We do not want its magnitude contributing to the rate of change of f. So be sure to always use a unit vector!!

Ex: $f(x,y) = x^2 + xy + y^2$. Find $D_{\vec{v}}f(1,0)$ for any unit vector $\vec{v}$.

Let $\vec{v} = \langle a,b \rangle$. Now $\vec{v}$ unit vector so $|\vec{v}| = \sqrt{a^2+b^2} = 1$.

$$D_{\vec{v}} f(1,0) = \lim_{h\to 0} \frac{f((1,0)+h(a,b)) - f(1,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+ha, hb) - f(1,0)}{h}$$

$$= \lim_{h\to 0} \frac{((1+ha)^2 + (1+ha)(hb) + (hb)^2) - (1+0+0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2b^2 + h^2ab + h^2a^2 + 2ha + hb}{h}$$

$$= 2a+b$$

Notice for $\vec{v} = \langle 1,0 \rangle = \hat{\imath} = \hat{x}$ and $\vec{v} = \langle 0,1 \rangle = \hat{\jmath} = \hat{y}$, we have

$$D_{\hat{x}} f(1,0) = 2 = f_x(1,0) \checkmark$$

$$D_{\hat{y}} f(1,0) = 1 = f_y(1,0) \checkmark$$

For $\vec{v} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$, we have $D_{\vec{v}} f(1,0) = 3/\sqrt{2}$.

* In 2D, you can write any unit vector as $\langle \cos\theta, \sin\theta \rangle$, so you may see

$$D_u f(x,y) = \lim_{t\to 0} \frac{f(x+t\cos\theta, y+t\sin\theta) - f(x,y)}{t}$$

and similar things here and in future formulas.

Luckily, we will rarely ever need to compute directional derivatives using the definition.

Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable. Let $\vec{x}(t) = \vec{x}_0 + t\vec{v}$. Then by the chain rule,

$$\frac{d}{dt} f(\vec{x}_0 + t\vec{v}) = Df(\vec{x})\, D\vec{x}(t) = Df(\vec{x})\, \vec{v}$$

Evaluating at $t=0$, we find...

$$\boxed{D_{\vec{v}} f(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{v}}$$

* BE SURE YOU USE A UNIT VECTOR, YOU WANT A PURE DIRECTION!

Student Ex: Find rate of change of $f(x,y) = x^3 + y^2 + 1$ at $(1,2)$ in the direction of $\langle 1, -1 \rangle$.

$$\nabla f = \langle 3x^2, 2y \rangle$$

$$\nabla f(1,2) = \langle 3, 4 \rangle$$

$$|\langle 1,-1 \rangle| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$D_{\vec{u}} f(1,2) = \langle 3,4 \rangle \cdot \frac{\langle 1,-1 \rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \left(3(1) + 4(-1) \right)$$

$$= -1/\sqrt{2}$$

* Could always use $\nabla . f(\vec{x}) \cdot \frac{\vec{v}}{|\vec{v}|}$ generally if f diff.

Student Ex: $f(x,y,z) = \dfrac{xe^y}{z+1}$. Find $D_{\vec{u}} f(2,1,0)$ for $\vec{u} \parallel$ to $\vec{v} = \hat{i} + \hat{j} + \hat{k}$.

$$\nabla f = \left\langle \frac{e^y}{z+1}, \frac{xe^y}{z+1}, \frac{-xe^y}{(z+1)^2} \right\rangle$$

$\hat{i} + \hat{j} + \hat{k} = \langle 1,1,1 \rangle$

$|\vec{v}| = \sqrt{1^2 + 1^2 + 1^2}$

$= \sqrt{3}$

$\vec{u} = \dfrac{\langle 1,1,1 \rangle}{\sqrt{3}}$

$$\nabla f(2,1,0) = \langle e, 2e, -2e \rangle$$

$$D_{\vec{u}} f(2,1,0) = \langle e, 2e, -2e \rangle \cdot \frac{\langle 1,1,1 \rangle}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \left(e(1) + 2e(1) - 2e(1) \right)$$

$$= e/\sqrt{3}$$

Student Ex: $f(x,y,z) = x \arctan(yz)$. Find $D_{\vec{u}} f$ at $P(4,1,1)$ for $\vec{u}$ in direction of $\langle 1,2,-1 \rangle$.

$$\nabla f = \left\langle \arctan(yz), \frac{xz}{1+(yz)^2}, \frac{xy}{1+(yz)^2} \right\rangle$$

$|\langle 1,2,-1 \rangle| =$

$\sqrt{1^2 + 2^2 + (-1)^2}$

$\sqrt{6}$

$$\nabla f(4,1,1) = \left\langle \frac{\pi}{4}, 2, 2 \right\rangle$$

$$D_{\vec{u}} f(4,1,1) = \left\langle \frac{\pi}{4}, 2, 2 \right\rangle \cdot \frac{\langle 1,2,-1 \rangle}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} \left(\frac{\pi}{4}(1) + 2(2) + 2(-1) \right)$$

$$= \frac{1}{\sqrt{6}} \left(\frac{\pi}{4} + 2 \right)$$

Note the above method will not always work. There are functions which have directional derivatives everywhere but are not diff. For example,

$$f(x,y) = \begin{cases} \dfrac{yx^2}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

The directional derivative tells us more:

$$D_{\vec{u}} f(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}$$

Then...

$$D_{\vec{u}} f(\vec{x}_0) = |\nabla f(\vec{x}_0)| \, |\vec{u}| \cos\theta$$
$$= |\nabla f(\vec{x}_0)| \cos\theta$$

But $-1 \leq \cos\theta \leq 1$. Then...

$$-|\nabla f(\vec{x}_0)| \leq D_u f(\vec{x}_0) \leq |\nabla f(\vec{x}_0)|$$

Hence, $D_{\vec{u}} f(\vec{x}_0)$ is maximal when $\vec{u}$ is // to $\nabla f(\vec{x}_0)$ (ie $\theta = 0°$ so $\vec{u}$, $\nabla f(\vec{x}_0)$ point in the same direction) and $D_{\vec{u}} f(\vec{x}_0)$ minimal when $\vec{u}$ is antiparallel (ie opposite direction) to $\nabla f(\vec{x}_0)$ (ie $\theta = 180°$ so $\vec{u}$, $\nabla f(\vec{x}_0)$ point in opposite directions).

These are also called the direction of steepest ascent / descent.

Student Ex: $f(x,y,z) = xyz$. Find in which direction f increases the quickest / decreases the quickest at $(1,-2,1)$.

$$\nabla f = \langle yz, xz, xy \rangle$$
$$\nabla f(1,-2,1) = \langle -2, 1, -2 \rangle$$

Direction max inc:
$\langle -2, 1, -2 \rangle$

Direction max dec:
$\langle 2, -1, 2 \rangle$

The fact that these point in opposite directions should make sense: if you face the steepest incline, then if you 'turn around', you face the steepest decline.

Student Ex: Find the directions of max inc. | dec. for $f(x,y) = x^2 + 3y^2$ at $(1,-1)$.

$$\nabla f = \langle 2x, 6y \rangle$$

$$\nabla f(1,-1) = \langle 2, -6 \rangle = 2\langle 1, -3 \rangle$$

Dir. Max: $\langle 1, -3 \rangle$

Dir. Min: $\langle -1, 3 \rangle$

Just need direction, not actual vector. So anything in same direction of ∇f will work.

Student Ex: Temperature of metal plate given by...

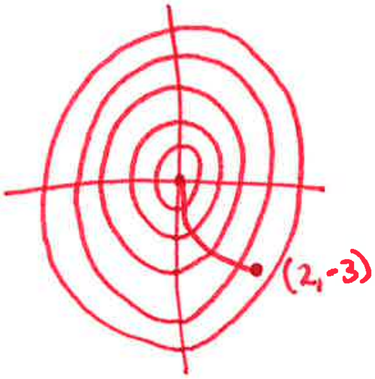
$$T(x,y) = 20 - 4x^2 - y^2$$

If a particle on the plate moves towards the regions of greatest temperature, find the path of the particle.

Say path $\vec{r}(t) = (x(t), y(t))$. The tangent vector is $\vec{r}\,'(t) = (x'(t), y'(t))$. Particle moves to region of highest heat, so in direction of $\nabla T(x,y) = \langle -8x, -2y \rangle$. These vectors are parallel, so there is K so that $K r' = \nabla T$:

$$-8x = K x' \qquad -2y = K y' \quad \rightsquigarrow \quad \frac{dt}{K} = \boxed{\frac{dx}{-8x} = \frac{dy}{-2y}}$$

Solving this differential equation, $x = Cy^4$. If particle were to begin at $(2,-3)$, we have $x = \frac{2}{81}y^4$.



Using a logic similar to the derivation that $D_{\vec{u}} f = \nabla f \cdot \vec{u}$, you can prove the following familiar result:

* If $X \subseteq \mathbb{R}^n$ is open and $f: X \to \mathbb{R}$ is C^1, $\vec{x}_0 \in S = \{\vec{x} \in X \mid f(\vec{x}) = c\}$, then $\nabla f(\vec{x}_0)$ is $\perp$ to S.

That is...

* ∇f is $\perp$ to the level curves of a surface given by $f = 0$.

This is why the tangent plane to a surface was what it was.

Student Ex: Find the tangent plane to $x^2 + y - \sin z = 3$ at $(2, -1, 0)$.

$$\nabla f = \langle 2x, 1, -\cos z \rangle$$

$$\nabla f(2,-1,0) = \langle 4, 1, -1 \rangle$$

$$\langle 4,1,-1 \rangle \cdot \langle x-2, y+1, z-0 \rangle = 0$$

$$4(x-2) + 1(y+1) + -1(z-0) = 0$$

$$4x + y - z = 7$$

But because ∇f is $\perp$ to the surface S at $(x_0, y_0, z_0) \in S$, we can find a line $\perp$ to S at this point.

Normal Line: A normal line to a surface S at P is a line passing through P and $\perp$ to the tangent plane at P.

Student Ex: Find the normal line to $xyz=12$ at $(2,-2,-3)$.

$F(x,y,z) = xyz - 12$

$\nabla F = \langle yz, xz, xy \rangle$

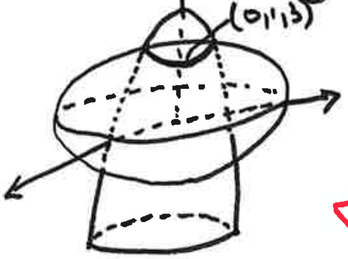
$\nabla F(2,-2,-3) = \langle 6,-6,-4 \rangle = 2\langle 3,-3,-2 \rangle$

$\ell(t) = (6t+2, -6t-2, -4t-3)$

or

$\ell(t) = (3t+2, -3t-2, -2t-3)$

Student Ex: Find line tangent to curve of intersection of $x^2+2y^2+2z^2=20$ and $x^2+y^2+z=4$ at $(0,1,3)$



$F = x^2+2y^2+2z^2-20$

$G = x^2+y^2+z-4$

$\nabla F = \langle 2x, 4y, 4z \rangle$

$\nabla F(0,1,3) = \langle 0,4,12 \rangle$

$\nabla G = \langle 2x, 2y, 1 \rangle$

$\nabla G(0,1,3) = \langle 0,2,1 \rangle$

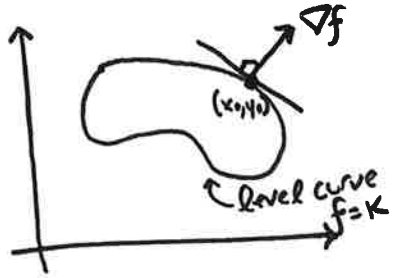
$\nabla F(0,1,3) \times \nabla G(0,1,3)$

$\langle -20,0,0 \rangle$

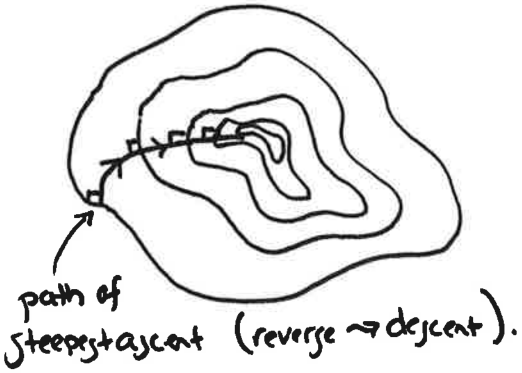
$-20\langle 1,0,0 \rangle$

$\ell(t) = (t+0, 1, 3)$

As a final note, because ∇f is $\perp$ to the level curves of S: $f=0$, we get the following graphical representation for paths of steepest ascent.



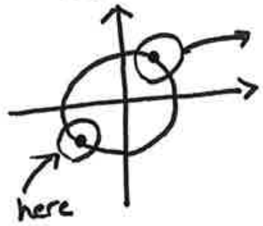
$\rightsquigarrow$



Before moving on, there are two other beautiful results that are very powerful & useful related to gradients, directional derivatives, etc.

Implicit function Theorem

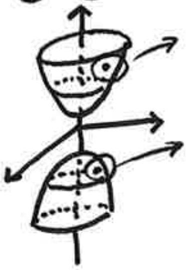
Note if $x^2+y^2=1$, we cannot solve for x, y as a function of the other, ie $x^2+y^2=1 \rightarrow y = \pm\sqrt{1-x^2}$, ie given x, do you choose + or − for y? There is no obvious choice, but you could associate y with one of them 'locally', ie



around here, $y = \sqrt{1-x^2}$

around here $y = -\sqrt{1-x^2}$

This appears in higher dimensions too. For example, if $\frac{z^2}{4} - x^2 - y^2 = 1$, then $z = \pm 2\sqrt{1+x^2+y^2}$



up here, $z = 2\sqrt{1+x^2+y^2}$

here, $z = -2\sqrt{1+x^2+y^2}$

The implicit function theorem tells you when can locally find one variable in terms of the others.

Implicit Function Theorem: Let $F: X \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^1 and $\vec{a} \in S = \{\vec{x} \in \mathbb{R}^n : F(\vec{x}) = c\}$. If $F_{x_n}(\vec{a}) \neq 0$, there is a neighborhood $\mathcal{U}$ of $(a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$, a neighborhood V of a_n in $\mathbb{R}$, and function $f: \mathcal{U} \subseteq \mathbb{R}^{n-1} \to V$ of class C^1 such that if $(x_1, \dots, x_{n-1}) \in \mathcal{U}$ and $x_n \in V$ satisfy $F(x_1, \dots, x_n) = c$, ie $(x_1, \dots, x_n) \in S$, then $x_n = f(x_1, \dots, x_{n-1})$.

This can be generalized.

The other result comes from the following question:

Given a system of equations....

$$\begin{cases} y_1 = f_1(x_1, \dots, x_n) \\ y_2 = f_2(x_1, \dots, x_n) \\ \vdots \\ y_n = f_n(x_1, \dots, x_n) \end{cases}$$

when can you solve for $\vec{x}$ in terms of $\vec{y}$? For example,

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

ie, if we switch to polar coordinates, can we go back? That is, can we find $r = r(x,y)$ and $\theta = \theta(x,y)$. The trick is to apply the Implicit Function Theorem to $\begin{cases} F_1(x_1, \dots, x_n, y_1, \dots, y_n) = 0 \\ \vdots \\ F_m(x_1, \dots, x_n, y_1, \dots, y_n) = 0 \end{cases}$

One then finds....

Inverse Function Theorem: Suppose $\vec{f} = (f_1, \dots, f_n)$ is of class C^1 on an open set $A \subseteq \mathbb{R}^n$. If

$$\det D\vec{f}(\vec{a}) = \left.\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)}\right|_{\vec{x} = \vec{a}} \neq 0$$

then there is an open set $U \subseteq \mathbb{R}^n$ containing $\vec{a}$ such that $\vec{f}$ is one-to-one on U, the set $V = \vec{f}(U)$ is open, and there is a unique inverse $\vec{g}: V \to U$ to $\vec{f}$ of class C^1.

So in the above,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$ so away from $(0,0)$ we can invert the coordinates.

§11.7: Max & Min Values

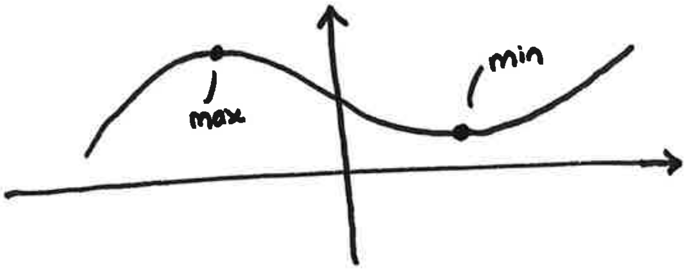
Topics

- Local Max | Min | Saddles
- Hessian
- Extrema over Regions

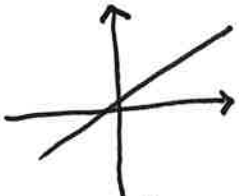
You Should Be Able to...

- Explain what a local max | min is.
- Explain (with a plot) what a saddle is.
- Find and classify the local max | min | saddles for a function.
- Find and classify the local max | min | saddles for a function where the Hessian method fails.
- Know the 'shortcut' formulas for 2 variable extrema methods.
- Find extrema for functions over specified regions.
- Be able to explain (with pictures) what a compact region is.

Recall from Calculus I that f has a max|min at x=a if f(a) is bigger|smaller than all the values f(x) for x values 'near' x=a.

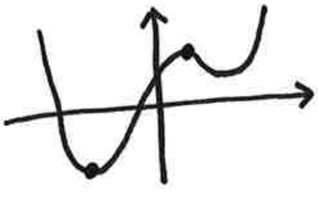


Strictly speaking, these are local max|min (ie near a particular value). A global max|min for f is a value as big|small or bigger|smaller than all other values for f(x). A function may not have any of there or many.


no max|min

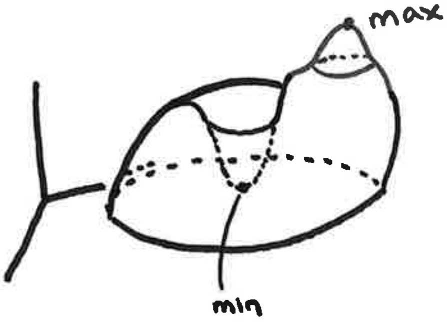

global|local min


local max|min
no global

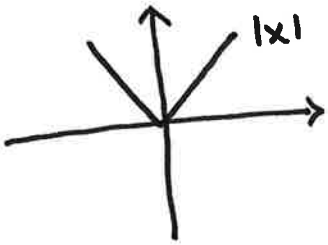

local max
global|local min
no global max

We want to generalize this to $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$

Local Max|Min: We say f has a local min at $\vec{x} = \vec{a} \in \mathbb{R}^n$ if there is some neighborhood U of $\vec{a}$ such that $f(\vec{x}) \geq f(\vec{a})$ for all $\vec{x} \in U$. Similarly, f has a local max at $\vec{x} = \vec{a}$ if there is such a U with $f(\vec{x}) \leq f(\vec{a})$ for all $\vec{x} \in U$.



Recall also from Calculus I that derivatives cannot detect all max/mins (if any), eg

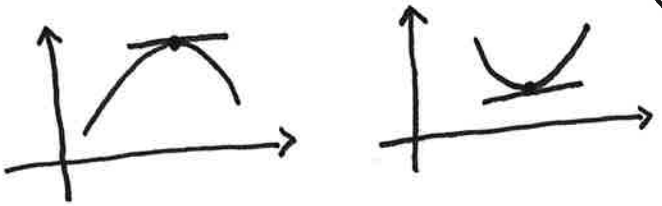


$f(x)=|x|$ not diff at $x=0$ but $f(x)$ has local min. at $x=0$

But if $f(x)$ is differentiable, a similar technique should work. As in Calculus I:

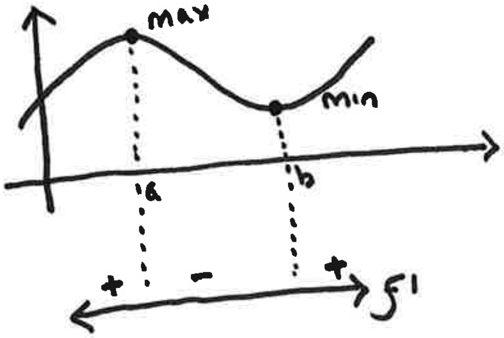
* If $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable and has a local extremum at $\vec{a} \in X$, then $Df(\vec{a})=0$.

critical point

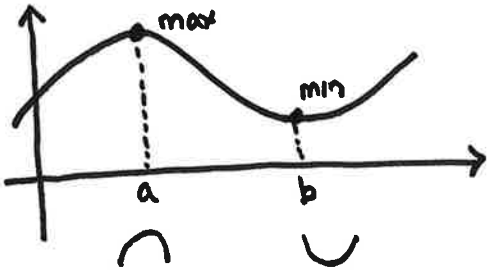


So again, we can use the derivative to find possible candidates for max/mins. But we need a way of testing these points. Before...

1st Deriv. Test



2nd Deriv. Test

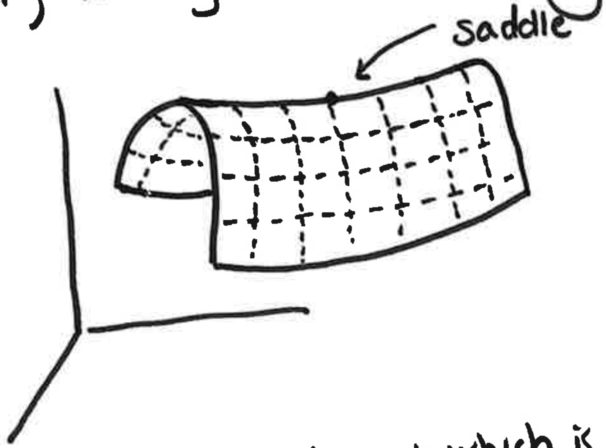


$f''(a)<0$ $f''(b)>0$

* $f''(a)=0 \rightsquigarrow$ a is...???

Of course, the situation is more complicated in $\mathbb{R}^n$. First, the 1st Derivative Test no longer applies. [It depended on direction, but now there are infinitely many.]

Moreover, there is a new possibility...



Saddle Point: A critical point which is neither a max or min.

Notice above, looking left/right, the point is a min. But looking 'forward'/'back', the point is a max. But overall, it is neither. It is a saddle point.

Because the 1st deriv test does not apply, we need a new 2nd deriv. test. We have such a result depending on something called the Hessian (whether it is $\underset{\text{min}}{\text{positive}}$ definite, $\underset{\text{max}}{\text{negative}}$ definite, or $\underset{\text{saddle}}{\text{neither}}$). We will not bother to explain why the method works, only how to do it and trust the reader will trust us. [The proof depends on second order n-variable Taylor approximations.]

Hessian:

$$Hf(\vec{a}) := \begin{pmatrix} f_{x_1x_1}(\vec{a}) & f_{x_1x_2}(\vec{a}) & \cdots & f_{x_1x_n}(\vec{a}) \\ f_{x_2x_1}(\vec{a}) & f_{x_2x_2}(\vec{a}) & \cdots & f_{x_2x_n}(\vec{a}) \\ \vdots & & \ddots & \vdots \\ f_{x_nx_1}(\vec{a}) & f_{x_nx_2}(\vec{a}) & \cdots & f_{x_nx_n}(\vec{a}) \end{pmatrix}$$

And define a sequence of principal minors...

$$d_1 = f_{x_1x_1}(\vec{a}), \quad d_2 = \begin{vmatrix} f_{x_1x_1}(\vec{a}) & f_{x_1x_2}(\vec{a}) \\ f_{x_2x_1}(\vec{a}) & f_{x_2x_2}(\vec{a}) \end{vmatrix}, \quad d_3 = \begin{vmatrix} f_{x_1x_1}(\vec{a}) & f_{x_1x_2}(\vec{a}) & f_{x_1x_3}(\vec{a}) \\ f_{x_2x_1}(\vec{a}) & f_{x_2x_2}(\vec{a}) & f_{x_2x_3}(\vec{a}) \\ f_{x_3x_1}(\vec{a}) & f_{x_3x_2}(\vec{a}) & f_{x_3x_3}(\vec{a}) \end{vmatrix}$$

etc. If $d_k > 0$ for all $k = 1, \dots, n$, then f has a local min at $\vec{x} = \vec{a}$. If $d_k < 0$ for odd k and $d_k > 0$ for even k, f has a local max at $\vec{x} = \vec{a}$. Otherwise, $\vec{x} = \vec{a}$ is a saddle point. If $\det Hf(\vec{a}) = 0$, $\vec{x} = \vec{a}$ is said to be degenerate and some other method is needed.

Generally, we proceed as follows.

1. Find the partials: $f_x, f_y, f_z, f_{xx}, f_{xy}, \dots$
2. Find the critical points, ie where $0 = f_x = f_y = f_z$.
3. Use the above to classify each critical value.

[4. Evaluate f at these points, if necessary.]

Although this looks nightmarish, it is easier than it appears. But it will never not be time consuming.

* It can be useful to note that the det of a diagonal max is the product of its diagonal entries. This is also true for upper/lower triangular matrices.

Student Ex: Find the local max|mins &saddles for $f(x,y) = x^2 + xy + y^2 + 2x - 2y + 5$

$f_x = 2x + y + 2$ $\qquad f_{xx} = 2$

$f_y = x + 2y - 2$ $\qquad f_{yy} = 2$

$f_{xy} = f_{yx} = 1$

We want $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \rightarrow \begin{cases} 2x+y+2=0 \\ x+2y-2=0 \end{cases} \rightarrow \begin{cases} 2x+y+2=0 \\ 2x+4y-4=0 \end{cases} \rightarrow$

$\{-3y+6=0 \rightarrow y=2 \rightarrow 2x+2+2=0 \rightarrow x=-2.$

The only critical point is $(-2,2)$.

$f_{xx}(-2,2)=2$

$f_{yy}(-2,2)=2$

$f_{xy}(-2,2)=f_{yx}(-2,2)=1$

$(-2,2): \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\qquad d_1 = 2 > 0$, $\quad d_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3 > 0$

Therefore, $(-2,2)$ is a min.

Student Ex: Find the local max|min &saddles for $f(x,y) = 2x - 3y + \ln xy$.

$f_x = 2 + \frac{1}{x}$ $\qquad f_{xx} = -1/x^2$

$f_y = -3 + \frac{1}{y}$ $\qquad f_{yy} = -1/y^2$

$f_{xy} = f_{yx} = 0$

$\begin{pmatrix} -1/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}$ or $d_1 = -1/x^2$, $d_2 = 1/(x^2y^2) = 1/(xy)^2$

$\begin{cases} 2 + 1/x = 0 \\ -3 + 1/y = 0 \end{cases} \rightarrow \begin{cases} x = -1/2 \\ y = +1/3 \end{cases} \rightarrow$ only critical point is $(-1/2, 1/3)$

$(-1/2, 1/3): \begin{pmatrix} -4 & 0 \\ 0 & -9 \end{pmatrix}$ $\qquad d_1 = -4 < 0$, $\quad d_2 = \begin{vmatrix} -4 & 0 \\ 0 & -9 \end{vmatrix} = 36 > 0 \rightarrow$ max

Therefore, $(-1/2, 1/3)$ is a local max for $f(x,y)$.

* It is possible to have no local max|min and it is possible to have no critical values. (Also possible to have infinitely many ☺)

* There is no single technique to find the critical points – it's experience and clever thinking. Just be sure to find them all.

Student Ex: Find the local max|min|saddles for

$f(x,y,z) = x^2 - xy + z^2 - 2xz + 6z$

$f_x = 2x - y - 2z$

$f_y = -x$

$f_z = 2z - 2x + 6$

$f_{xx} = 2 \quad f_{yy} = 0$

$f_{xy} = -1 \quad f_{zz} = 2$

$f_{xz} = -2 \quad f_{yz} = 0$

$$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \\ f_z = 0 \end{cases} \dashrightarrow -x = 0 \quad \boxed{x = 0}$$

$$\begin{cases} -y - 2z = 0 \\ 2z + 6 = 0 \end{cases} \rightarrow z = -3 \rightsquigarrow -y - 2(-3) = 0 \quad y = 6$$

Only critical point is $(0,6,-3)$

$(0,6,-3)$: $\begin{pmatrix} 2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 2 \end{pmatrix}$

$d_1 = 2$

$d_2 = \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = 0 - 1 = -1$

Done! $(0,6,-3)$ is a saddle

↳ no need for d_3

Now recall that if $\det Hf = 0$, the test is inconclusive, ie if any of our minors vanish. If this happens, most often, we will need logic to classify the critical point.

Student Ex: $f(x,y,z) = x^4 + x^2 + y^4$

$g(x,y) = -(x^2 + y^6)$

$h(x,y) = x^4 - x^2 + y^8$

Each of f, g, h have $(0,0,0)$ as a critical value but $\det Hf = 0$. Classify $(0,0,0)$ as a local max|min|saddle.

f: $(0,0,0)$ min (Why?)

g: $(0,0,0)$ max (Why?)

h: $(0,0,0)$ saddle (Why?)

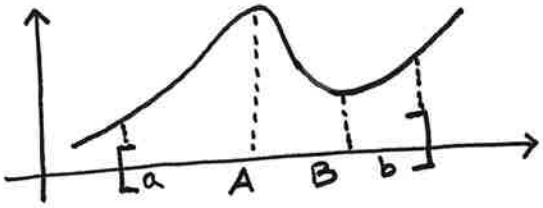
Finally, note that using our method in the case of two variables and using $f_{xy} = f_{yx}$, you can find the 'shortcut'

Min	Max	Saddle
$f_{xx} > 0$ & $f_{xx} f_{yy} - (f_{xy})^2 > 0$	$f_{xx} < 0$ & $f_{xx} f_{yy} - (f_{xy})^2 > 0$	$f_{xx} f_{yy} - (f_{xy})^2 < 0$

But this does not (easily/at all) generalize.

Extrema over Regions

Recall from Calculus I, we also found max/mins of functions over intervals. There were extra steps for this:



The derivative detects the max/min at A, B, respectively, but not at the endpoints a, b because they are not max/min for the entire function, just when we restrict to $[a,b]$. So we need to test the endpoints. We have to do something similar in higher dimensions.

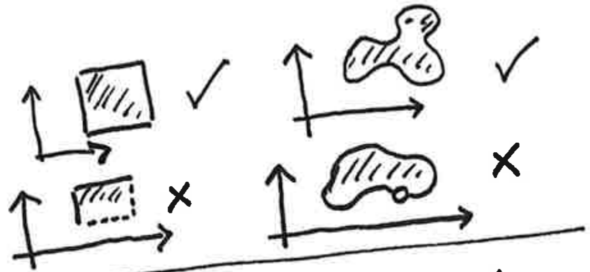
Extreme Value Theorem: If $X \subseteq \mathbb{R}^n$ is compact and $f: X \to \mathbb{R}$ is continuous, then f must have a global max & min somewhere on X, ie there exist $\vec{a}_{min}$ and $\vec{a}_{max}$ with

$$f(\vec{a}_{min}) \leq f(x) \leq f(\vec{a}_{max})$$

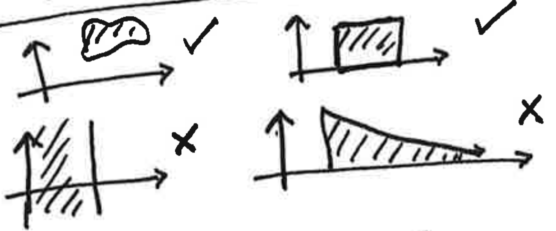
for all $x \in X$.

Compact (in $\mathbb{R}^n$): A region $X \subseteq \mathbb{R}^n$ is compact if it is closed & bounded

Closed

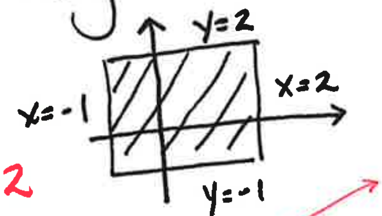


Bounded:



We proceed as usual: use the derivatives to find max/min inside X, test the boundary, and compare to find the global max/mins.

Student Ex: Find the global max/min for $f(x,y) = x^2 - xy + y^2 + 1$ on/in the region:



$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$f_x = 2x - y \quad f_{xx} = 2$

$f_y = -x + 2y \quad f_{yy} = 2$

$f_{xy} = f_{yx} = -1$

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 0 \end{cases} \rightarrow y = 2x \rightarrow \begin{array}{c} -x + 2(2x) = 0 \\ x = 0 \end{array} \rightarrow \begin{array}{c} 0 - y = 0 \\ y = 0 \end{array}$$

(0,0) only critical value

(0,0) min.

$$(0,0): \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{array}{l} d_1 = 2 > 0 \\ d_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0 \end{array}$$

Now we test the boundary:

Left Edge $x = -1$: $f(-1,y) = y^2 + y + 2 \rightarrow h'(y) = 2y + 1 \rightarrow (-1, -1/2)$

$h(y) = y^2 + y + 2$ $\quad h' = 0 \rightarrow y = -1/2$

Right Edge $x = 2$: $f(2,y) = y^2 - 2y + 5 \rightarrow h'(y) = 2y - 2 \rightarrow y = 1 \rightarrow (2,1)$

$h(y) = y^2 - 2y + 5$ $\quad h' = 0$

Top Edge $y = 2$: $f(x,2) = x^2 - 2x + 5 \rightarrow h'(x) = 2x - 2 \rightarrow (1,2)$

$h(x) = x^2 - 2x + 5$ $\quad h' = 0 \rightarrow x = 1$

Bottom Edge $y = -1$: $f(x,-1) = x^2 + x + 2 \rightarrow h'(x) = 2x + 1 \rightarrow (-1/2, -1)$

$h(x) = x^2 + x + 2$ $\quad h' = 0 \rightarrow x = -1/2$

We need also include the corners.

(x,y)	f(x,y)
(-1,2)	8
(2,2)	5
(2,-1)	8
(-1,-1)	2

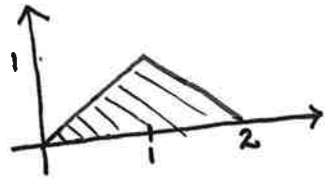
(x,y)	f(x,y)
(0,0)	1
(-1,-1/2)	7/4
(2,1)	4
(1,2)	4
(-1/2,-1)	7/4

Global Max: 8 at (-1,2) & (2,-1)

Global Min: 1 at (0,0).

Again, these problems are not difficult but very, very time consuming. Just a note of how to handle more general regions

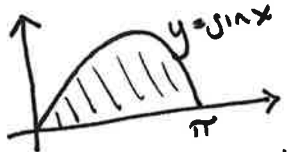
Ex: $f(x,y)$ on



Consider points/curves:

1) Critical values for f inside/on the region.
2) Left side (the line $y=x$). Max/mins of $f(x,x)$ on $[0,2]$
3) Right side (the line $y=2-x$) Max/min of $f(x,2-x)$ or $f(2-y,y)$ on $[0,2]$ or $[0,1]$, resp.
4) Points $(0,0)$, $(2,0)$, $(1,1)$

Ex: $f(x,y)$ on



1) Crit. values for f inside/on the region
2) Bottom edge ($y=0$). Max/min of $f(x,0)$ on $[0,\pi]$.
3) Top edge (curve $y=\sin x$). Max/min of $f(x,\sin x)$ on $[0,\pi]$.
4) Points $(0,0)$ & $(\pi,0)$.

§11.8: Lagrange Multipliers

Topics

- Lagrange multipliers

You Should Be Able to...

- Explain the method of Lagrange Multipliers.
- Find max/min of functions subject to constraints.
- Find max/min of functions on/in regions.

We have seen how to find max/min values for functions of many variables. But thus far the problems have lacked a 'real world flavor'!

Ex: Suppose you sell x units at price y. The revenue is given by $P(x,y) = xy$. The max revenue is clearly when $x = y = \infty$, but this is not possible.

Clearly, $x = y = \infty$ is not 'real world'. But examples do not need to be so extreme.

Ex: $P(x,y) = 10 + 200x + 300y - 0.03x^{3/2} - 0.02y^{3/2}$
Max at $x \approx 19{,}753{,}100$, $y = 100{,}000{,}000$. Though these values are finite, they are unreasonable.

'Real world' problems have limitations: supply, demand, money, time, energy, space, etc.

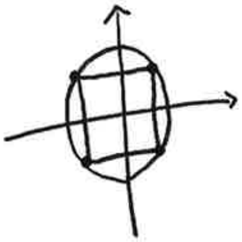
How do we find max/mins of functions relative to a list of constraints, ie limitations?

This will be the method of Lagrange multipliers.

$$\begin{cases} \text{Maximize/Minimize } f(\vec{x}) \\ \text{Constraint: } g(\vec{x}) = c \end{cases}$$

Idea of the Method:

Suppose we want to find rectangle of maximum area inscribed in the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, ie



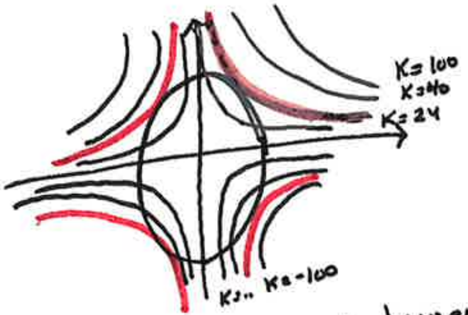
$$\begin{cases} f(x,y) = 4xy \\ g(x,y) = 1 \end{cases}$$

$$\underbrace{\underbrace{g(x,y)}_{}-1=0}_{}$$

$$G(x,y) = \frac{x^2}{9}+\frac{y^2}{4}-1$$

⟵ Rect. must have one vertex with $x,y > 0$. But it must be symmetric in x & y as ellipse is
Then area rect. is
$A = lw = (2x)(2y) = 4xy$

Call the possible outputs of $f(x,y)$ K. Then we want to consider $f(x,y)=K$. But these are level curves for $f(x,y)$. But we have a constraint $g(x,y)=1$, ie the points (x,y) must lie on the ellipse. We can plot this constraint & the level curves together.



* Level curves $f(x,y)=K$ are hyperbolas

We want to max. $f(x,y)$. So we want hyperbola, ie level curve, that lies on/along/intersects ellipse.... but just barely, ie we want level curve tangent to ellipse. That way we satisfy constraint but are 'free' to choose 'largest' possible level curve. The tangents have slopes/directions ∇f, ∇g. To be parallel, they are scalars of each other. So we want:

$$\nabla f = \lambda \nabla g$$

* Could write $\nabla g = \lambda' \nabla f$, as then $\nabla f = 1/\lambda' \, \nabla g = \lambda \nabla g$, where $\lambda = 1/\lambda'$.

Thm: Suppose $X \subseteq \mathbb{R}^n$ is open and $f, g : X \to \mathbb{R}$ are C^1 functions. Let $S = \{\vec{x} \in X : g(\vec{x}) = c\}$ are the level sets of g at c. If $f|_S$, the restriction of f to S, has an extremum at $\vec{x}_0 \in S$ and $\nabla g(\vec{x}_0) \neq \vec{0}$, there is a scalar λ such that

$$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$$

Rem: The scalar λ is called the Lagrange multiplier, after Joseph Lagrange (1736-1813) who developed the method.

* Be slow, methodical, and organized when solving these problems.

* There is no general method to solve these, and often they can be solved only numerically. Again, slow & steady wins here.

Ex:
$$\begin{cases} f(x,y) = 4xy \\ g(x,y) = \frac{x^2}{9} + \frac{y^2}{16} = 1 \end{cases}$$

$$\nabla f = \lambda \nabla g$$
$$\langle 4y, 4x \rangle = \lambda \langle \tfrac{2}{9}x, \tfrac{1}{8}y \rangle$$

Then we have ....

$$\begin{cases} 4y = \frac{2}{9}\lambda x \\ 4x = \frac{1}{8}\lambda y \\ \frac{x^2}{9} + \frac{y^2}{16} = 1 \end{cases} \qquad \longrightarrow \lambda = \frac{18y}{x} \qquad \longrightarrow \qquad \begin{aligned} 4x &= \tfrac{1}{8}\left(\tfrac{18y}{x}\right)y \\ 4x &= \tfrac{9}{4}\tfrac{y^2}{x} \\ x^2 &= \tfrac{9}{16}y^2 \end{aligned}$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \rightarrow \frac{1}{9}\cdot\frac{9}{16}y^2 + \frac{y^2}{16} = 1 \rightarrow \frac{1}{8}y^2 = 1$$

$$y^2 = 8$$
$$y = \pm 2\sqrt{2} \;;\; y > 0$$
$$y = 2\sqrt{2}$$

……

$$x^2 = \frac{9}{16}y^2 = \frac{9}{16}(8) = 9/2$$

$$x = 3/\sqrt{2} \;;\; x > 0$$

Max Area: $f(3/\sqrt{2}, 2\sqrt{2}) = 4(3/\sqrt{2})(2\sqrt{2}) = 24$

* (Could use $g(x,y) = \frac{x^2}{9} + \frac{y^2}{16}$ or $g(x,y) = \frac{x^2}{9} + \frac{y^2}{16} - 1$ because same after taking the gradient.

Student Ex: The Cobb-Douglas production function is used in Economics for production of single goods with 2 factors: $Y = AL^{\beta}K^{\alpha}$. Suppose production given by $f(x,y) = 100x^{3/4}y^{1/4}$; x units labor, at \$150/unit, y units capital, \$250/unit. Total expenses at most \$50,000. Find max production level. Interpret Lagrange mult.

$$\begin{cases} f(x,y) = 100x^{3/4}y^{1/4} \\ g(x,y) = 150x + 250y = 50000 \end{cases}$$

* why not ≤? why can we simply use =?

$$\nabla f = \lambda \nabla g$$

$$\left\langle 75\frac{1}{x^{1/4}}y^{1/4},\ 25x^{3/4}\frac{1}{y^{3/4}}\right\rangle = \lambda\langle 150, 250\rangle$$

$$\begin{cases} 75\sqrt[4]{\frac{y}{x}} = 150\lambda \\ 25\left(\frac{x}{y}\right)^{3/4} = 250\lambda \\ 150x + 250y = 50000 \end{cases}$$

$$\lambda = \frac{1}{2}\sqrt[4]{\frac{y}{x}} = \frac{1}{10}\left(\frac{x}{y}\right)^{3/4} \to \sqrt[4]{\frac{y}{x}} = \frac{1}{5}\left(\frac{x}{y}\right)^{3/4} \to \frac{y}{x} = \frac{1}{625}\frac{x^3}{y^3}$$

$$5^4y^4 = x^4$$

$$x = \pm 5y$$

$$x = 5y$$

$$150x + 250y = 50000 \to 150(5y) + 250y = 50000$$

$$1000y = 50000$$

$$y = 50$$

$$y = 50 \to x = 250 \longrightarrow f(250, 50) \approx 16{,}719 \text{ units}$$

$$\lambda = \frac{1}{2}\sqrt[4]{\frac{y}{x}} = \frac{1}{2}\sqrt[4]{\frac{1}{5}} \approx 0.334$$

$$\nabla f = \lambda \nabla g$$

$$\Delta f = \lambda \Delta g$$

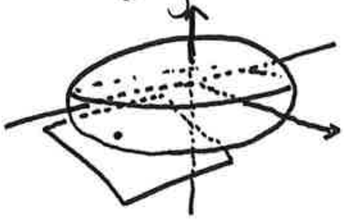
$$\text{unit } \Delta f = \lambda \cdot \text{unit } \Delta g$$

For every \$1 spent on production, approximately 0.334 additional units of product can be made, ie every \$1000 spent means 334 more units.

* Called marginal productivity of money

* Could divide first two equations right away (another common trick) to find $3 \cdot \frac{y}{x} = \frac{3}{5} \Rightarrow x = 5y$. This technique works well when everything is given by products.

Student Example: Find min. value of $f(x,y,z)=2x^2+y^2+3z^2$ subject to $2x-3y-4z=49$.



$$\begin{cases} f(x,y,z)=2x^2+y^2+3z^2 \\ g(x,y,z)=2x-3y-4z=49 \end{cases}$$

$$\nabla f=\lambda\nabla g$$

$$\langle 4x,2y,6z\rangle=\lambda\langle 2,-3,-4\rangle$$

$$\begin{cases} 4x=2\lambda \\ 2y=-3\lambda \\ 6z=-4\lambda \\ 2x-3y-4z=49 \end{cases}$$

$x=\frac{1}{2}\lambda$; $y=-\frac{3}{2}\lambda$; $z=-\frac{2}{3}\lambda$ → $2(\frac{1}{2}\lambda)-3(-\frac{3}{2}\lambda)-4(-\frac{2}{3}\lambda)=49$

$\lambda+\frac{9}{2}\lambda+\frac{8}{3}\lambda=49$

$6\lambda+27\lambda+16\lambda=294$

$49\lambda=294$

$\lambda=6$

$x=\frac{1}{2}(6)=3$

$y=-\frac{3}{2}(6)=-9$

$z=-\frac{2}{3}(6)=-4$

$(3,-9,-4)$

$f(3,-9,-4)=$
$2(3)^2+(-9)^2+3(-4)^2=147$

OR

$\lambda=2x=-\frac{2}{3}y=-\frac{3}{2}z$

$x=-\frac{1}{3}y$ $\quad z=\frac{4}{9}y$

$2x-3y-4z=49$

$2(-\frac{1}{3}y)-3y-4(\frac{4}{9}y)=49$

$-\frac{2}{3}y-3y-\frac{16}{9}y=49$

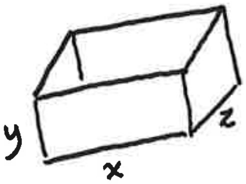
$-6y-27y-16y=441$

$-49y=441$

$y=-9$

But then $x=-\frac{1}{3}(-9)=3$ and $z=\frac{4}{9}(-9)=-4$.

Student Ex: An open rectangular box needs to have a volume of $4\,ft^3$. What box dim. minimizes the materials required?



$$V = V(x,y,z) = xyz$$

$$V(x,y,z) = 4$$

$$A = A(x,y,z) = \underbrace{2xy}_{\text{front/back}} + \underbrace{2yz}_{\text{sides}} + \underbrace{xz}_{\text{bottom}}$$

$$\begin{cases} A(x,y,z) = 2xy + 2yz + xz \\ V(x,y,z) = xyz = 4 \end{cases}$$

$$\nabla A = \lambda \nabla V$$

$$\langle 2y+z,\ 2x+2z,\ 2y+x\rangle = \lambda\langle yz,\ xz,\ xy\rangle$$

$$\begin{cases} 2y+z = \lambda yz \\ 2x+2z = \lambda xz \\ 2y+x = \lambda xy \\ xyz = 4 \end{cases}$$

$$\lambda = \frac{2y+z}{yz} = \frac{2x+2z}{xz} = \frac{2y+x}{xy}$$

$$\lambda = \underbrace{\frac{2}{z} + \frac{1}{y} = \frac{2}{z}} + \underbrace{\frac{2}{x} = \frac{2}{x}} + \frac{1}{y}$$

$$\frac{1}{y} = \frac{2}{x} \qquad \frac{2}{z} = \frac{1}{y}$$

$$x = 2y \xleftrightarrow[x=z]{} z = 2y$$

$$V(x,y,z) = 4$$
$$xyz = 4$$
$$(2y)y(2y) = 4$$
$$4y^3 = 4$$
$$y^3 = 1 \Rightarrow y = 1$$

$\underline{y=1}$
$x = 2(1) = 2$
$z = 2(1) = 2$

$\boxed{2ft \times 2ft \times 1ft}$

OR

$$V = 4$$
$$xyz = 4$$
$$z = \frac{4}{xy}$$

$$A(x,y) = A(x,y,4/xy)$$
$$= 2xy + 2y\left(\frac{4}{xy}\right) + x\left(\frac{4}{xy}\right)$$
$$= 2xy + \frac{8}{x} + \frac{4}{y}$$

$$\begin{cases} A_x = 2y - \frac{8}{x^2} = 0 \\ A_y = 2x - \frac{4}{y^2} = 0 \end{cases}$$

$$y = 4/x^2$$

$$2x - 4\left(\frac{x^4}{16}\right) = 0 \Rightarrow x\left(1 - \frac{1}{8}x^3\right) = 0$$

$x = 0$ or $x = 2$
$\Downarrow$
$y = 1$

$$\begin{pmatrix} 16/x^3 & 2 \\ 2 & 8/y^3 \end{pmatrix}\Bigg|_{(2,1)} = \begin{pmatrix} 2 & 2 \\ 2 & 8 \end{pmatrix}$$

$d_1 = 2 > 0$
$d_2 = 16 - 4 = 12 > 0$

The theorem|method generalizes to an arbitrary number of constraints: $g_1(\vec{x}) = c_1$, $g_2(\vec{x}) = c_2$, $\dots$, $g_n(\vec{x}) = c_n$ via

$$\nabla f = \sum_{i=1}^{n} \lambda_i \nabla g_i$$

Ex: Find the points on the cone formed by the intersection of the cone $z^2 = x^2+y^2$ and the plane $z = x+y+2$, ie a conic section, closest|farthest from the origin.

Distance: $\sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$, which is minimum if and only if $f(x,y,z) = x^2+y^2+z^2$ is minimized.

$$\begin{cases} f(x,y,z) = x^2+y^2+z^2 \\ g_1(x,y,z) = x^2+y^2-z^2 = 0 \\ g_2(x,y,z) = x+y-z = -2 \end{cases}$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla g_1 = \langle 2x, 2y, -2z \rangle$$

$$\nabla g_2 = \langle 1, 1, -1 \rangle$$

$$\begin{cases} \left.\begin{array}{l} 2x = 2\lambda_1 x + \lambda_2 \\ 2y = 2\lambda_1 y + \lambda_2 \end{array}\right\} \longrightarrow \lambda_2 = 2x - 2\lambda_1 x = 2y - 2\lambda_1 y \\ 2z = -2\lambda_1 z - \lambda_2 \\ x^2+y^2-z^2 = 0 \\ x+y-z = -2 \end{cases}$$

$$2(x-y)(1-\lambda_1) = 0$$

$$x = y \text{ or } \lambda_1 = 1$$

If $\lambda_1 = 1$, then $\lambda_2 = 0$. Third equation then gives $2z = -2z$ so $z=0$. Then fourth equation shows $x=y=0$. But $(0,0,0)$ is not on plane $x+y-z=-2$.

$x=y$: last two equations

$$\begin{cases} 2x^2 - z^2 = 0 \\ 2x - z = -2 \end{cases} \longrightarrow z = 2x+2 \longrightarrow 2x^2 - (2x+2)^2 = 0$$

$$2x^2 + 8x + 4 = 0$$

$$x = -2 \pm \sqrt{2}$$

$$y = -2 \pm \sqrt{2}$$

$$z = 2(-2 \pm \sqrt{2}) + 2 = -2 \pm 2\sqrt{2}$$

$$f('+','+','+') = 24 - 16\sqrt{2}$$

$$f('-','-','-') = 24 + 16\sqrt{2}$$

Lagrange Mult

&

Extrema over regions ?

§12.1-12.2: Double Integrals

Topics

- Double Integrals

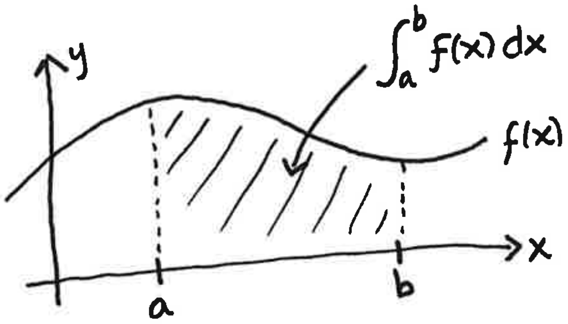
You Should Be Able to......

- Explain how the double integral is defined.
- Interpret the values of double integrals.
- Approximate the value of double integrals.
- Integrate double integrals.
- Set-up double integrals.
- Reverse the order of integration.
- Find areas of regions using double integrals.
- Explain why double integrals can compute areas & volumes.
- Compute volumes using double integrals.

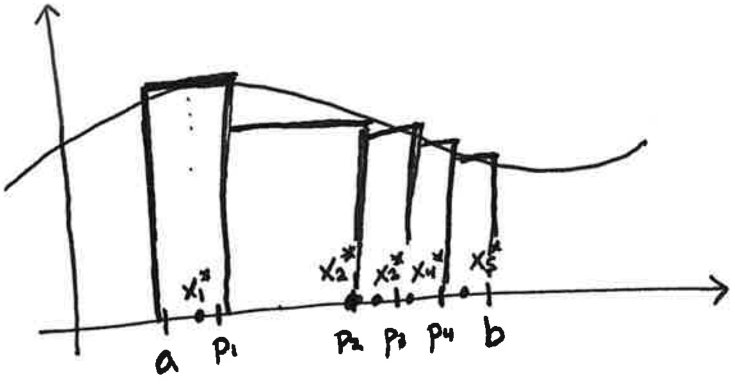
Before defining double, triple, ... integrals, let's review the ordinary Riemann integral.

Review

We shall think of $\int_a^b f(x)\,dx$ as the (signed) area of $f(x)$ between a & b.



We would approximate this area by using sums of areas of rectangles. To do this, we broke up (partioned) the interval $[a,b]$, and chose x-values x_i^* (usually left/right/midpoint) from each piece, then found heights $f(x_i^*)$. The approximation was then $\sum$ Rect Area$_i$ $= \sum f(x_i^*)\,\Delta x_i$



$$\int_a^b f(x)\,dx \approx f(x_1^*)(p_1-a) + f(x_2^*)(p_2-p_1) + f(x_3^*)(p_3-p_2) + f(x_4^*)(p_4-p_3) + f(x_5^*)(b-p_4)$$

$$= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3 + f(x_4^*)\Delta x_4 + f(x_5^*)\Delta x_5$$

$$= \sum_i f(x_i^*)\,\Delta x_i$$

* The x_i^* (called tags) were usually consistently chosen to be some special values, eg. left endpoints, right endpoints, midpoints, etc.

* The breakdown of $[a,b]$ (called the partition of $[a,b]$) was usually specially chosen so that each Δx_i was the same, eg $\Delta x = \frac{b-a}{n}$.

The value $\max_{0 \le i \le n} \{\Delta x_i\}$ is called the mesh of the partition of $[a,b]$, eg $\frac{b-a}{n}$. Call the mesh M_i. We defined the integral to be...

$$\int_a^b f(x)\,dx := \lim_{M_i \to 0} \sum f(x_i^*)\Delta x_i$$

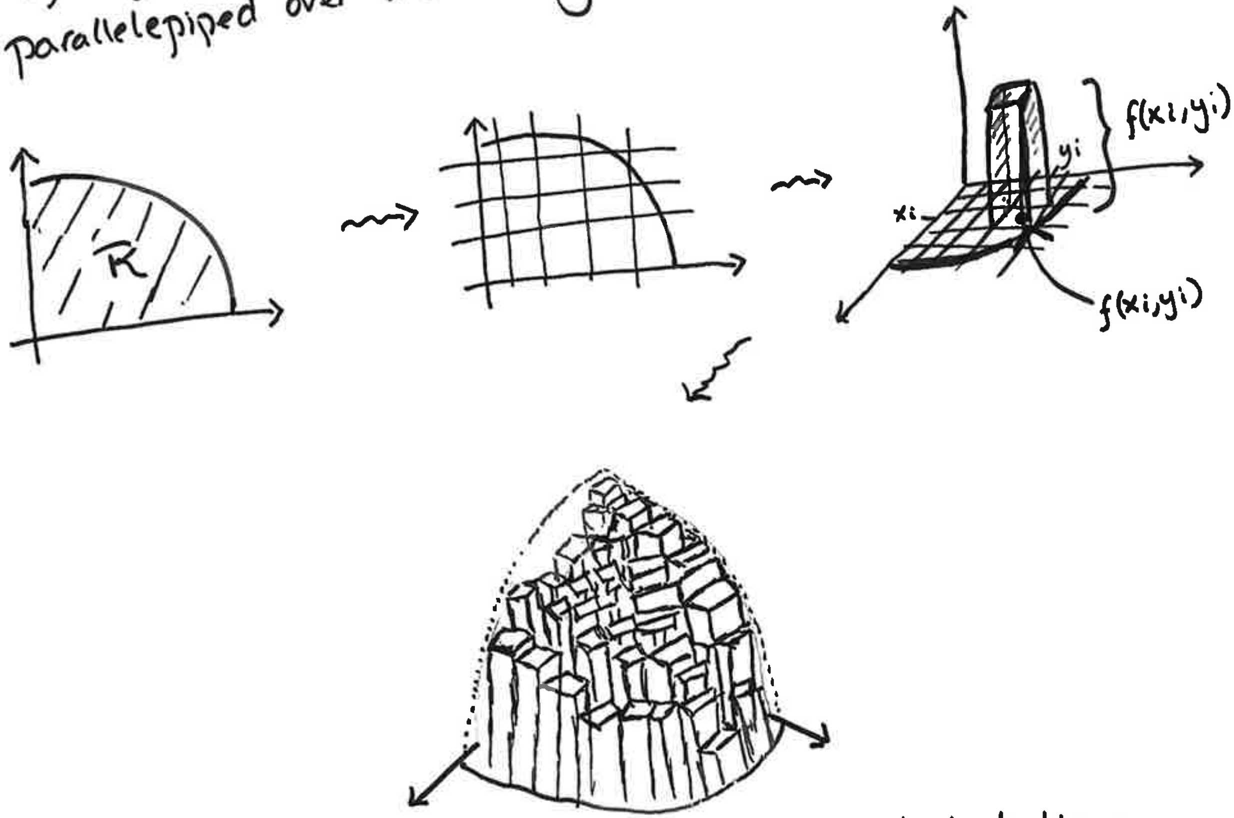
That is, $\int_a^b f(x)\,dx$ existed if the values obtained from $\sum_i f(x_i^*)\Delta x_i$ are 'nearly' always the same no matter how you broke up $[a,b]$ and no matter what x_i^* you chose so long as the width of each piece of the interval $[a,b]$ was 'small'. The 'common value' of these sums was the value of the integral.

We want to do the same for more variables, ie more dimensions. Note before we integrated over an interval $[a,b]$, which has length. Next, we want to integrate over $[a,b] \times [c,d]$, which has area.

Notice from the definition of the ordinary Riemann integral, we can define what we mean by approximate, then define the integral to be the limit of these approximations.

$\iint_R f(x,y)\,dA$

What do we mean by $\iint_R f(x,y)\,dA$? By R we mean some region, f(x,y) will be a function. The differential dA will mean some 'small' chunk of area. So an approximote definition of $\iint_R f(x,y)\,dA$ will be to break R into rectangles, choose points from each rectangle, plug these points into f(x,y), use this as a height. Then we can find the volume of some parallelepiped over the rectangle. We then add these up.



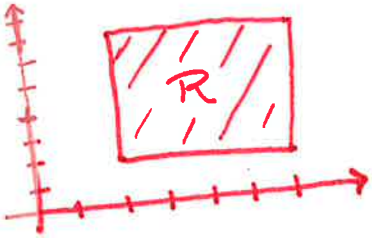
Notice this seems to be computing a volume. Indeed, this is the case:

	Region	Interp.	Integrand	Output
$\int_a^b f(x)\,dx$:	[a,b]	lengths	height f(x)	(Signed) Area
$\iint_R f(x)\,dA$:	R	area's	height f(x)	(Signed) Volume

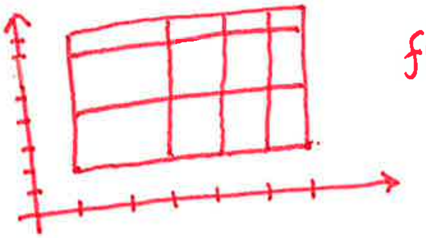
Of course, $\int_a^b f(x)\,dx$ did not have to represent area. If $f(x)$ gave income at time x, $\int_a^b f(x)\,dx$ represented total income between time $x=a$ and $x=b$. Similarly, if income $f(x,y)$ was given by x,y, say representing sales of products x,y, then $\iint_R f(x,y)\,dA$ is total income for $(x,y)\in R$, where R is some region in the plane.

Now let's approximate $\iint_R f(x,y)\,dA$, which we have still yet to formally define.

Ex: $\iint_R (x+y)\,dA$, where $R=\{(x,y) : 1\le x\le 6,\ 2\le y\le 7\}$



We break up R into pieces

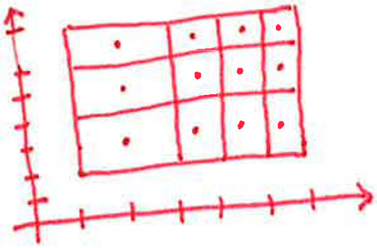


$f(x,y)=x+y$

We need to choose special values from each piece of this breakdown of R. For example, the middle or bottom left.

$$\sum \overbrace{f(x_i,y_i)}^{\text{height}} * \overbrace{(\Delta x\,\Delta y)}^{\text{Area}} = \text{Volume}$$

Middle:



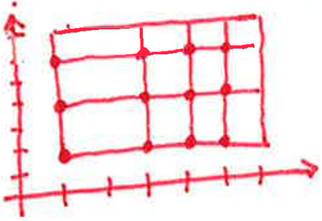
Points

(2,3)
(2,5)
(2,6.5)
(3.5,3)
(3.5,5)
(3.5,6.5)
(4.5,3)
(4.5,5)
(4.5,6.5)
(5.5,3)
(5.5,5)
(5.5,6.5)

$5(4) + 10(1) + 10.5(2)$
$+ 7(4) + 7.5(2) + 12(1)$
$+ 8.5(2) + 9.5(2)$
$+ 6.5(2) + 11(1)$
$+ 8.5(2) + 8.5(2)$

$\boxed{200}$

Bottom Left:



Points

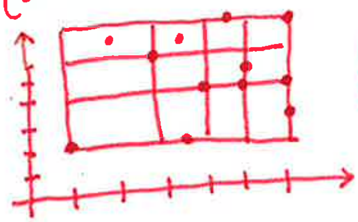
(1,2)
(1,4)
(1,6)
(3,2)
(3,4)
(3,6)
(4,2)
(4,4)
(4,6)
(5,2)
(5,4)
(5,6)

$3(4) + 7(2) + 10(1)$
$+ 5(4) + 9(1) + 7(2)$
$+ 7(2) + 6(2) + 9(2)$
$+ 5(2) + 8(2) + 11(1)$

$\boxed{1160}$

Or we can even choose however we wish...

'Arbitrary':



Points

(1,2)
(3,6)
(2,6.5)
(3.5,2)
(4,4)
(3.5,6.5)
(5,4)
(5,5)
(4.5,7)
(6,3)
(6,4)
(6,7)

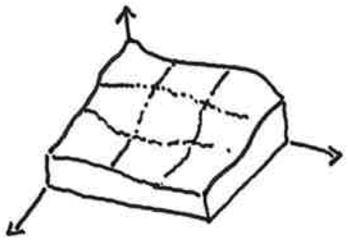
$3(4) + 8(2) + 11.5(1)$
$+ 9(4) + 10(1) + 9(2)$
$+ 8.5(2) + 9(2) + 10(2)$
$+ 5.5(2) + 10(2) + 13(1)$

$\boxed{202.5}$

For integral to exist, need all these values 'close' if □'s chosen 'small'

Ex: Table of values gives depth of dirt in pile over a region (in ft).
Approximate $\iint_R f(x,y)\,dA$.

x\y	1	3	5
1	10	9	7
3	7	7	4
4	5	5	4
10	4	5	3



Looking at table, we could produce (probably) an over- & under-estimate.

Over: $\sum \underbrace{f(x_i,y_i)}_{\text{height}}\underbrace{(\Delta x\Delta y)}_{\text{Area}}$

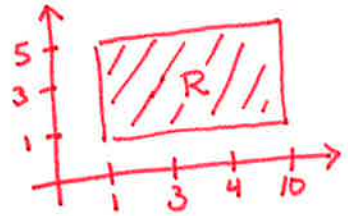
$10(2\cdot 2)+9(2\cdot 2)+7(1\cdot 2)+7(1\cdot 2)+5(6\cdot 2)+5(6\cdot 2)$

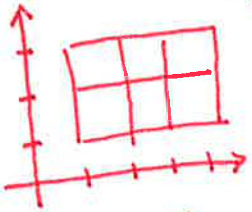
$\boxed{224}$

Under: $\sum_i \overbrace{f(x_i,y_i)}^{\text{height}}\cdot\overbrace{(\Delta x\Delta y)}^{\text{Area}}$

$7(2\cdot 2)+4(2\cdot 2)+5(1\cdot 2)+4(1\cdot 2)+4(6\cdot 2)+3(6\cdot 2)$

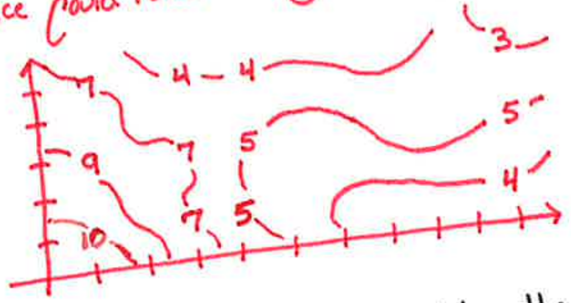
$\boxed{146}$





Between 146 & 224 ft^3 of dirt.

Notice could have been given diagram ↺ or even a contour plot and much of the problem would have remained the same (can you see why?).



* Notice as we changed the points, the areas were the same, because the region was the same. This is a general concept: in $\iint_R f(x,y)\,dA$, R & f(x,y) have 'nothing' to do with each other. We can integrate any function over R (assuming it is defined over R.)

* We formed areas by breaking up x & y, this gave us 'tiny' areas $\Delta A_i = \Delta x_i \Delta y_i$.

Then we can define the double integral: Break R into rectangles (rect. region (cont. R), by breaking up x's & y's. Choose $\vec{x}^*_{i,j}$ from each rect in R, let M be area of biggest rect., and define...

$$\iint_R f(x,y)\,dA = \lim_{M\to 0}\sum_{i,j} f(\vec{x}^*_{i,j})\,\overset{\Delta x_i \Delta y_i}{\Delta A_i}$$

If the limit exists. We can define $\iint\cdots\int f(\vec{x})\,d-$ similarly.

Note: With some modification, something called Fobini's Theorem shows all this works as expected. But we shall gloss this over.

All the usual properties of the ordinary Riemann integral still hold:

1) $\iint_R (f \pm g)\, dA = \iint_R f\, dA \pm \iint_R g\, dA$

2) $\iint_R cf\, dA = c \iint_R f\, dA$

3) If $f \leq g$ on R, then $\iint_R f\, dA \leq \iint_R g\, dA$

4) $\left| \iint_R f\, dA \right| \leq \iint_R |f|\, dA$

5) $\frac{1}{\text{area}(R)} \iint_R f\, dA =$ average value of f on R.

↳ often a double integral itself, as we shall see: $\text{area}(R) = \iint_R dA$

How do we compute double integrals? We work inside out, treating the variables we are not integrating as constant, just as with partial derivatives.

Ex: $\int_2^7 \int_1^6 (x+y)\, dx\, dy$

Integrate w.r.t. x first $\rightsquigarrow$ y's constant. 'See'...

$$\int_2^7 \int_1^6 (x + \square)\, dx\, dy$$

$$\int_2^7 \left(\frac{x^2}{2} + \square x \right) \Big|_1^6 dy \longrightarrow$$

Could write $\Big|_{x=1}^{x=6}$ for emphasis or to help keep track of things

$$\int_2^7 \left(\frac{x^2}{2} + yx \right) \Big|_1^6 dy$$

$$\int_2^7 (6y+18) - (y+1/2)\, dy$$

$$\int_2^7 (5y + 35/2)\, dy = \frac{5y^2}{2} + \frac{35y}{2} \Big|_2^7 = 245 - 45$$

$$= 200$$

Which aligns with our estimate from before!

Ex: $\int_0^2\int_1^2 \frac{y}{x}\,dx\,dy$

$'\int_0^2\int_1^2 \frac{\square}{x}\,dx\,dy'$

$'\int_0^2 \square \ln|x| \Big|_1^2\,dy'$

$\int_0^2 y \ln|x| \Big|_{x=1}^{x=2}\,dy$

$\int_0^2 y(\ln 2 - \ln 1)\,dy$

$\ln 2 \int_0^2 y\,dy = \ln 2 \cdot \frac{y^2}{2}\Big|_0^2 = 2\ln 2$

Ex: $\int_0^1\int_0^x x e^y\,dy\,dx$

$\int_0^1\int_0^x \square\, e^y\,dy\,dx$

$\int_0^1 x e^y \Big|_0^x\,dx$

$\int_0^1 x e^x - x e^0\,dx$

$\int_0^1 x e^x - x\,dx$

$x e^x - e^x - \frac{x^2}{2}\Big|_0^1$

$-\frac{1}{2} - -1 = 1/2$

Note: Sometimes you can separate the integrals entirely!

Ex: $\int_0^1\int_0^1 x y\,dy\,dx = \int_0^1\int_0^1 \square\, y\,dy\,dx = \int_0^1 \square \int_0^1 y\,dy\,dx =$

$\int_0^1 x \underbrace{\int_0^1 y\,dy}_{\text{depends only on } y}\,dx = \left(\int_0^1 y\,dy\right)\left(\int_0^1 x\,dx\right)$

You want to be very careful doing this! Note would not work with

$\int_0^1\int_0^1 x+y\,dx\,dy$ or $\int_0^1\int_0^x x y\,dy\,dx$.

* Be careful with indefinite integrals: $\iint x+y\,dy\,dx = \int xy + \frac{y^2}{2} + C_1\,dx = \frac{x^2}{2}y + \frac{xy^2}{2} + C_1 x + C_2$

Two Important Things:

* The integrand and limits of integration have nothing to do with each other. So long as f is defined over a region R, we can stick f into any $\iint_R - \, dA$. Different f's just make the problem harder/easier – but we can still choose how to set-up the integral to maybe make things easier. This leads to...

* Iterated Integrals are all about the setup, ie the 'order' to dA and the limits of integration. Here is something incredibly important

✳✳ Though we evaluate integrals inside-out, they are ✳✳ defined from outside-in!!

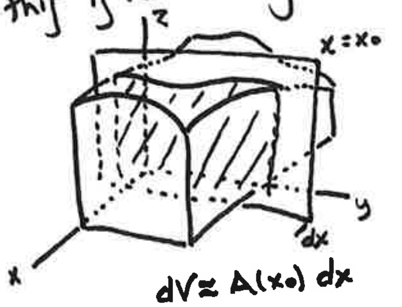
Meaning, we use 'fancy' Fundamental Theorem of Calculus ideas to compute our integrals. But from the definition, it is really limits of iterated sums. Think about how you would approximate an integral by hand or get a computer to do it....

$$\int_0^1 \int_0^x \underbrace{xy}_{} \, dy \, dx \approx \sum \sum \underbrace{xy}_{} \Delta y \, \Delta x$$

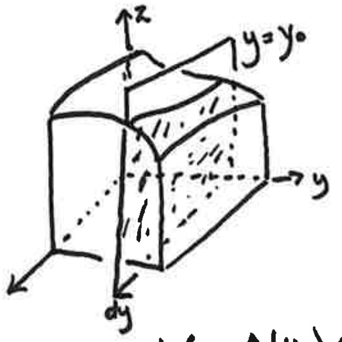
plugging in y's but integrate up to x... but don't know where to stop unless x has been fixed...

plugging in for y's but can't get numbers unless x's also numbers...

So while we think inside → out, integrals work|defined outside → in. This is how you will set them up, take a 'slice' in one variable and see how other, more inner variable(s) vary. Indeed, this is how we get volumes:

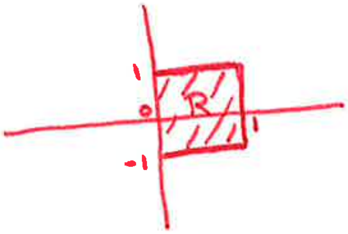


$dV \approx A(x_0)\, dx$



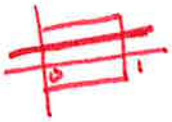
$dV \approx A(y_0)\, dy$

Ex: $\iint_R f(x,y)\,dA$; $R = \{ (x,y) : 0 \le x \le 1, -1 \le y \le 1 \}$



Sketch R first.

dx dy: Slice in 'y' and see how 'x' varies



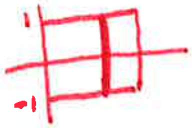
x goes from 0 to 1.

$\int_?^? \int_0^1 f(x,y)\,dx\,dy$

What 'y' slices can be chosen? Any from -1 to 1.

$$\int_{-1}^1 \int_0^1 f(x,y)\,dx\,dy$$

dy dx: Slice in 'x' and see how 'y' varies



y goes from -1 to 1.

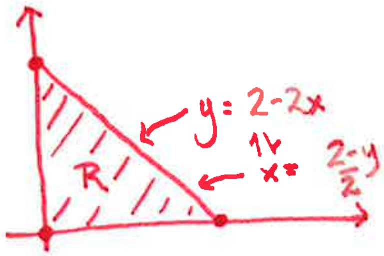
$\int_?^? \int_{-1}^1 f(x,y)\,dy\,dx$

What 'x' slices can be chosen? Any from 0 to 1

$$\int_0^1 \int_{-1}^1 f(x,y)\,dy\,dx$$

* Notice none of this setup at all involved the function f(x,y). It was just 'there for the ride!

Ex: $\iint_R f(x,y)\,dA$ R is the triangular region in the xy-plane with vertices (0,0), (1,0), and (0,2).



$m = \frac{2-0}{0-1} = \frac{2}{-1} = -2$

$y = mx+b$

$y = -2x+b$

$2 = -1(0)+b$

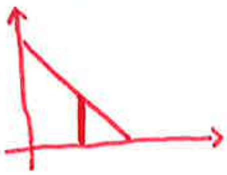
$b=2$

$\rightarrow$ $y = 2-2x$

$\Updownarrow$

$x = \frac{2-y}{2}$

dy dx: Slice in 'x' and see how 'y' varies.



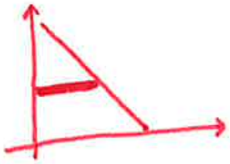
'y' goes from 0 to whatever 2-2x is, ie up to line

$$\int_?^? \int_0^{2-2x} f(x,y)\,dy\,dx$$

What 'x' slices can we take? Any from 0 to 1.

$$\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$$

dx dy: Slice in 'y' and see how 'x' varies



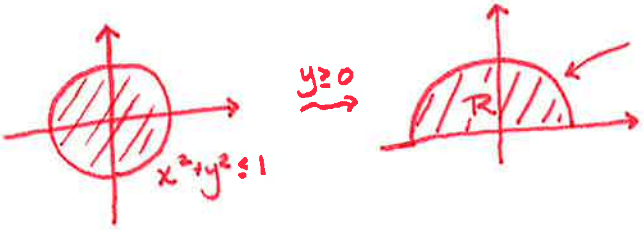
'x' goes from 0 to line, ie $\frac{2-y}{2}$.

$$\int_?^? \int_0^{\frac{2-y}{2}} f(x,y)\,dx\,dy$$

What 'y' slices possible? Any from 0 to 2.

$$\int_0^2 \int_0^{\frac{2-y}{2}} f(x,y)\,dx\,dy$$

Ex: $\iint_R f(x,y)\,dA$ $R = \{(x,y) : x^2+y^2 \le 1,\ y \ge 0\}$



$x^2+y^2=1 \Rightarrow x^2 = 1-y^2$

$x = \pm\sqrt{1-y^2}$

OR

$y^2 = 1-x^2$

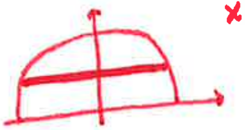
$y = \sqrt{1-x^2}$ ← No ± because know $y \ge 0$.

dy dx: Slice in x and see how y varies:

y goes from 0 to circle, ie $\sqrt{1-x^2}$ → $\int_?^? \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$

x slices from -1 to 1 → $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$

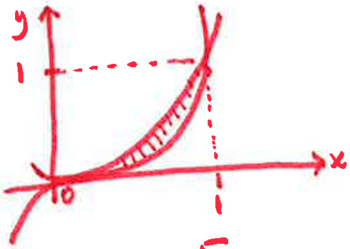
dx dy: Slice in y and see how x varies:



x goes from $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$ → $\int_?^? \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$

y slices from 0 to 1 → $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$

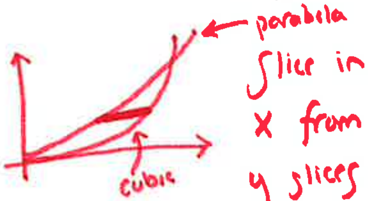
Ex: $\iint_R f(x,y)\,dA$; $R=\{(x,y): 0\le x\le 1,\ x^3\le y\le x^2\}$



$y=x^3 \rightleftharpoons x=\sqrt[3]{y}$

$y=x^2 \rightleftharpoons x=\underbrace{\sqrt{y}}_{x\ge 0}$

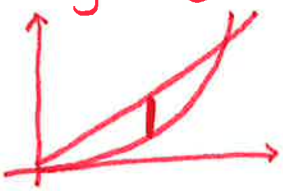
dx dy:



Slice in y, see how x varies: x from $\sqrt{y}$ to $\sqrt[3]{y}$ y slices from 0 to 1.

$$\int_0^1\int_{\sqrt{y}}^{\sqrt[3]{y}} f(x,y)\,dx\,dy$$

dy dx: Slice in x, see how y varies:



y from x^3 to x^2. x slices from 0 to 1.

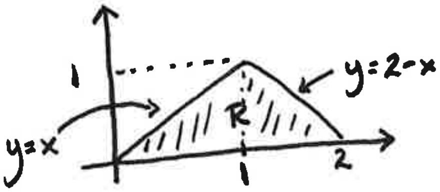
$$\int_0^1\int_{x^3}^{x^2} f(x,y)\,dy\,dx$$

* If you have a variable in its own limits of integration or later, you are $\underline{\text{wrong}}$. Because once evaluated, it is never reevaluated, so if it appears again it will never be 'turned into a number.'

No ⇝ $\int_0^1\int_0^y xy\,dy\,dx$, $\int_0^x\int_0^1 xy\,dx\,dy$, ...

* You have the choice of setup and some are easier than others....

Ex: $\iint_R f(x,y)\,dA$, where R is shown below



dx dy: Slice in y, see how x varies x from y to 2-y. y slices from 0 to 1

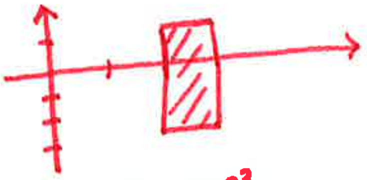
$$\int_0^1\int_y^{2-y} f(x,y)\,dx\,dy$$

dy dx: Slice in x and see how y varies. Notice on left half from 0 to x but on right side from 0 to 2-x. Slices in x from 0 to 1, 1 to 2, respectfully

$$\int_0^1\int_0^x f(x,y)\,dx\,dy + \int_1^2\int_0^{2-x} f(x,y)\,dx\,dy$$

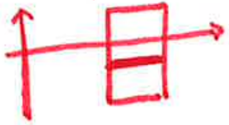
We need to be able to go other way as well, ie from integral to region R. This will allow us to change the order of integration.

Ex: Sketch R in $\int_2^3 \int_{-3}^1 f(x,y)\,dy\,dx$ and reverse the order of int.



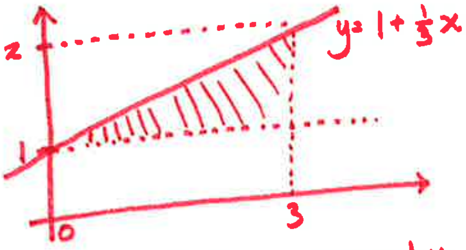
x=2 y=-3
x=3 y=1

dxdy: slice y, see how x varies



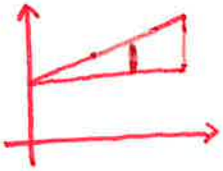
$$\int_{-3}^1 \int_2^3 f(x,y)\,dx\,dy$$

Ex: Sketch R in $\int_1^2 \int_{3y-3}^3 f(x,y)\,dx\,dy$ and reverse the order of int.



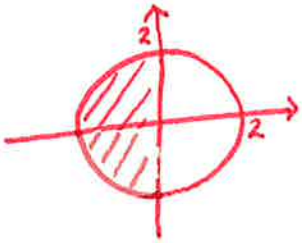
y=1 x=3y-3 → $y = 1 + \frac{1}{3}x$
y=2 x=3

dydx: Slice in x and see how y varies



$$\int_0^3 \int_1^{1+\frac{1}{3}x} f(x,y)\,dy\,dx$$

Ex: Sketch R in $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^0 f(x,y)\,dx\,dy$ and reverse order of int.

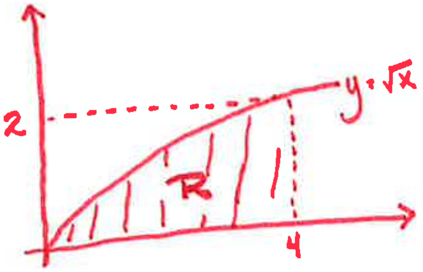


y=-2 x=0
y=2 $x = -\sqrt{4-y^2}$
$x^2 = 4-y^2$
$x^2+y^2=4$

dydx: slice in x, see how y varies

$$\int_{-2}^0 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

Ex: Sketch R in $\int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$ and reverse order of integration.



dx dy: Slice in y and see how x varies

$$\int_0^2 \int_{y^2}^4 f(x,y)\,dx\,dy$$

y = 0
y = √x
x = 0
x = 4

$y = \sqrt{x} \rightleftharpoons x = y^2$

How does this differ from $\int_0^4 \int_{\sqrt{x}}^2 f(x,y)\,dy\,dx$?

$$\int_0^2 \int_0^{x^2} f(x,y)\,dx\,dy$$

We have seen that the order of integration can make the problem easier/harder. Sometimes changing the order of integration can make impossible integrals possible.

Ex: $\int_0^1 \int_y^1 e^{-x^2}\,dx\,dy$ $\quad \int e^{-x^2}dx$ Has no elementary antiderivative,

so as written, the problem is impossible, but....

$$\int_0^1 \int_y^1 e^{-x^2}dx\,dy = \int_0^1 \int_0^x e^{-x^2}\,dy\,dx$$

$$= \int_0^1 y e^{-x^2}\Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 x e^{-x^2} dx$$

$$= -\frac{1}{2} e^{-x^2}\Big|_{x=0}^{x=1}$$

$$= \frac{e-1}{2e}$$

x = y
x = 1
y = 0
y = 1

Ex: $\int_0^{e^2}\int_{e^x}^{2} \frac{dy\,dx}{\ln y}$

$\int_1^2\int_0^{\ln y} \frac{dx\,dy}{\ln y}$

$y = e^x \rightleftarrows x = \ln y$
$y = 2$
$x = 0$
$x = e^2$

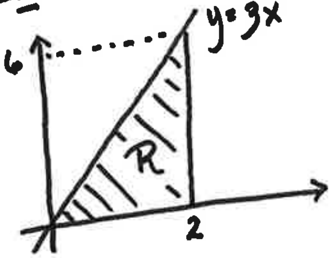
What are the classic examples of non-integrable functions?

$\int e^{-x^2}dx$, $\int \sin x^2\,dx$, $\int \cos x^2\,dx$, $\int \frac{\sin x}{x}dx$, $\int \frac{\cos x}{x}dx$

Now if $f \equiv 1$, then $\iint_R 1\,dA$ adds up all the infinitesimal small area pieces dA. So this should give the area of R.

$$\text{Area } R = \iint_R dA$$

Ex: Find the area of....



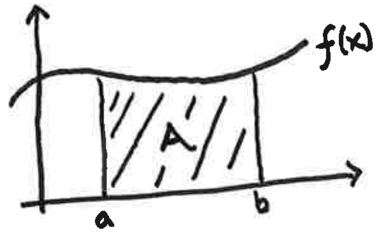
$A = \frac{1}{2}bh$
$= \frac{1}{2}(2)6$
$= 6$

$\text{Area } R = \iint_R dA$
$= \int_0^2\int_0^{3x} dy\,dx$
$= \int_0^2 y\Big|_{y=0}^{y=3x} dx$
$= \int_0^2 3x\,dx$
$= \frac{3x^2}{2}\Big|_{x=0}^{x=2}$
$= \frac{3}{2}(2^2 - 0^2)$
$= \frac{3}{2}\cdot 4 = \frac{3}{1}\cdot 2 = 6$

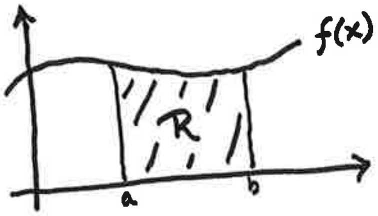
This is the area under the curve $y = 3x$ between $x = 0$ and $x = 2$. From Calculus I, we know this to also be...

$$A = \int_0^2 3x\, dx$$

How do we reconcile these results?



$$A = \int_a^b f(x)\, dx$$



$$\text{area } R = \iint_R dA$$

$$= \int_a^b \int_0^{f(x)} dy\, dx$$

$$= \int_a^b f(x) - 0 \; dy\, dx$$

$$= \int_a^b f(x)\, dx$$

Notice that the concepts align. In fact, the double integral allows us to compute areas of more general regions with simpler logic. In fact, this generalizes...

$$\int_{[a,b]} dx = \text{length } [a,b] \longleftarrow b-a$$

$$\iint_R dA = \text{area } R$$

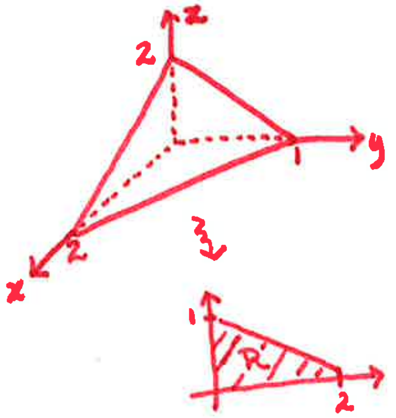
$$\iiint_R dV = \text{volume } R \longleftarrow V = \iiint dV = \iint_R \int_0^f dV = \iint_R f\, dA$$

$$\vdots$$

Thinking about things in this way is 'nice' because the dimensions of the differential matches the dimensions of the output. We shall see more about the third case when we look at triple integrals. But double integrals can still give volumes, as we have seen.

Ex: Find volume enclosed by plane $x + 2y + z = 2$ and the coordinate planes.

$$x + 2y + z = 2 \rightleftharpoons z = 2 - x - 2y$$



$$\iint_R f(x,y)\,dA = \int_0^2 \int_0^{\frac{2-x}{2}} (2 - x - 2y)\,dy\,dx$$

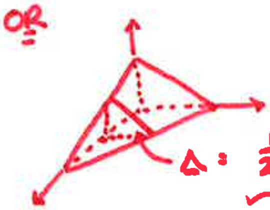
$$= \int_0^2 (2y - xy - y^2)\Big|_{y=0}^{y=\frac{2-x}{2}} dx$$

$$= \int_0^2 \tfrac{1}{4}(2-x)^2\,dx = \tfrac{-1}{12}(2-x)^3\Big|_0^2 = \tfrac{2}{3}$$

<u>OR</u>

Volume Tetrahedron Formula $= \frac{1}{3}A_b h = \frac{1}{3}\left(\frac{1}{2}\cdot 2\cdot 1\right)\cdot 2 = \frac{2}{3}$

<u>OR</u>



By slices: $V = \int A(x)\,dx$;

$$\Delta = \underbrace{\tfrac{1}{2}bh}_{\text{dep on } x} = \tfrac{1}{2}\cdot\left(\tfrac{2-x}{2}\right)(2-x) = \tfrac{(2-x)^2}{4}$$

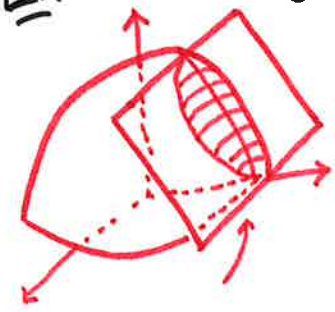
$$V = \int_0^2 \frac{(2-x)^2}{4}\,dx$$

$$= 2/3$$

So all of our notions of volume agree.

Ex: Volume of solid bound by paraboloid $z = 1 - x^2 - y^2$ and plane $z = 1 - y$.

Same as with ordinary integrals, we can do $\iint_R \text{top} - \text{bottom}\,dA$.

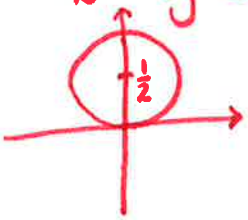


$$z = z$$
$$1 - x^2 - y^2 = 1 - y$$
$$x^2 + y^2 - y = 0$$
$$x^2 + \left(y - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} = 0$$
$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

$$\int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1 - x^2 - y^2) - (1 - y)\,dx\,dy$$

$$= \frac{\pi}{32}$$

Takes a bit of work to show this.



As a final note, what kinds of functions over which types of regions do these techniques apply, ie which functions over 'nice' regions are integrable?

<u>Thm</u>: If $f(\vec{x})$ is continuous for all but finitely many $\vec{x} \in R \subseteq \mathbb{R}^n$, where $R \subseteq \mathbb{R}^n$ is compact, then $\iint \cdots \int_R f(\vec{x})\, d\vec{x}$ exists.

§ 12.5: Triple Integrals

Topics

- Triple Integrals

You Should Be Able to....

- Explain the construction of triple integrals.
- Compute triple integrals.
- Explain why the triple integral computes volume.
- Set-up triple integrals.
- Sketch regions bounded by different surfaces.
- Sketch regions of integration
- Interchange the order of integration.
- Compute volumes using triple integrals.

We define $\iiint_R f\,dV$ just as we did with double integrals

1) Break up R into cubes
2) Choose a point (x_i^*, y_i^*, z_i^*) in each cube.
3) Use $f(x_i^*, y_i^*, z_i^*)$ as a 'height', and form a (hyper) 'volume' $f(x_i^*, y_i^*, z_i^*)\,\Delta V_i \longleftarrow \Delta x_i\,\Delta y_i\,\Delta z_i$
4) Sum our approximations: $\sum_i f(x_i^*, y_i^*, z_i^*)\,\Delta V_i$
5) Define the integral to be the limit (if it exists) of this sum if you force the cubes to be smaller and smaller.

We will not go through this in detail because we have already done this for double integrals, and because to give a visual representation we would need 4-dimensions.... However, we will most often use this to compute volumes, $V = \iiint_R dV$, where we can plot. First, note these are computed just as in double integrals:

Ex: $\int_0^2 \int_0^y \int_1^e \frac{y+z}{x}\,dx\,dz\,dy$

$$\int_0^2 \int_0^y (y+z)\ln x \Big|_{x=1}^{x=e} dz\,dy$$

$$\int_0^2 \int_0^y (y+z)(\ln e - \ln 1)\,dz\,dy$$

$$\int_0^2 \int_0^y (y+z)\,dz\,dy$$

$$\int_0^2 \left(yz + \frac{z^2}{2}\right)\Big|_{z=0}^{z=y} dy$$

$$\int_0^2 \left(y^2 + \frac{y^2}{2}\right) - (0+0)\,dy$$

$$\frac{3}{2}\int_0^2 y^2\,dy$$

$$\frac{3}{2}\cdot\frac{y^3}{3}\Big|_0^2$$

$$\frac{1}{2}\cdot(2^3 - 0)$$

$$4$$

Student Ex: $\int_0^1 \int_0^\pi \int_0^{\sqrt{z}} 2y \cos(xz)\, dy\, dx\, dz$

$$\int_0^1 \int_0^\pi y^2 \cos(xz) \Big|_{y=0}^{y=\sqrt{z}} dx\, dz$$

$$\int_0^1 \int_0^\pi \cos(xz)\,(z-0)\, dx\, dz$$

$$\int_0^1 \int_0^\pi z \cos(xz)\, dx\, dz$$

$$\int_0^1 z \cdot \frac{\sin(xz)}{z} \Big|_{x=0}^{x=\pi} dz$$

$$\int_0^1 \sin(xz) \Big|_{x=0}^{x=\pi} dz$$

$$\int_0^1 \sin(\pi z) - \sin(0)\, dz$$

$$\int_0^1 \sin(\pi z)\, dz$$

$$-\frac{\cos(\pi z)}{\pi} \Big|_{z=0}^{z=1}$$

$$-\frac{1}{\pi}\left(\cos \pi - \cos 0\right)$$

$$-\frac{1}{\pi}\left(-1 - 1\right)$$

$$\frac{2}{\pi}$$

* The same caution about indefinite integrals (if you should ever even see one) applies just as it did with double integrals.

* When it comes to triple integrals, work slowly and methodically, there is a lot that can go wrong.

* Once again, a variable cannot appear anywhere after it has been integrated with respect to, just as with double integrals.

* 'All' the properties of double integrals still apply.

For now, we will mostly use these to compute volumes. V = volume $R = \iiint_R dV$, ie '$\iiint$' sums up all the 'dV' to obtain the total volume. But we already knew that the volume under f over R was $\iint_R f\,dA$, or the volume between f and g over R was $\iint_R (f-g)\,dA$, how do these agree?

$$V = \iiint_R dV = \iint_{R'} \int_g^f dz\,\overset{\swarrow \text{or } dy\,dx}{dx\,dy}$$

$$= \iint_{R'} (f-g)\ dx\,dy \qquad \leftarrow \text{Similar for } \iint_{R'} \int_0^f dz\,dA$$

$$= \iint_{R'} (f-g)\ dA$$

as expected.

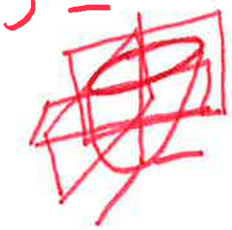
As commented before, $V = \iiint_R dV$ is perhaps more flexible. Moreover, the units of volume more naturally align with the units of the differential so that it is easier to see that this indeed gives you volume.

As with double integrals, the hard part is setting up the correct integral. To set these up, sketches are essential. Your sketchs do not have to be perfect, just good enough to convey the essential pieces to set up the integral. Do this in a series of quick sketchs:

1) Really crappy
2) Clean up so bit clearer
3) Good enough to set-up.

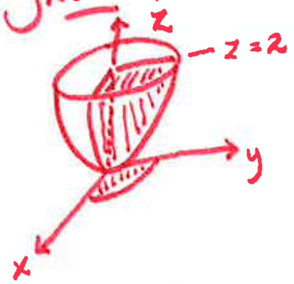
Ex: Let R be the region bounded by $x=0$, $y=0$, $z=2$, and $z=x^2+y^2$. Find the volume of R.

Sketch 1



Sketch 2

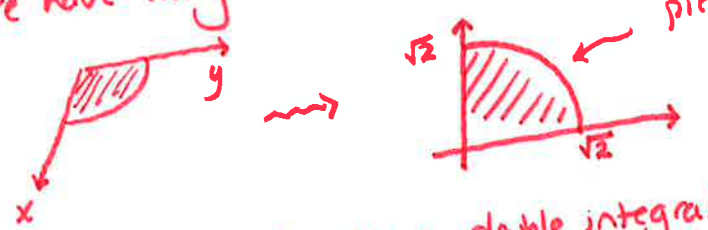
Sketch 3



If we 'slice' in x + y, we see z varies by always going from paraboloid up to $z=2$.

$$\int_{?}^{?}\int_{?}^{?}\int_{x^2+y^2}^{2} dz\, d?\, d?$$

After this, 'collapse' the surface down to the plane, because we have integrated z, we are done with it.



piece of $x^2+y^2=2$ $\underbrace{\text{from } z=x^2+y^2}$
$\downarrow$
$2=x^2+y^2$

Now we are reduced down to a double integral, which we already know how to set-up.

$$\frac{\pi}{2} = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-y^2}}\int_{x^2+y^2}^{2} dz\,dx\,dy \quad \underline{\text{or}} \quad \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{x^2+y^2}^{2} dz\,dy\,dx = \frac{\pi}{2}$$

* Notice the first step is the hardest, because it is 3D. But after, we are back to an ordinary double integral.

* Again, notice the region R and the integrand do not interact, the set-up is unchanged if we wanted $\iiint_R xz\,dV$. The bulk of these problems is setting up the integral in the first place. So we focus on the case of $\iiint_R dV$.

Student Ex: Find the volume of the ellipsoid $x^2+4y^2+z^2=16$.

Sketch 1

⇝

Sketch 2

⇝

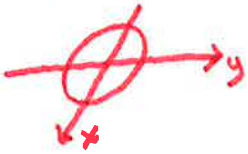
Sketch 3

Notice this is symmetric in x, y, z. So we only compute volume of the upper half.

Slicing in x & y, we see that z varies from z = 0 to ellipsoid.

$$\int_?^? \int_?^? \int_0^{\sqrt{16-x^2-4y^2}} dz\, d?\, d?$$

Projecting to plane, ie what is the 'boundary' of the integral



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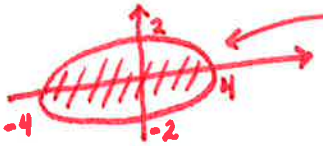


$x^2+4y^2=16$

Then...

$$V = 2\int_{-4}^{4}\int_{-\sqrt{\frac{16-x^2}{4}}}^{\sqrt{\frac{16-x^2}{4}}}\int_0^{\sqrt{16-x^2-4y^2}} dz\,dy\,dx = \frac{128}{3}\pi$$

Check for yourself that these also work....

$$V = 2\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_0^{\sqrt{\frac{16-x^2}{4}-z^2}} dy\,dz\,dx = \frac{128\pi}{3}$$

$$V = 8\int_0^4\int_0^{\sqrt{16-x^2}}\int_0^{\sqrt{\frac{16-x^2}{4}-z^2}} dy\,dz\,dx = \frac{128\pi}{3}$$

$$V = \int_{-2}^{2}\int_{-\sqrt{16-4y^2}}^{\sqrt{16-4y^2}}\int_{-\sqrt{16-4y^2-z^2}}^{\sqrt{16-4y^2-z^2}} dx\,dz\,dy = \frac{128\pi}{3}$$

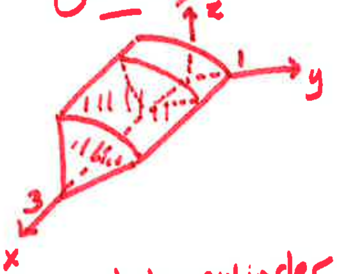
In fact, there should be at least 18 'obvious' ways of setting this up !

Student Ex: Find the volume in the first octant of the region bound by $z=1-y^2$, $x+y=1$, and $x+y=3$

Sketch 1

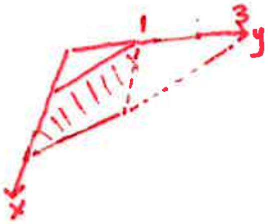
Sketch 2

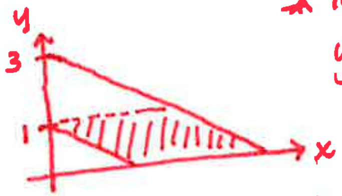
Sketch 3



Slicing in x & y, we see z goes from plane up to parabolic cylinder

Projecting to the plane....





* In plane, $z=0$. Then $y^2=1 \rightarrow y=\pm 1 \rightarrow y=1$ Because $x,y,z \geq 0$.

To avoid 2 integrals, do dx dy

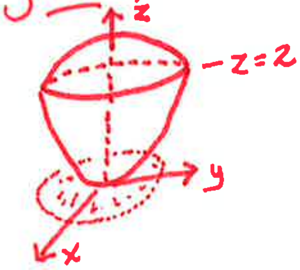
$$V=\int_0^1\int_{1-y}^{3-y}\int_0^{1-y^2} dz\,dx\,dy = 4/3$$

Student Ex: Find volume of region bound by $x^2+y^2+z^2=6$ and $z=x^2+y^2$.

Sketch 1

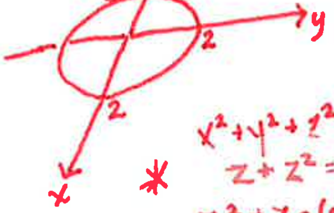
Sketch 2

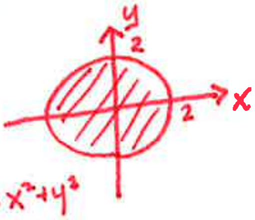
Sketch 3



Slicing in x & y, z goes from paraboloid up to sphere.

Projecting to xy-plane:





$$V=\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz\,dy\,dx$$

$$=\frac{2\pi}{3}\left(6\sqrt{6}-11\right)$$

* $x^2+y^2+z^2=6$; $z=x^2+y^2$

$z+z^2=6$

$z^2+z-6=0$

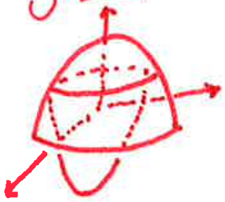
$(z-2)(z+3)=0$

$z=2$ or $z=-3$

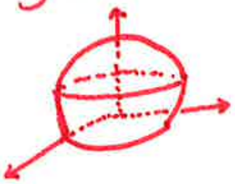
Student Ex: Find the volume enclosed by $z = 9 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 16$.

Sketch 1

Sketch 2

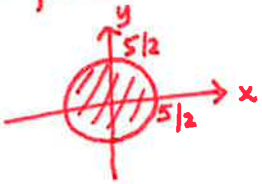


Sketch 3



Slicing in x & y, z goes from 'bottom' paraboloid to the 'top' one.

$$9 - x^2 - y^2 = 3x^2 + 3y^2 - 16$$
$$4x^2 + 4y^2 = 25$$
$$x^2 + y^2 = \frac{25}{4} = \left(\frac{5}{2}\right)^2$$



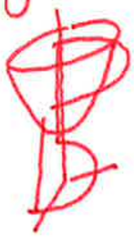
$$V = \int_{-5/2}^{5/2} \int_{-\sqrt{25/4 - x^2}}^{\sqrt{25/4 - x^2}} \int_{3x^2+3y^2-16}^{9-x^2-y^2} dz\,dy\,dx$$

$$= 4 \int_0^{5/2} \int_0^{\sqrt{25/4 - x^2}} \int_{3x^2+3y^2-16}^{9-x^2-y^2} dz\,dy\,dx = \frac{625}{8}\pi$$

A lot of work.

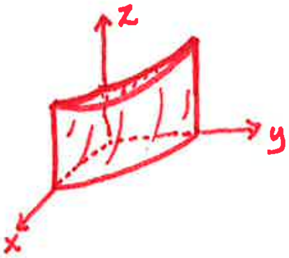
Student Ex: Find volume in first octant bound by $z = x^2 + y^2 + 9$ and $y = 4 - x^2$.

Sketch 1

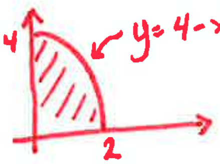


Sketch 2

Sketch 3



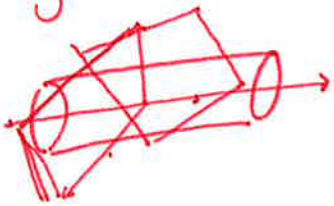
Slicing in x & y, z goes from z=0 to paraboloid.



$$V = \int_0^2 \int_0^{4-x^2} \int_0^{x^2+y^2+9} dz\,dy\,dx = \frac{2512}{35}$$

Student Ex: Compute $\iiint_R y\,dV$, where R is the region bounded by $x+y+z=2$, $x^2+z^2=1$, $y=0$.

Sketch 1



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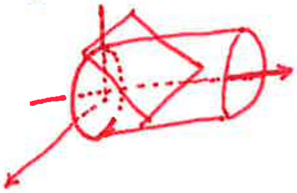
Sketch 2



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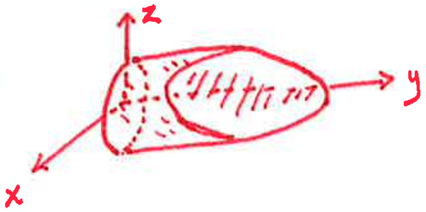
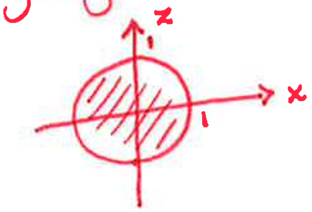
Sketch 3



Slicing in x & z, we see y goes from y=0 to plane.

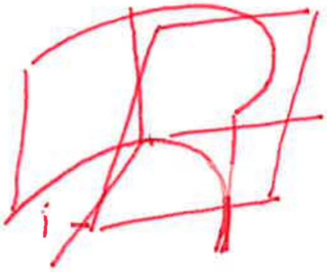


$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{2-x-z} y\;dy\,dz\,dx = \frac{9\pi}{4}$$

Student Ex: Sketch the region of integration for $\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} f(x,y,z)\,dz\,dx\,dy$

Boundaries:
$z=0$ $\quad x=1$ $\quad y=-1$
$z=1-x$ $\quad x=y^2$ $\quad y=1$

Sketch 1



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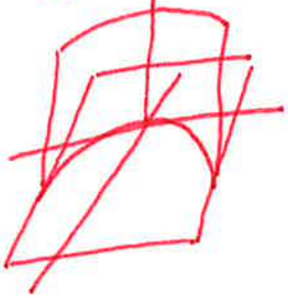
Sketch 2



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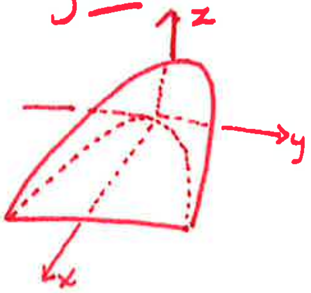
Sketch 3



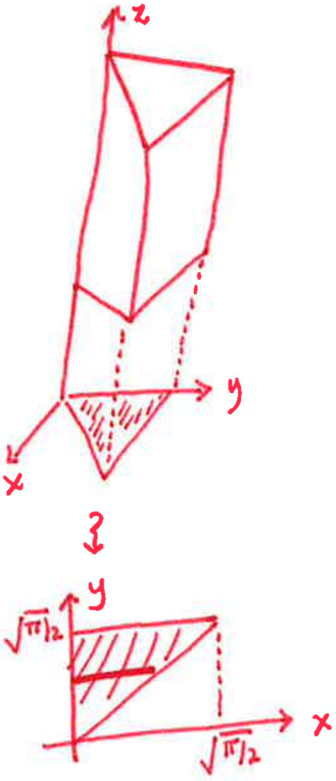
Note: As with double integrals, sometimes one will need to interchange the order of integration in order to evaluate.

Example: $\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\, dz\, dy\, dx$

Eventually, we need to integrate w.r.t. y, but we cannot. We will put y last and try to pick up a 'y' term along the way for a u-sub.

Boundary: $z=1$ $\quad y=x$ $\quad x=0$
$z=3$ $\quad y=\sqrt{\pi/2}$ $\quad x=\sqrt{\pi/2}$

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_0^{y} \int_1^3 \sin(y^2)\, dz\, dx\, dy = 1$$



§12.8: Change of Variables

Topics

- Change of Variables

You Should Be Able to...

- Describe the method of change of variables.
- Set-up the matrix of a transformation.
- Find the transformation given a matrix.
- Find images of regions under a given transformation.
- Interpret the determinant of a transformation
- Compute the Jacobian of a transformation.
- Compute integrals by changing variables.

Some integrals are cumbersome to work with. 'Usually' this for one of two reasons:

1) Bad Regions:

There is no way to evaluate $\iint_R f(x,y)\,dA$ in x and y without having to use two integrals. It would be 'nicer' if the region R could look more like □.

2) Terrible Limits:

To find the volume of the ellipsoid $x^2+4y^2+z^2=16$, we compute

$$\int_{-4}^{4}\int_{-\sqrt{\frac{16-x^2}{4}}}^{\sqrt{\frac{16-x^2}{4}}}\int_{-\sqrt{16-x^2-4y^2}}^{\sqrt{16-x^2-4y^2}} dz\,dy\,dx$$

These are terrible limits of integration! It would be nice if we could make these easier to deal with. We shall see this is the case by exploiting the symmetry of the region - something we already did a bit.

Both of these issues can be dealt with via a transformation of coordinates. This will be our multivariable calculus analog of change of variables. So let's review this first.

Suppose we had the integral $\int_a^b f(x)\,dx$ and we needed | wanted to make a change of variables.

$x = g(u)$

$dx = g'(u)\,du$

$$\int_a^b f(x)\,dx = \int_c^d f(g(u)) \cdot g'(u)\,du$$

where $a = g(c)$ and $b = g(d)$. Let's break down these pieces.

$$\underbrace{\int_c^d}_{\text{I}} \underbrace{f(g(u))}_{\text{II}} \cdot \underbrace{g'(u)\,du}_{\text{III}}$$

I : The limits of integration in our new variable.

II : The new variables 'plugged into' the original integrand

III : The adjusted differential in our new variable.

Let's see this in an example, Keeping in mind what the integral computes.

Ex: $\int_0^1 (2x+1)^2\,dx$ $\quad x = \frac{u-1}{2} \leftarrow$ ie $u = 2x+1$

$dx = \frac{1}{2}\,du$

If $x = 0 \rightsquigarrow u = 2(0) + 1 = 1$

If $x = 1 \rightsquigarrow u = 2(1) + 1 = 3$

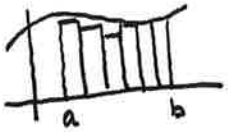
$$\Downarrow$$

$\int_1^3 u^2 \cdot \frac{1}{2}\,du$ Let's examine the pieces one by one.

$\int_{\boxed{1}}^{\boxed{3}} \ldots$: The new limits of integration after 'transforming' from x's to u's.

$\int_1^3 \boxed{u^2}$: The integrand using the new variable.

$\int_1^3 u^2 \cdot \boxed{\frac{1}{2}\,du}$: The new differential. Remember : $\int_a^b \underbrace{f(x)}_{\text{heights}} \underbrace{dx}_{\text{widths}}$



There is no reason to think that taking a 'small' step in x will be the same as taking a 'small' step in u. This affects the widths, hence areas, hence the integral value. We have to take this effect into account when changing variables. In our example, □'s in u have to be half the width as in x's. becuse u 'moves' 'twice as fast.' This scaling need not be constant but can depend on u. Here it is constant, sometimes not, ie $u = x^2$, $dx = 2x\,du = 2\sqrt{u}\,du$. We need to build parallels of these pieces in multiple variables, this will be the notion of transformations.

The notion of a transformation is very general. But to treat it dignifiedly, we would need linear algebra, which is beyond our scope. So we shall define things and discuss them with an eye towards our $\mathbb{R}^2$ or $\mathbb{R}^3$ case.

Coordinate Transformations

A point $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is just a collection of data values. But they do not need to be measured in the same way. For example, $\vec{x}$ may be distance from you, eg $(3, -1, 2)$ = 3ft to right, 1ft back, and 2ft up. If $\vec{x}$ was USD\$, CD\$, Yuan values, then $(100, 2, 500)$ = \$100, \$2 CD, 500 Yuan. But I, too, may try to describe where an object is. Unless I am standing where you are, even measuring the location of the same object, we do not obtain equivalent coordinates. Similarly, you may measure the money in single 'dollars' while I may use tens of thousands of dollars. But ultimately, we are measuring the same things. Even though we have different values, there 'must' be a way of converting from your measurements to mine and vice versa. This is the idea of a coordinate transformation.

Coordinate Transformation: A coordinate transformation is a C^1 function $T: \mathbb{R}^n \to \mathbb{R}^n$.

Let's see some examples.

Ex: You measure where a pole is located relative to where you are standing, ie $(x,y) \in \mathbb{R}^2$ says the pole is x ft left/right and y ft forward/backward.

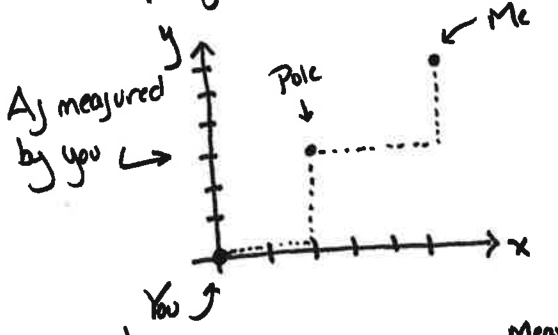
$(2,3)$ = 2 ft right and 3 ft ahead
$(-1,-2)$ = 1 ft left and 2 ft behind
$(1,-1)$ = 1 ft right and 1 ft behind

But I am measuring the pole's location relative to myself as well. I use coordinates (u,v), ie $(u,v) \in \mathbb{R}^2$.

$(u,v) = (2,3)$ = 2 ft right and 3 ft ahead " ← Of ME, not you.
"

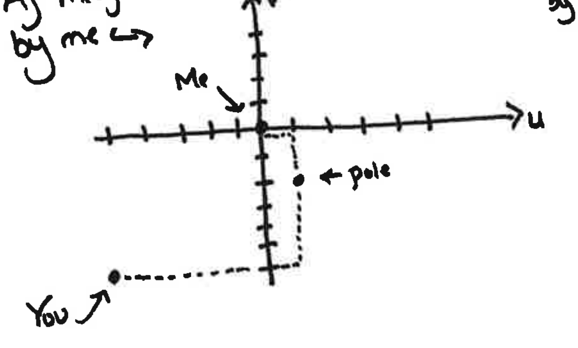
But we are both measuring the location of the same pole, so there should be a way of converting back and forth between our measurements.

Suppose I am 5ft to your right and 6 ft forward. Suppose you measure the pole as $(2,3)$, ie $(x,y) = (2,3)$, what do I measure?



I measure $(-3,-3)$. What if I measured $(1,-2)$, ie $(u,v) = (1,-2)$. What do you measure? You measure $(6,4)$. What is the transformation taking your data to mine?

$$\underset{\text{Measured by you}}{T(x,y)} = \left(\underbrace{x-5}_{u}, \underbrace{y-6}_{v}\right) \leftarrow \text{Measured by me}$$



Check: $T(5,6) = (0,0)$ ✓ ← Right where I am
$T(2,3) = (-3,-3)$ ✓

What about taking my data to yours?

$$\underset{\text{measured by me}}{\tilde{T}(u,v)} = \left(\underbrace{u+5}_{x}, \underbrace{v+6}_{y}\right) \leftarrow \text{Measured by you}$$

Check: $\tilde{T}(-5,-6) = (0,0)$ ✓
$\tilde{T}(1,-2) = (6,4)$ ✓

What makes these coordinate transformations? We check the definition.

T: 1) Function $\mathbb{R}^2 \to \mathbb{R}^2$? ✓ ⇜ $T(x,y) = (x-5, y-6)$

2) Class C^1? ✓ ⟵ Each coordinate is differentiable in x/y.

$$(\underbrace{x-5}_{\text{diff in } x/y}, \underbrace{y-6}_{\text{diff in } x/y})$$

$\tilde{T}$: 1) Function $\mathbb{R}^2 \to \mathbb{R}^2$? ✓ ⟵ $\tilde{T}(u,v) = (u+5, v+6)$

2) Class C^1? ✓ ⟵ $\tilde{T}(u,v) = (\underbrace{u+5}_{\substack{\text{diff in}\\ u\,\&\,v}}, \underbrace{v+6}_{\substack{\text{diff in}\\ u\,\&\,v.}})$

Therefore, these are coordinate transformations.

* We really, for now, want each coordinate transformation to be linear.

Ex: $T(a,b) = (2a+b, a-b)$

What does T do to points?

$T(0,0) = (0+0, 0-0) = (0,0)$ ⟵ Fixes origin

$T(1,1) = (2+1, 1-1) = (3,0)$

$T(2,1) = (4+1, 2-1) = (5,1)$

Is this a (linear) coordinate transformation?

1) Function $\mathbb{R}^2 \to \mathbb{R}^2$? ⟵ $T\underbrace{(a,b)}_{\in\mathbb{R}^2} = \underbrace{(2a+b, a-b)}_{\in\mathbb{R}^2}$ ✓

2) Class C^1? $T = \underbrace{(\underbrace{2a+b}_{\substack{\text{diff in a}\\ \text{and b}}}, \underbrace{a-b}_{\substack{\text{diff. in a}\\ \text{and b}}})}_{\text{in fact, lin. in a and b.}}$ ✓

Ex: $T(x,y,z) = (2x+y,\ x-3y+z,\ 4z-x)$

1) $T: \mathbb{R}^3 \to \mathbb{R}^3$ ✓

2) Class C^1: $(\underbrace{2x+y,\ x-3y+z,\ 4z-x}_{})$ ✓

Each coord a diff (C^1) function of x, y, z.

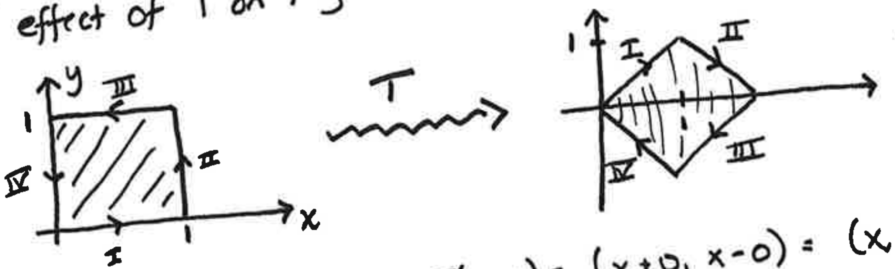
So T is a coordinate transformation.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (\underbrace{x+y}_{u},\ \underbrace{x-y}_{v})$.

1) $T: \mathbb{R}^2 \to \mathbb{R}^2$ ✓

2) Class C^1: Each coord. is a differentiable (class C^1) function of x & y.

Therefore, T is a (linear) coordinate function. Suppose you are measuring things in (x,y)-space and I in (u,v)-space. Then T converts your data to mine. Suppose you take the box (in the xy plane) shown below. What is the effect of T on this box?



Reversed orientation ‼

Bottom Edge: $(x,0)$; $0 \le x \le 1$; $T(x,0) = (x+0,\ x-0) = (x,x)$

Right Edge: $(1,y)$; $0 \le y \le 1$; $T(1,y) = (y+1,\ 1-y)$ '=' $(1,1) + t\langle 1,-1\rangle$

Top Edge: $(x,1)$; $0 \le x \le 1$; $T(x,1) = (x+1,\ x-1)$ '=' $(1,-1) + t\langle 1,1\rangle$

Left Edge: $(0,y)$; $0 \le y \le 1$; $T(0,y) = (0+y,\ 0-y) = (y,-y)$

Notice the transformation T in the previous example tilts the square (as well as reverses orientation). So the 'reverse' should untilt squares. But this is exactly what we wanted to do ‼

↑ Not nec. all squares.

Before going further, we need to represent transformations as matrices.

Ex: $T(x,y) = (\underbrace{x}_{u}, \underbrace{y}_{v})$. Then T is the matrix $\begin{array}{c|cc} & x & y \\ \hline u & 1 & 0 \\ v & 0 & 1 \end{array}$

Ex: $T(x,y) = (\underbrace{x+y}_{u}, \underbrace{x-y}_{v})$. Then T is the matrix $\begin{array}{c|cc} & x & y \\ \hline u & 1 & 1 \\ v & 1 & -1 \end{array}$

Ex: $T(x,y) = (2y, 3x+4y)$. Then T is the matrix $\begin{array}{c|cc} & x & y \\ \hline u & 0 & 2 \\ v & 3 & 4 \end{array}$

Ex: $T(x,y,z) = (x-y+z, x-2z, 3y+5z)$. Then T is the matrix

$$\begin{array}{c|ccc} & x & y & z \\ \hline u & 1 & -1 & 1 \\ v & 1 & 0 & -2 \\ w & 0 & 3 & 5 \end{array}$$

Ex: $T(x,y,z) = (2x+y-z, x+2y+3z, -x+4y-2z)$. Then T is the matrix

$$\begin{array}{c|ccc} & x & y & z \\ \hline u & 2 & 1 & -1 \\ v & 1 & 2 & 3 \\ w & -1 & 4 & -2 \end{array}$$

* For those that know some Linear Algebra, the labels on the top are the basis for the domain and the labels on the side are the basis for the codomain. Things are chosen this way so they work out as below:

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \to (u,v)$

$T(x,y) = (2x-y, x+3y)$

$T(x,y) = T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-y \\ x+3y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$T(1,3) = (2-3, 1+3(3)) = (-1, 10)$

$T: \begin{array}{c|cc} & x & y \\ \hline u & 2 & -1 \\ v & 1 & 3 \end{array}$

$\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-3 \\ 1+3(3) \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \end{pmatrix}$

Notice the entries for T are the partial derivatives of the new coordinates with respect to the old coordinates.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (\underbrace{2x-y}_{u}, \underbrace{x+3y}_{v})$. Then T is $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

$u(x,y) = 2x - y \qquad u_x = 2 \quad u_y = -1$

$v(x,y) = x + 3y \qquad v_x = 1 \quad v_y = 3$

$$\begin{array}{c} \\ u \\ v \end{array} \!\!\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \end{array} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

We can also go the other way.

Ex: $\begin{array}{c|cc} & x & y \\ \hline u & 3 & 0 \\ v & 0 & 4 \end{array} \rightsquigarrow T(x,y) = (3x, 4y)$

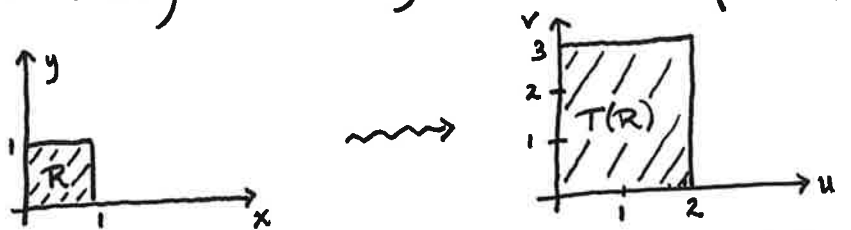
Ex: $\begin{array}{c|cc} & x & y \\ \hline u & 1 & 2 \\ v & -3 & 4 \end{array} \rightsquigarrow T(x,y) = (x+2y, -3x+4y)$

Ex: $\begin{array}{c|cc} & x & y \\ \hline u & 0 & 3 \\ v & 1 & -2 \end{array} \rightsquigarrow T(x,y) = (3y, x-2y)$

Ex: $\begin{array}{c|ccc} & x & y & z \\ \hline u & 1 & 1 & 1 \\ v & 2 & 0 & -1 \\ w & 3 & 4 & 0 \end{array} \rightsquigarrow T(x,y,z) = (x+y+z, 2x-z, 3x+4y)$

Thus far, we know how to create transformations of coordinates, ie the T matrices. We also know how to apply these transformations to regions, this will allow us to set-up the limits of integration in our new coordinates. There is only one final piece, the one corresponding to $g'(u)\,du$. We motivate this via an example.

Ex: $T(x,y) = (2x, 3y) \rightsquigarrow T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$



Area $R = 1(1) = 1$ Area $T(R) = 2(3) = 6$

Because integrals depend on area differentials, volume diff., etc, we need a way of tracking how our transformation affects these. Note the transformation above scaled by a factor of 6. Notice also

$$\det T = \det \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = 6 - 0 = 6$$

This is no accident.

'Thm': A linear transformation scales the 'volumes' of 'boxes' by $|\det T|$.

So in the above, it should scale parallelogram areas by $|6| = 6$.

* The sign of the determinant indicates whether the transformation preserves (+) or reverses (−) orientation.

The object that tracks this for us generally is the Jacobian of the transformation.

Jacobian: Let $T(x_1, x_2, \dots, x_n) = (y_1, \overset{\text{All } y_i \text{ depend on } x_1, \dots, x_n}{y_2}, \dots, y_m)$ be a transformation.

The Jacobian matrix is…

$$J = J(x_1, \dots, x_n) = \overset{\text{Just a notation}}{\frac{\partial(y_1, \dots, y_m)}{\partial(x_1, \dots, x_n)}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots\cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & & \ddots & \frac{\partial y_m}{\partial x_n} \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

The Jacobian of T is $|\det J|$.

Also denoted $\left| \frac{\partial(y_1, \dots, y_m)}{\partial(x_1, \dots, x_n)} \right|$.

* Confusingly, the term Jacobian is also sometimes used for the Jacobian matrix.

Ex: $T(x,y) = (\underbrace{2x-y}_{A}, \underbrace{3x+y}_{B})$

$$J = \frac{\partial(A,B)}{\partial(x,y)} = \begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$$

$$|\det J| = \left| \frac{\partial(A,B)}{\partial(x,y)} \right| = \left|\det \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\right| = |2 - {}^{-}3| = 5$$

Ex: $T(x,y) = (\underbrace{x^2-y}_{A}, \underbrace{y^3+x}_{B})$

$$J = \frac{\partial(A,B)}{\partial(x,y)} = \begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ 1 & 3y^2 \end{pmatrix}$$

$$|\det J| = \left| \frac{\partial(A,B)}{\partial(x,y)} \right| = \left|\det \begin{pmatrix} 2x & -1 \\ 1 & 3y^2 \end{pmatrix}\right| = |6xy^2 + 1|$$

Ex: $T(x,y,z) = (\underbrace{x+y}_{A}, \underbrace{y+z}_{B}, \underbrace{z+x}_{C})$

$$J = \frac{\partial(A,B,C)}{\partial(x,y,z)} = \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$|\det J| = \left| \frac{\partial(A,B,C)}{\partial(x,y,z)} \right| = \left|\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\right| = \left| 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right| = 2$$

$$= |1(1) - 1(-1) + 0|$$

Ex: $T(x,y) = (\cos x, \sin y)$

$$J = \frac{\partial(A,B)}{\partial(x,y)} = \begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix} = \begin{pmatrix} -\sin x & 0 \\ 0 & \cos y \end{pmatrix}$$

$$|\det J| = \left| \frac{\partial(A,B)}{\partial(x,y)} \right| = \left|\det \begin{pmatrix} -\sin x & 0 \\ 0 & \cos y \end{pmatrix}\right| = |-\sin x \cos y|$$

* The Jacobian matrix generally need not be a square matrix.

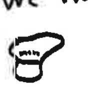
The Jacobian keeps track at every point in the region how much we need to scale our differentials when going from dx dy (dz) to whatever coordinates we are using. We now have everything we need. We state our theorem in the 3-dimensional case:

Change of Variables: Let R and $\tilde{R}$ be 'nice' regions and $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a 'nice' coordinate transformation with $T(R') = R$. Then if under T we have $x = x(u,v,w)$, $y = y(u,v,w)$, and $z = z(u,v,w)$, we have....

$$\iiint_R f(x,y,z)\, dV = \iiint_{\tilde{R}} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| d\tilde{V}$$

More generally, if $U \subseteq \mathbb{R}^n$ is open and $\Phi: U \to \mathbb{R}^n$ is injective and differentiable with continuous partials and has nonzero Jacobian for all $\vec{x} \in U$, then for 'nice' continuous functions $f(\vec{x})$,

$$\iint \cdots \int_{\Phi(U)} f(\vec{x})\, d\vec{x} = \iint \cdots \int_U f(\Phi(\vec{u}))\, |\det J(\vec{u})|\, d\vec{u}$$

* We have glossed over some details, obviously. For instance, what is 'nice'? Suffice to say, the problems you will have consist of 'nice' regions and 'nice' functions. To be more specific takes us too far astray. Suffice to say we want f to be C^1 and the regions to be 'reasonable blobs', eg  , , , etc.

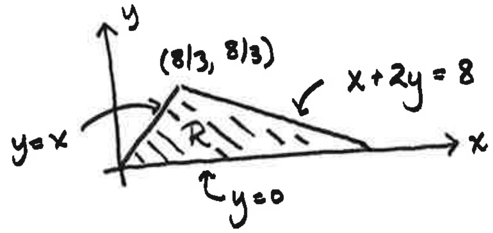
* One detail we will comment on is the injective | 'nice' part. For each output for f, we want only one input (at most) that gives it on R, ie we want to avoid the analog of...

$\int_{-1}^{1} x^2 dx \to$ Clearly not $= 0$. But $u = x^2 \to du = 2x\, dx \to \frac{1}{2}\int_1^1 \sqrt{u}\, du = 0$.... what?

Notice $u = 1 = (-1)^2 = 1^2$ on $[-1, 1]$.... whoops.

All this looks much worse than it is. Let's see some examples.

Ex: $\iint_R \cos(x+2y)\sin(x-y)\,dx\,dy$



Make the change of variables:

$u = x+2y$

$v = x-y$

First, solve for x & y.

$$\begin{array}{r} u = x+2y \\ -\ v = x-y \\ \hline u-v = 3y \end{array} \rightarrow y = \frac{u-v}{3}$$

$$v = x-y$$
$$3v = 3x-3y$$
$$3v = 3x-(u-v)$$
$$u+2v = 3x$$
$$x = \frac{u+2v}{3}$$

$$\begin{cases} x = \frac{u+2v}{3} \\ y = \frac{u-v}{3} \end{cases}$$

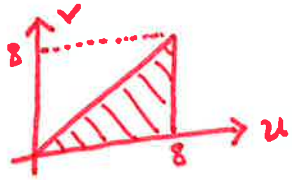
Now find our transformed region in the (u,v)-plane so we know the limits of integration.

$y=x$: $\quad y = x \rightsquigarrow \frac{u+2v}{3} = \frac{u-v}{3} \rightsquigarrow u+2v = u-v \rightsquigarrow 3v=0 \rightsquigarrow v=0$

$y=0$: $\quad y=0 \rightsquigarrow \frac{u-v}{3}=0 \rightsquigarrow u-v=0 \rightsquigarrow u=v$

$x+2y=8$: $\quad x+2y=8 \longrightarrow \frac{u+2v}{3} + 2\left(\frac{u-v}{3}\right) = 8 \rightarrow \frac{3u+0v}{3} = 8 \rightarrow u=8$

The new region is...



Now compute the Jacobian...

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det \begin{array}{c|cc} & u & v \\ x & 1/3 & 2/3 \\ y & 1/3 & -1/3 \end{array}\right| = \left|-\frac{1}{9} - \frac{2}{9}\right| = \left|-\frac{1}{3}\right| = 1/3$$

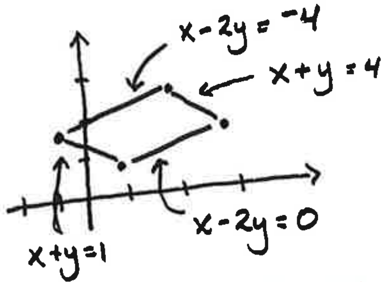
Then...

$$\iint_R \cos(x+2y)\sin(x-y)\,dx\,dy = \int_0^8\int_0^u \cos u \sin v \cdot \frac{1}{3}\,du\,dv$$

$$= \vdots$$

$$= \frac{1}{3}\left(\sin 8 - 4 - \frac{\sin 16}{4}\right)$$

Ex: $\iint_R 27xy\ dA$



* Would require several integrals.

Let $u = x+y$
$v = x-2y$

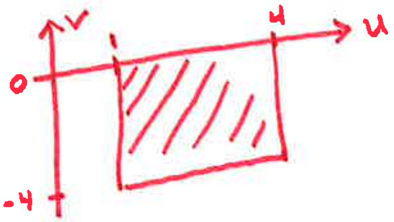
Solving for x & y, we find $x = \frac{2u+v}{3}$, $y = \frac{u-v}{3}$. Now we find our new region:

$x+y=1$: $x+y=1 \rightarrow \frac{2u+v}{3} + \frac{u-v}{3} = 1 \rightarrow \frac{3u}{3} = 1 \rightarrow u = 1$

$x+y=4$: $x+y=4 \rightarrow \frac{2u+v}{3} + \frac{u-v}{3} = 4 \rightarrow \frac{3u}{3} = 4 \rightarrow u = 4$

$x-2y=0$: $x-2y=0 \hookrightarrow \frac{2u+v}{3} - 2\left(\frac{u-v}{3}\right) = 0 \rightarrow \frac{3v}{3} = 0 \rightarrow v = 0$

$x-2y=-4$: $x-2y=-4 \rightarrow \frac{2u+v}{3} - 2\left(\frac{u-v}{3}\right) = -4 \rightarrow \frac{3v}{3} = -4 \rightarrow v = -4$



Now we find the Jacobian:

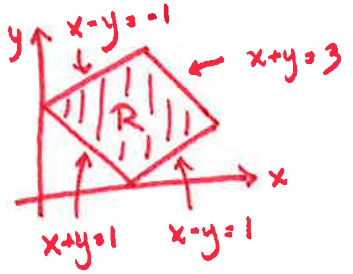
$$|\det J| = \left| \det \begin{array}{c|cc} & u & v \\ x & 2/3 & 1/3 \\ y & 1/3 & -1/3 \end{array} \right| = \left| -\frac{2}{9} - \frac{1}{9} \right| = \left| -\frac{1}{3} \right| = 1/3$$

$$\iint_R 27xy\ dA = \int_1^4 \int_{-4}^0 27\left(\frac{2u+v}{3}\right)\left(\frac{u-v}{3}\right) \cdot \frac{1}{3}\ dv\,du$$

$= \vdots$

$= 164$

Ex: $\iint_R (x+y)^2 \sin^2(x-y)\, dA$, where R is the square with vertices (0,1), (1,2), (2,1), and (1,0).



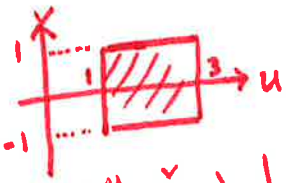
Let $u = x+y$
$v = x-y$

Then $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

$x+y=3 \rightarrow u=3$; $x+y=1 \rightarrow u=1$

$x-y=-1 \rightarrow v=-1$;

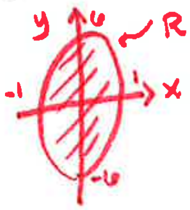
$x-y=1 \rightarrow v=1$



$$|\det J| = \left|\det \begin{array}{c|cc} & u & v \\ \hline x & 1/2 & 1/2 \\ y & 1/2 & -1/2 \end{array}\right| = \left|-\tfrac{1}{4} - \tfrac{1}{4}\right| = \left|-\tfrac{1}{2}\right| = 1/2$$

$$\iint_R (x+y)^2 \sin^2(x-y)\, dA = \int_{-1}^{1}\int_{1}^{3} u^2 \sin^2 v \cdot \tfrac{1}{2}\, du\, dv = \tfrac{13}{6}(2-\sin 2)$$

Ex: Make a change of variables in $\iint_R f(x,y)\, dA$, where R is $x^2 + \frac{y^2}{36} = 1$.



$x = \frac{u}{2}$

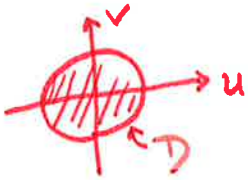
$y = 3v$

$x^2 + \frac{y^2}{36} = 1$

$\left(\frac{u}{2}\right)^2 + \frac{1}{36}(3v)^2 = 1$

$\frac{u^2}{4} + \frac{9v^2}{36} = 1$

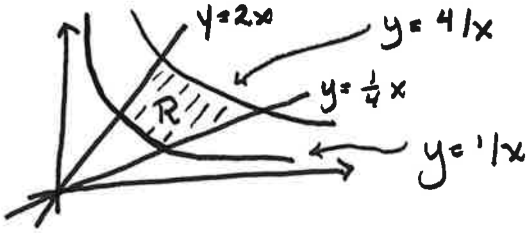
$u^2 + v^2 = 1$



$$|\det J| = \left|\det \begin{array}{c|cc} & u & v \\ \hline x & 1/2 & 0 \\ y & 0 & 3 \end{array}\right| = |3/2| = 3/2$$

$$\iint_R f(x,y)\, dA = \iint_D f(u/2, 3v) \cdot \tfrac{3}{2}\, d\tilde{A}$$

Ex: $\iint_R e^{-xy/2}\,dA$



Let $u = y/x$
$v = xy$

Then $u \cdot v = \frac{y}{x} \cdot xy = y^2 \rightarrow y = \sqrt{uv}$

$v = xy = x \cdot \sqrt{uv} \rightarrow x = \frac{v}{\sqrt{uv}} = \frac{\sqrt{v}\sqrt{v}}{\sqrt{u}\sqrt{v}} \rightarrow x = \sqrt{\frac{u}{v}}$

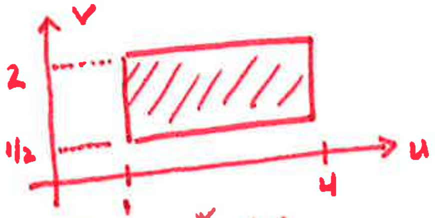
$y = 2x$: $\sqrt{uv} = 2\sqrt{\frac{u}{v}} \rightarrow uv = 4\frac{u}{v} \rightarrow v^2 = 4 \rightarrow v = \pm 2 \rightarrow v = 2$

$y = \frac{1}{4}x$: $\sqrt{uv} = \frac{1}{4}\sqrt{\frac{u}{v}} \rightarrow uv = \frac{1}{4}\frac{u}{v} \rightarrow v^2 = \frac{1}{4} \rightarrow v = \pm 1/2 \rightarrow v = 1/2$

$y = 4/x$: $\sqrt{uv} = 4 \cdot \sqrt{\frac{v}{u}} \rightarrow uv = 16 \cdot \frac{v}{u} \rightarrow u^2 = 16 \rightarrow u = \pm 4 \rightarrow u = 4$

$y = 1/x$: $\sqrt{uv} = \sqrt{v/u} \rightarrow uv = v/u \rightarrow u^2 = 1 \rightarrow u = \pm 1 \rightarrow u = 1$

$* \; -\frac{xy}{2} = -\frac{1}{2}\sqrt{\frac{u}{v}}\sqrt{uv}$
$= -\frac{1}{2}u$



$$|\det J| = \left| \det \begin{array}{c|cc} & u & v \\ \hline x & \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ y & \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{array} \right| = \left| \frac{1}{4}\frac{1}{\sqrt{uv}}\sqrt{\frac{u}{v}} - -\frac{1}{4}\sqrt{\frac{v}{u}}\sqrt{\frac{u}{v^3}} \right|$$

$$= \left| \frac{1}{4v} + \frac{1}{4v} \right|$$

$$= \left| \frac{1}{2v} \right| = \frac{1}{2v}$$

$$\iint_R e^{-xy/2}\,dA = \int_1^4 \int_{1/2}^2 e^{-u/2} \cdot \frac{1}{2v}\,dv\,du$$

$$= \vdots$$

$$= \left(e^{-1/2} - e^{-2}\right)\ln 4$$

$$= \frac{\left(e^{3/2} - 1\right)\ln 4}{e^2}$$

Try more yourself!

Ex: $\iint_R e^{\frac{x+y}{x-y}}\, dA$, R parallelogram with vertices (1, 0), (2, 0), (0, −2), and (0, −1).

[Hint: $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$, do you see why?]

Ex: $\int_0^3 \int_0^4 \int_{y/2}^{y/2+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$ using

$u = \frac{2x-y}{2}$

$v = y/2$

$w = z/3$

While these methods are useful, we will most often use them in 3 specific cases, where we can better exploit the symmtries involved (these are also the more common integrals you 'meet walking down the street'):

1) Polar Integrals
2) Cylindrical Integrals
3) Spherical Integrals

§12.3: Polar Integrals

Topics

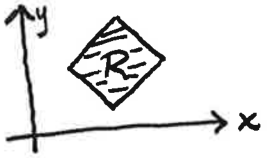
- Polar Coordinates
- Polar Integrals

You Should Be Able to....

- Derive the change of variables to polar coordinates
- Convert curves and regions between Cartesian and polar coordinates.
- Graph in polar coordinates.
- State the Jacobian for polar integrals.
- Integrate by changing to polar coordinates.
- Find areas using polar integrals.

We have seen how change of variables helps us 'deal with' integrals over 'ugly' regions, eg

$$\iint_R f(x,y)\, dA$$

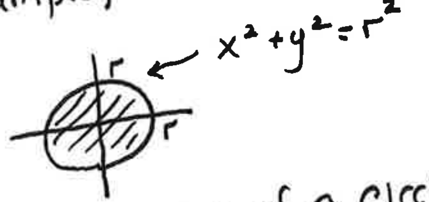


and how to deal with things we could not have otherwise integrated, eg

$$\iint_R e^{\frac{x+y}{x-y}}\, dA$$

But there will be a whole class of integrals which will be more easily integrated using a very specific change of variables. For example,

$$\iint_R dA$$



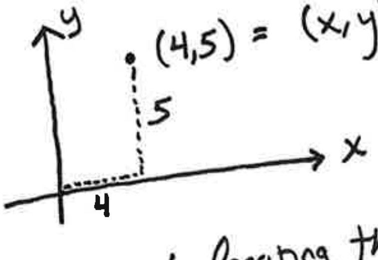
This integral computes the area of a circle of radius r, which we know is πr^2. But setting up the integral...

$$\iint_R dA = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\, dx$$

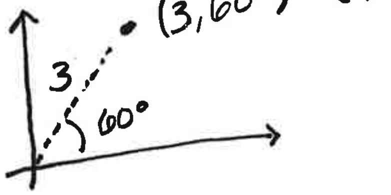
This is rather ugly for what should be a rather simple problem. There should be an easier way. Indeed, there will be by integrating over coordinates which take advantage of all the circular symmetry.

Polar Coordinates

A coordinate system is just a systematic method of describing the location of objects. For example, using Cartesian coordinates, you locate objects by left/right and up/down distances relative to some 'center' (the origin).

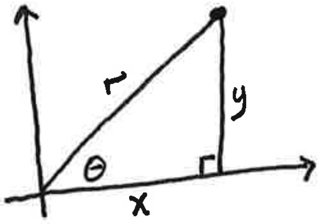


But this is not the only way of locating the 'object'. We could give an angle relative to some direction (+x-direction) and then the distance of the object. Think from the movies, "Bogie at 3 o'clock".



* Always use radians.

We can relate these coordinate systems using simple geometry/trig.



$r^2 = x^2 + y^2$

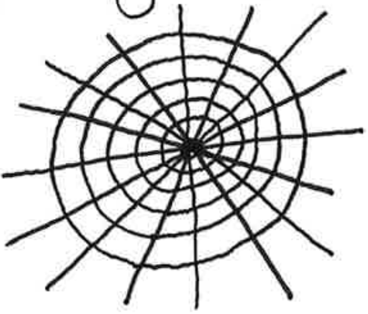
$\theta = \tan^{-1}(y/x)$

- - - - - - - - - -

$x = r\cos\theta$

$y = r\sin\theta$

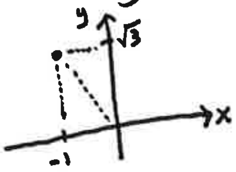
In fact, there is even graph paper for polar coordinates



* Notice all the radial symmetry !

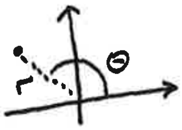
We need to be able to convert coordinates

Ex: $(x,y) = (-1, \sqrt{3})$



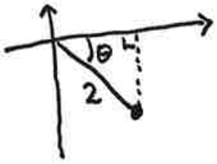
$$r^2 = (-1)^2 + (\sqrt{3})^2$$
$$= 1 + 3$$
$$= 4$$
$$r = 2$$

$$\theta = \tan^{-1}(\sqrt{3}/-1)$$
$$= \tan^{-1}(-\sqrt{3})$$
$$= -\frac{\pi}{3}$$



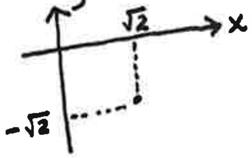
* Quad II → $\frac{\pi}{2} + \frac{\pi}{3} = \frac{2\pi}{3}$

Ex: $(r,\theta) = (2, -\frac{\pi}{4})$



$$x = 2\cos(-\tfrac{\pi}{4})$$
$$= 2 \cdot \frac{1}{\sqrt{2}}$$
$$= \sqrt{2}$$

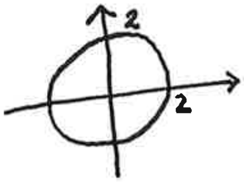
$$y = 2\sin(-\tfrac{\pi}{4})$$
$$= 2 \cdot -\frac{1}{\sqrt{2}}$$
$$= -\sqrt{2}$$



We should also be able to go back and forth between curves/regions in the different coordinates.

* Formula is $r^2 = x^2 + y^2$

Ex: $r = 2$ ⇝



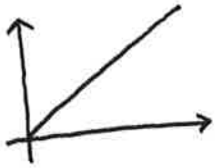
Radius fixed at 2 and independent of θ

Expect $x^2 + y^2 = 4$

$$r = 2$$
$$r^2 = 4$$
$$x^2 + y^2 = 4$$

Ex: $\theta = \frac{\pi}{4}$ →

Fixed ∡ and independent of r



Not quite a line but a ray. Expect something linear.

$\theta = \frac{\pi}{4}$

$\tan\theta = \tan\frac{\pi}{4}$

$y/x = 1$

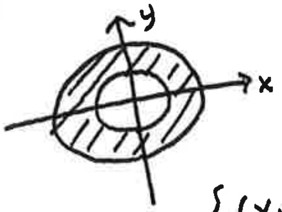
$y = x$ ← But want only part of the line

* Know $\tan\theta = y/x$

$\{(x,y) : y = x, x \geq 0\} \longrightarrow y = x$ & $x \geq 0$

Ex: $1 \leq r \leq 2$
$0 \leq \theta \leq 2\pi$

$r = 1$ & $r = 2$ circles. $0 \leq \theta \leq 2\pi$ means entire circle. So between two circles.

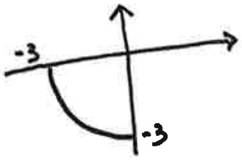


$r = 1 \rightarrow r^2 = 1 \rightarrow x^2 + y^2 = 1$

$r = 2 \rightarrow r^2 = 4 \rightarrow x^2 + y^2 = 4$

$\{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$

Ex: $r = 3$, $\pi \leq \theta \leq \frac{3\pi}{2}$



$r = 3 \rightarrow r^2 = 9 \rightarrow x^2 + y^2 = 9$. But only want part of the circle.

$\{(x,y) : x^2 + y^2 = 9, x \leq 0, y \leq 0\}$

Ex: $r = \cos\theta$

$r \cdot r = r\cos\theta$

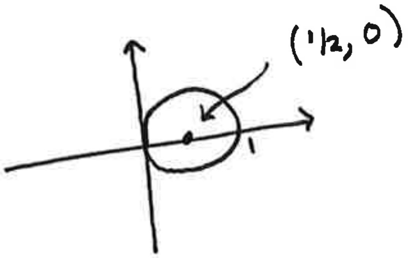
$r^2 = r\cos\theta$

$x^2 + y^2 = x$

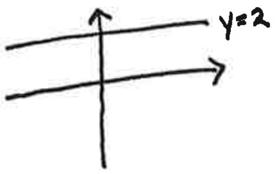
$x^2 - x + y^2 = 0$

$x^2 - x + 1/4 + y^2 = 1/4$

$(x - 1/2)^2 + y^2 = (1/2)^2$

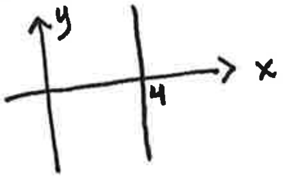


Ex: $y = 2$



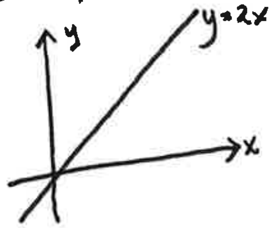
$$y = 2$$
$$r\sin\theta = 2$$
$$r = \frac{2}{\sin\theta}$$
$$r = 2\csc\theta$$

Ex: $x = 4$



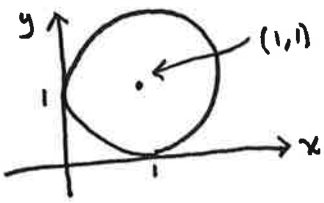
$$x = 4$$
$$r\cos\theta = 4$$
$$r = \frac{4}{\cos\theta}$$
$$r = 4\sec\theta$$

Ex: $y = 2x$



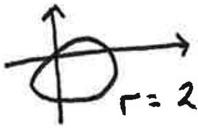
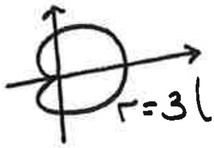
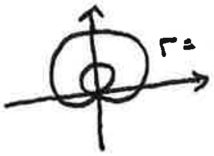
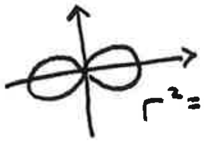
$$y = 2x$$
$$r\sin\theta = 2r\cos\theta$$
$$\sin\theta = 2\cos\theta \;;\; r \neq 0$$
$$\frac{\sin\theta}{\cos\theta} = 2$$
$$\tan\theta = 2$$

Ex: $(x-1)^2 + (y-1)^2 = 1$



$$(x-1)^2 + (y-1)^2 = 1$$
$$(r\cos\theta - 1)^2 + (r\sin\theta - 1)^2 = 1$$
$$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta - 2r\sin\theta + 1 = 1$$
$$r^2(\sin^2\theta + \cos^2\theta) - 2r(\cos\theta + \sin\theta) = -1$$
$$r^2 + 1 = 2r(\cos\theta + \sin\theta)$$

We should make note of a few common curves.

Name	Equation(s)	Example
Ray (line) ⊂ Slope C	$\theta = C$	 $\theta = \frac{\pi}{6}$
Circle Through origin	$r = a\cos\theta + b\sin\theta$	 $r = 2\cos\theta - 3\sin\theta$
Spiral	$r = a + b\theta$	 $r = \theta/2$
Cardioid	$r = a(1 \pm \cos\theta)$ ← Along x $r = a(1 \pm \sin\theta)$ ← Along y	 $r = 3(1+\cos\theta)$
Limaçon	$r = a\cos\theta + b$ ← Along x $r = a\sin\theta + b$ ← Along y	 $r = 2 + 4\sin\theta$
Rose Curve	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$ b irrational → ∞ 'petals' b integer → even → 2b petals odd → b petals	
Lemniscates	$r^2 = a^2\cos 2\theta$ $r^2 = a^2\sin 2\theta$	 $r^2 = 4\cos 2\theta$

Of course, this is all review from Calculus II. Recall we also had...

Area : $A = \frac{1}{2}\int_{\alpha}^{\beta} (r(\theta))^2 \, d\theta$

Arc length: $\mathcal{L} = \int_{\alpha}^{\beta} \sqrt{(r(\theta))^2 + (r'(\theta))^2} \, d\theta$

Surface Area of Revolution about.... : $SA = 2\pi \int_{\alpha}^{\beta} r(\theta) R(\theta) \sqrt{(r(\theta))^2 + (r'(\theta))^2} \, d\theta$

polar axis → x-axis : $R(\theta) = \sin\theta$

line $\theta = \frac{\pi}{2}$ → y-axis : $R(\theta) = \cos\theta$

We want to exploit polar coordinates to help us compute integrals involving a lot of radial symmetry. We use

$$x = r\cos\theta$$
$$y = r\sin\theta$$

But we need to compute the Jacobian to find the scaling factor going from $dA = dx\,dy$ to $dA = dr\,d\theta$.

$x_r = \cos\theta$ $\quad y_r = \sin\theta$
$x_\theta = -r\sin\theta$ $\quad y_\theta = r\cos\theta$

Thus, the Jacobian is....

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det \left(\begin{array}{ccc} & x & y \\ r & \cos\theta & -r\sin\theta \\ \theta & \sin\theta & r\cos\theta \end{array}\right)\right| = \left|r\cos^2\theta + r\sin^2\theta\right| = |r| = r$$

Thus, we have...

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\cdot r\,d\tilde{A}$$

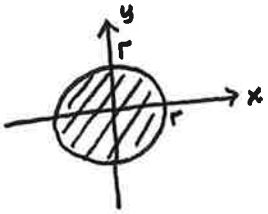
For instance,

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

To remember the conversion factor $r\,dr\,d\theta$, think like a pirate, "RRRRRRRRRR $dr\,d\theta$."

Let's see some examples.

Ex: $\iint_R dA$



In Cartesian Coordinates, we had...

$$\iint_R dA = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\,dx = \int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx\,dy$$

Now we convert to polar...

$$\iint_R \underbrace{dA}_{\text{Cartesian}} = \iint_R r\,\underbrace{dA}_{\text{polar}}$$

To set this integral up, we 'slice' in θ and see how r varies

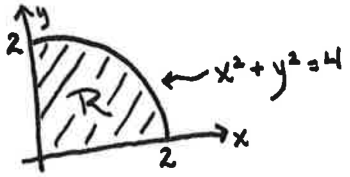
r from 0 to 'r' in each θ slice

$$\int_0^{2\pi}\int_0^r r\,dr\,d\theta$$

$$\int_0^{2\pi}\frac{r^2}{2}\,d\theta = \frac{1}{2}r^2\cdot 2\pi = \pi r^2.$$

Whoa!

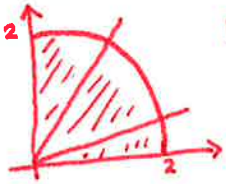
Ex: $\iint_R \sqrt{1+x^2+y^2}\,dxdy$



$$R = \{(x,y) : 0 \le y \le \sqrt{4-x^2},\ 0 \le x \le 2\}$$
$$= \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

$$\iint_R \sqrt{1+x^2+y^2}\,dA = \iint_R \sqrt{1+r^2}\cdot r\,dr\,d\theta$$

Slice in θ and see how r varies....



Every θ slice r ranges from 0 to 2.

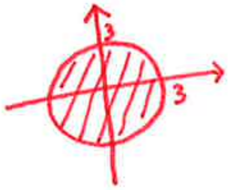
Choice of θ slices: 0 to $\frac{\pi}{2}$.

$$\int_0^{2\pi}\int_0^2 r\sqrt{1+r^2}\,dr\,d\theta$$

$$\frac{\pi}{6}\left(5^{3/2}-1\right)$$

These are much easier in polar coordinates than they are in Cartesian coordinates!

Ex: $\iint_R (x^2+y^2)^{3/2}\, dA \qquad R = \{(x,y) : 0 \le x^2+y^2 \le 9\}$

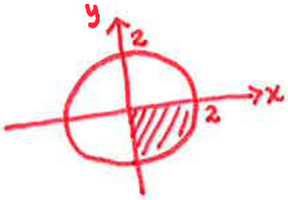


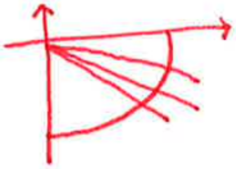
$$\iint_R (x^2+y^2)^{3/2} dA = \int_0^{2\pi}\int_0^3 (r^2)^{3/2} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 r^4\, dr\, d\theta$$

$$= \frac{486\pi}{5}$$

Ex: $\int_{-2}^{0}\int_0^{\sqrt{4-y^2}} e^{x^2+y^2}\, dx\, dy$





Sketch region

$x = 0$

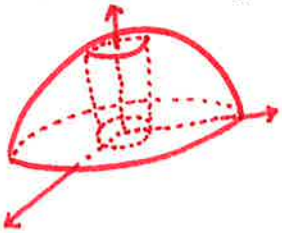
$x = \sqrt{4-y^2} \rightarrow x^2 = 4-y^2 \rightarrow x^2+y^2=4$

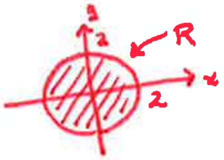
$y = 0$

$y = -4$

$$\int_{-\frac{\pi}{2}}^{0}\int_0^2 e^{r^2} \cdot r\, dr\, d\theta = \int_{\frac{3\pi}{2}}^{2\pi}\int_0^2 e^{r^2} \cdot r\, dr\, d\theta = \frac{\pi}{4}(e^4-1)$$

Ex: Find the volume of the surface bound above by the upper hemisphere $z = \sqrt{16-x^2-y^2}$, below by the xy-plane, over the region $x^2+y^2 \le 16$





$$V = \iint_R f(x,y)\, dA$$

$$= \iint_R \sqrt{16-x^2-y^2}\, dA$$

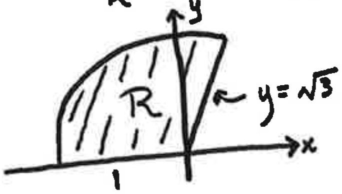
$$\sqrt{16-x^2-y^2} = \sqrt{16-(x^2+y^2)}$$

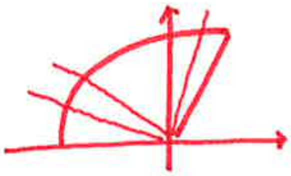
$$= \int_0^{2\pi}\int_0^4 \sqrt{16-r^2} \cdot r\, dr\, d\theta$$

$$= \frac{128\pi}{3}$$

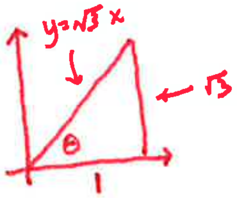
Sometimes, more work is required to set-up the integral, especially when the region is not 'circular' but the integrand calls for polar coordinates.

Ex: $\iint_R \cos(x^2+y^2)\,dA$





θ 'slices'
r varies from 0 to 1.



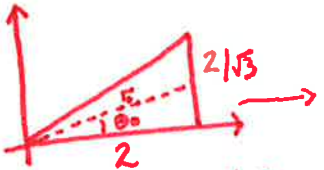
$\tan\theta = \frac{\sqrt{3}}{1}$

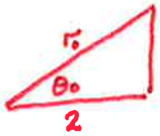
$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$$\iint_R \cos(x^2+y^2)\,dA = \int_0^{\frac{\pi}{3}}\int_0^1 \cos(r^2)\cdot r\,dr\,d\theta = \frac{\pi\sin 1}{3}$$

Ex: $\iint_R \sqrt{x^2+y^2}\,dA$

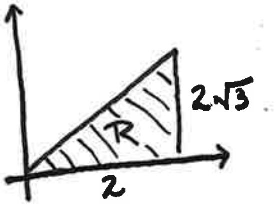


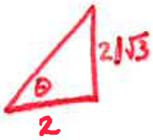


$\cos\theta_0 = \frac{2}{r_0}$

$r_0 = \frac{2}{\cos\theta_0}$

$r_0 = 2\sec\theta_0$





$\tan\theta = \frac{2\sqrt{3}}{2}$

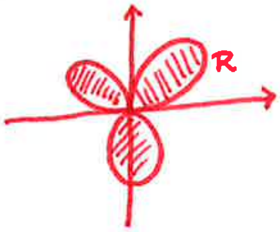
$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \pi/6$

$$\iint_R \sqrt{x^2+y^2}\,dA = \int_0^{\frac{\pi}{6}}\int_0^{2\sec\theta}\sqrt{r^2}\cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{6}}\int_0^{2\sec\theta} r^2\,dr\,d\theta = \frac{2}{9}(4+3\ln 3)$$

Ex: Find the area inside rose curve $r = \sin 3\theta$.

Whole curve appears over $\theta \in [0,\pi]$

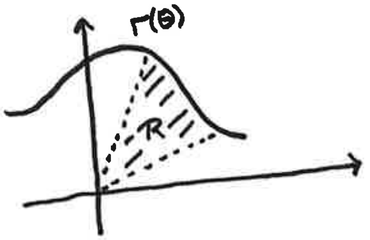


$$A = \iint_R dA = \int_0^{\pi}\int_0^{\sin 3\theta} r\,dr\,d\theta = \frac{\pi}{4}$$

We saw in Calculus II that

$$A = \frac{1}{2}\int_\alpha^\beta (r(\theta))^2\,d\theta$$

How does this align with our double integral approach?



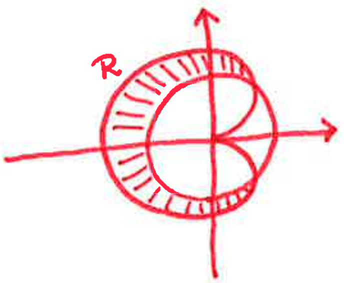
Suppose the region is given by a curve $r(\theta)$ from $\theta = \alpha$ and $\theta = \beta$, $\alpha < \beta$. Then

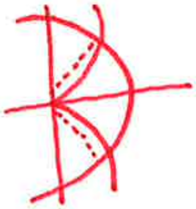
$$A = \iint_R dA = \int_\alpha^\beta \int_0^{r(\theta)} r\,dr\,d\theta$$

$$= \int_\alpha^\beta \frac{r^2}{2}\Big|_0^{r(\theta)} d\theta$$

$$= \frac{1}{2}\int_\alpha^\beta (r(\theta))^2\,d\theta$$

Ex: Find the area between $r = 1 - \cos\theta$ and $r = 1$.



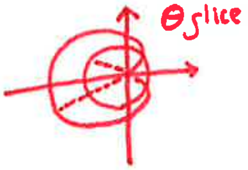


$r = r$

$1 - \cos\theta = 1$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

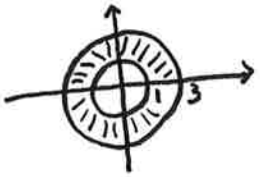


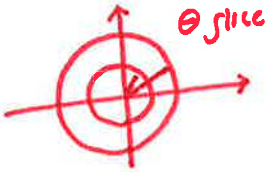
$$A = \iint_R dA$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^{1-\cos\theta} r\,dr\,d\theta$$

$$= 2 + \frac{\pi}{4}$$

Ex: $\iint_R (x^2 + y)\,dA$

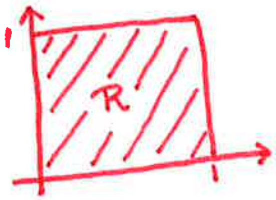




$$\iint_R (x^2+y)\,dA = \int_0^{2\pi}\int_1^3 (r^2\cos^2\theta + r\sin\theta)\cdot r\,dr\,d\theta$$

$$= 20\pi$$

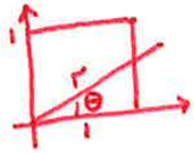
Ex. $\int_0^1 \int_0^1 \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$



$$\int_0^1\int_0^1 \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\frac{\pi}{4}}\int_0^{\sec\theta} \frac{r\cos\theta}{\sqrt{r^2}}\,dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\csc\theta} \frac{r\cos\theta}{\sqrt{r^2}}\,dr\,d\theta$$

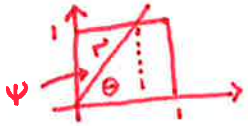
$$= \frac{\ln(\sqrt{2}+1) + \sqrt{2} - 1}{2}$$

θ slices



$\cos\theta = \frac{1}{r}$

$r = \sec\theta$



$\cos\psi = \frac{1}{r}$

$\cos(\frac{\pi}{2} - \theta) = \frac{1}{r}$

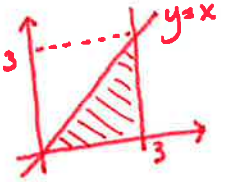
$\sin\theta = \frac{1}{r}$

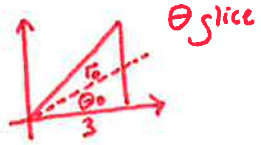
$r = \csc\theta$

Ex: $\int_0^3 \int_0^x \frac{dy\,dx}{\sqrt{x^2+y^2}}$

Plot the region:

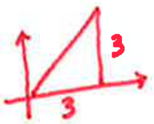
$y=0$ $x=0$
$y=x$ $x=3$





$\cos\theta_0 = \frac{3}{r_0}$

$r_0 = 3\sec\theta_0$

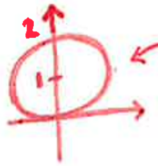


$\int_0^3 \int_0^x \frac{dy\,dx}{\sqrt{x^2+y^2}}$

$\int_0^{\frac{\pi}{4}} \int_0^{3\sec\theta} \frac{r\,dr\,d\theta}{\sqrt{r^2}}$

$\ln(1+\sqrt{2})$

Ex: $\iint_R \frac{dA}{\sqrt{4-x^2-y^2}}$, where R is the circle with radius 1 and center (0,1).



$x^2+(y-1)^2=1$

$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 1$

$r^2 = 2r\sin\theta$; $r \neq 0 \leftarrow 0$ does work

$r = 2\sin\theta$

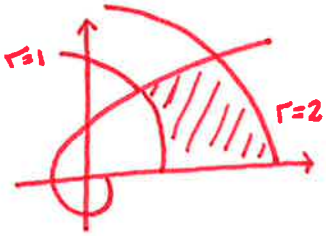
$R = \{(r,\theta): r \leq 2\sin\theta,\ 0 \leq \theta \leq \pi\}$

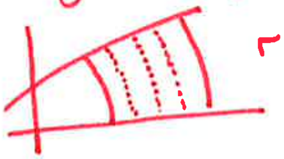
$\iint_R \frac{dA}{\sqrt{4-x^2-y^2}} = \int_0^{\pi} \int_0^{2\sin\theta} \frac{1}{\sqrt{4-r^2}} \cdot r\,dr\,d\theta = 2(\pi - 2)$

Sometimes, it is necessary to take r 'slices' and see how θ varies.

Ex: Find the area of the region bounded by $r = \frac{\pi}{3\theta}$, $r=1$, $r=2$, and the polar axis.

To integrate using $dr\,d\theta$ would require more than one integral. Instead, we slice in r and see how θ varies.





r slices θ from θ = 0 to $\theta = \frac{\pi}{3r}$

$A = \iint_R dA = \int_1^2 \int_0^{\frac{\pi}{3r}} r\,d\theta\,dr = \frac{\pi}{3}$

Gaussian Integral

One of the most important functions in all of Mathematics is the Gaussian function $f(x) = e^{-x^2}$ and its generalization $f(x) = Ae^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

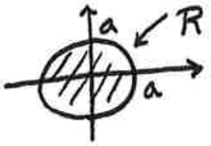
This function is used in....

- Statistics : This function gives the normal distribution: This is the limiting distribution for sample group means by the Central Limit Theorem.
- Partial Differential Equations: e^{-x^2} is the Green functions for the heat equation
- Quantum Mechanics : e^{-x^2} is the wave equation for the ground state of the quantum harmonic oscillator. They also appear in vacuum states.
- Chemistry: e^{-x^2} appears in molecular orbits for atoms.
- Computer Vision: e^{-x^2} is used as a smoothing Kernels.
- Computer Learning & AI: They appear in machine learning algorithms.
- Signal Processing: They appear in Gaussian filters.
- Social Sciences: e^{-x^2} appears nearly everywhere Statistics does, and hence in the social sciences.

The integral $\int_{-\infty}^{\infty} e^{-x^2}dx$, in context, Represents the integral over all of space, so its value is important. But $\int e^{-x^2}dx$ has no solution in elementary functions, so how do we find its value? Polar Integrals to the rescue!

$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

Proof: We will compute this using $\iint_R e^{-(x^2+y^2)} dA$, where



Using polar coordinates, this is...

$$\iint_R e^{-(x^2+y^2)}dA = \int_0^{2\pi}\int_0^a e^{-r^2}\cdot r\,dr\,d\theta = \frac{-1}{2}\int_0^{2\pi}\left(e^{-a^2}-1\right)d\theta = \pi\left(1-e^{-a^2}\right)$$

Then $\lim_{a\to\infty}\iint_R e^{-(x^2+y^2)}dA = \lim_{a\to\infty}\pi\left(1-e^{-a^2}\right) = \pi$. But we also know..

$$** \iint_{\tilde{R}} e^{-(x^2+y^2)}dA = \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}dx\,dy = \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}dx\,dy$$

$$= \left(\int_{-a}^{a} e^{-x^2}dx\right)\left(\int_{-a}^{a} e^{-y^2}dy\right)$$

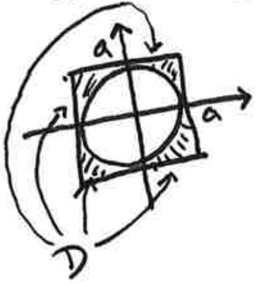
$$* = \left(\int_{-a}^{a} e^{-x^2}dx\right)^2$$

Then $\left(\int_{-a}^{a} e^{-x^2}dx\right)^2 = \pi$. The result follows.

---

* The only difference in these integrals is the variable used, everything else is the same, so they must have the same value.

** If you were watching closely, we switched regions to $\tilde{R}$ - a square with sides 'a'. Why are these integrals, ie over R and $\hat{R}$, the same? After, $a \to \infty$, they cover the entire plane $\mathbb{R}^2$, but this does not mean that their limiting value is the same. We show that their difference, the integral over the region between the square and circle, tends to zero as required.



$$\lim_{a\to\infty}\left(\iint_{\tilde{R}} e^{-(x^2+y^2)}dA - \iint_R e^{-(x^2+y^2)}dA\right)$$

$\lim_{a\to\infty}\iint_D e^{-(x^2+y^2)}dA$; On D, $x^2+y^2 \geq a^2$ because we are outside the circle. Then $e^{-(x^2+y^2)} \leq e^{-a^2}$

$$0 \leq \iint_D e^{-(x^2+y^2)}dx\,dy \leq \iint_D e^{-a^2}dx\,dy = e^{-a^2}\iint_D dA$$

$= e^{-a^2}$ Area D $= e^{-a^2}(4a^2-\pi a^2)$. But $\lim_{a\to\infty} e^{-a^2}(4a^2-\pi a^2) = 0$.

§12.6: Cylindrical Integrals

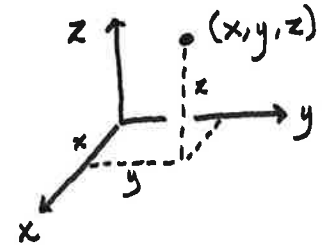
Topics

- Cylindrical Coordinates
- Cylindrical Integrals

You Shoud Be Able to....

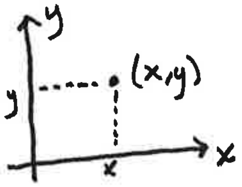
- 'Derive' cylindrical coordinates
- Convert between Cartesian and cylindrical coordinates
- Sketch surfaces given in cylindrical form
- Convert surfaces to/from Cartesian and cylindrical forms
- State the Jacobian for cylindrical integrals
- Integrate by changing to cylindrical coordinates.
- Compute volumes using cylindrical coordinates.

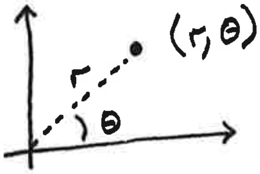
Just like in the plane $\mathbb{R}^2$, we have a systematic way of locating points in 3-space using Cartesian coordinates.



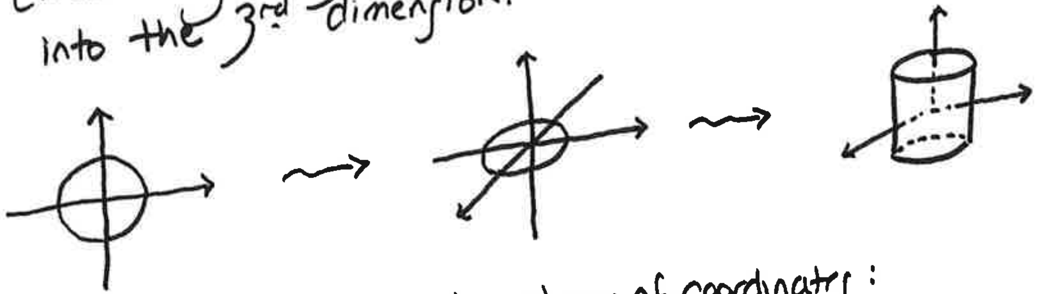
x = distance left/right
y = distance forward/back
z = distance up/down

In the plane, we had polar coordinates as an alternative to Cartesian coordinates.



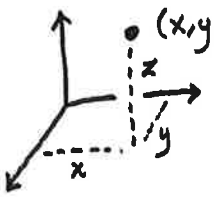


We create a 3-dimensional version of polar coordinates – cylindrical coordinates, so named because we drag the circular symmetry from polar coordinates and 'drag' them into the 3rd dimension.

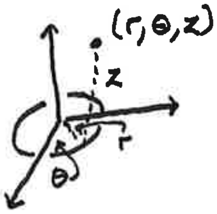


This immediately gives the change of coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \\ z = z \end{cases}$$



Cartesian Coordinates



Cylindrical Coordinates

Ex: Convert Coordinates

(a) $(x,y,z) = (-1,0,2)$

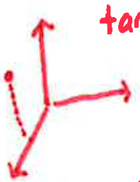
(b) $(x,y,z) = (-1,\sqrt{3},13)$

(c) $(r,\theta,z) = (4,\frac{2\pi}{3},-2)$

(d) $(r,\theta,z) = (5,\frac{\pi}{6},4)$

(a) $r^2 = (-1)^2 + 0^2 = 1 \to r = 1$

$z = 2$



$\tan\theta = \frac{0}{-1}$

$= 0$

$\theta = 0$

$* \tan\theta = \frac{\#}{0}$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

(b) $r^2 = (-1)^2 + (\sqrt{3})^2$

$r^2 = 1+3$

$r^2 = 4$

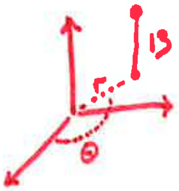
$r = 2$

$z = 13$

$\tan\theta = \frac{\sqrt{3}}{-1}$

$\theta = \frac{-\pi}{3}$

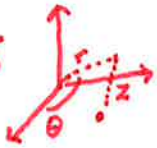
$\theta = \frac{2\pi}{3}$



(c) $x = 4\cos(\frac{2\pi}{3}) = -2$

$y = 4\sin(\frac{2\pi}{3}) = 2\sqrt{3}$

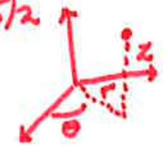
$z = -2$



(d) $x = 5\cos(\frac{\pi}{6}) = \frac{5\sqrt{3}}{2}$

$y = 5\sin(\frac{\pi}{6}) = 5/2$

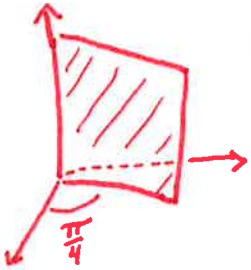
$z = 4$



We can also plot surfaces in these coordinate system.

Ex: $\theta = \frac{\pi}{4}$, ie $R = \{(r,\theta,z) : \theta = \frac{\pi}{4}\}$

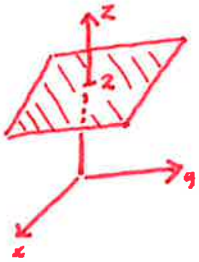
r, z independent of θ, fixed plane angle.



'Half' plane

Ex: $z = 2$, ie $R = \{(r,\theta,z) : z = 2\}$

Just like in Cartesian coordinates



Plane

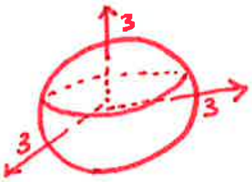
Ex: $r^2+z^2=9$, ie $R=\{(r,\theta,z): r^2+z^2=9\}$

$r^2+z^2=9$

$(x^2+y^2)+z^2=9$

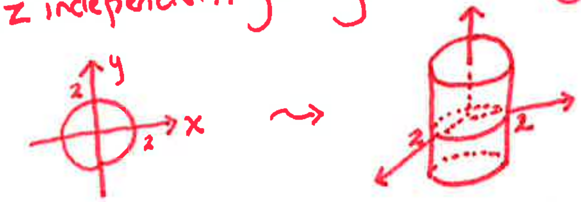
$x^2+y^2+z^2=9$

Sphere



Ex: $r=2$, ie $R=\{(r,\theta,z): r=2\}$

θ, z independent. So any $\measuredangle$ and height.



Cylinder

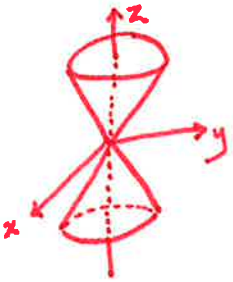
Ex: $z=r$, ie $R=\{(r,\theta,z): z=r\}$

$z=r$

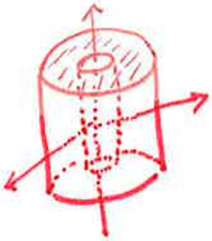
$z^2=r^2$

$z^2=x^2+y^2$

Cone



Ex: $1\le r\le 2$, ie $R=\{(r,\theta,z): 1\le r\le 2\}$



Region between two cylinders

$1\le r\le 2$

$1\le r^2\le 4$

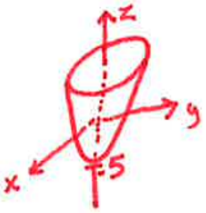
$1\le x^2+y^2\le 4$ ← z-independent

Ex: $z = r^2 - 5$, ie $R = \{(r,\theta,z) : z = r^2 - 5\}$

$z = r^2 - 5$

$z = x^2 + y^2 - 5$

paraboloid



Ex: Find cylindrical equation(s) for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$

$$4x^2 + 4y^2 + z^2 = 1$$
$$4(x^2+y^2) + z^2 = 1$$
$$4r^2 + z^2 = 1$$

Ex: Find a cylindrical description of the ellipsoid $x^2 + 4y^2 + z^2 = 1$

$$x^2 + 4y^2 + z^2 = 1$$
$$x^2 + y^2 + 3y^2 + z^2 = 1$$
$$r^2 + 3r^2\sin^2\theta + z^2 = 1$$

* Note: It is not the z-direction you have to 'drag' circles through space. Any one of x, y, or z will do. For example, in the previous example choosing y as the axis of symmetry

$$x^2 + 4y^2 + z^2 = 1$$
$$x^2 + z^2 + 4y^2 = 1$$
$$r^2 + 4y^2 = 1$$

Where y is playing the role of our 'z', ie

$$\begin{cases} r^2 = x^2 + z^2 \\ \tan\theta = z/x \\ y = y \end{cases}$$

We can similarly work with x as our axis of symmetry.

Why do we care? Just as in polar integrals, cylindrical coordinates will give us a way of exploiting symmetry to simplify certain triple integrals. But we first need to find the Jacobian.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$x_r = \cos\theta \qquad x_\theta = r\sin\theta \qquad x_z = 0$$
$$y_r = \sin\theta \qquad y_\theta = r\cos\theta \qquad y_z = 0$$
$$z_r = 0 \qquad z_\theta = 0 \qquad z_z = 1$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \left|\det \begin{array}{c|ccc} & r & \theta & z \\ \hline x & \cos\theta & -r\sin\theta & 0 \\ y & \sin\theta & r\cos\theta & 0 \\ z & 0 & 0 & 1 \end{array}\right|$$

expanding along 3rd column

$$= \left| 0 - 0 + 1 \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \right|$$
$$= \left| r\cos^2\theta + r\sin^2\theta \right|$$
$$= |r|$$
$$= r$$

Then we have...

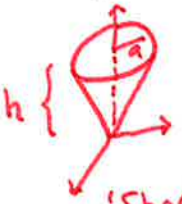
$$dV = r\,dr\,d\theta\,dz$$

Not a surprise, as all we have done is 'add' $z = z$. Note, depending on the axis of symmetry, we could have $dV = r\,dr\,d\theta\,dx$ or $dV = r\,dr\,d\theta\,dy$.

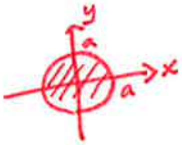
Now for some examples 

Ex: Compute the volume of a right circular cone with base radius 'a' and height h.

In Cartesian coordinates, this would be...



'Shadow in xy-plane



$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{\frac{h}{a}\sqrt{x^2+y^2}}^{h} dz\,dy\,dx$$

The cone is $\frac{a^2}{h^2}z^2 = x^2+y^2$ (why?). Notice every z-slice is a circle, hence the cylindrical coordinates.

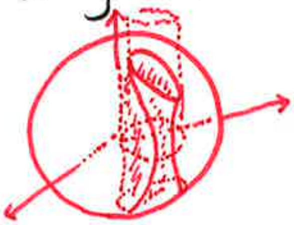
$$V = \iiint_R dV = \int_0^{2\pi}\int_0^a\int_{\frac{h}{a}r}^{h} r\,dr\,d\theta\,dz = \frac{1}{3}\pi a^2 h$$

Ex: $\iiint_R z^2(x^2+y^2)\,dV$, where R is the region $R = \left\{(x,y,z) : \begin{matrix} 0 \le x^2+y^2 \le 1 \\ -1 \le z \le 1 \end{matrix}\right\}$

Notice by the description of R, every z-slice is a circular segment !

$$\iiint_R z^2(x^2+y^2)\,dV = \int_{-1}^{1}\int_0^{2\pi} z^2(r^2)\cdot r\,dr\,d\theta\,dz = \frac{\pi}{3}$$

Ex: Find the volume of the region bound by $x^2+y^2+z^2 = 4$ and $x^2+y^2 = 2y$, ie $x^2+(y-1)^2 = 1$.



Because the region is bounded by a cylinder with an axis of symmetry // to z-axis, slicing in the z-direction should be circular.

$$V = \iiint_R dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$= \frac{16}{9}(3\pi-4)$$

$x^2+y^2 = 2y$

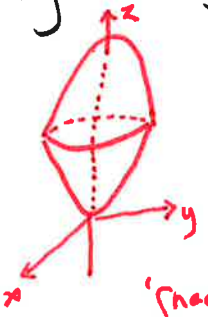
$r^2 = 2r\sin\theta$

$r = 2\sin\theta$

$4 = x^2+y^2+z^2 = r^2+z^2$

$-\sqrt{4-r^2} \le z \le \sqrt{4-r^2}$

Ex: Compute $\iiint_R xyz\,dV$, where R is the region enclosed by $z=x^2+y^2$ and $z=8-(x^2+y^2)$.



$z=z$

$x^2+y^2=8-(x^2+y^2)$

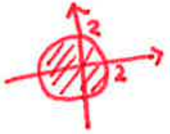
$2(x^2+y^2)=8$

$x^2+y^2=4 \Rightarrow z=4$

$$\iiint_R xyz\,dV$$

$$\int_0^{2\pi}\int_0^{2}\int_{r^2}^{8-r^2}(r\cos\theta)(r\sin\theta)z\cdot r\,dz\,dr\,d\theta$$

'shadow' in xy-plane



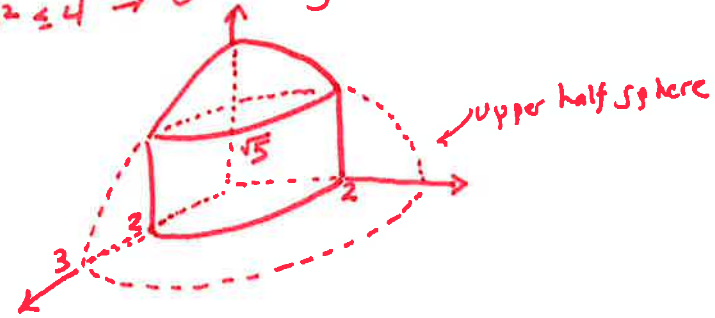
Ex: Sketch the region with volume $\int_0^{\frac{\pi}{2}}\int_0^{2}\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$

$z=0$

$z=\sqrt{9-r^2} \rightarrow z^2=9-r^2 \rightarrow z^2=9-x^2-y^2 \rightarrow x^2+y^2+z^2=9$

$0\le r\le 2 \rightarrow 0\le r^2\le 4 \rightarrow 0\le x^2+y^2\le 4$

$\theta\in[0,\pi/2]$



$z^2=9-r^2$

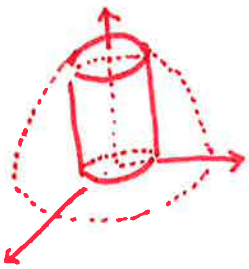
$r=2$ max

$z^2=9-4$

$z^2=5$

$z=\sqrt{5}$

Ex: Find volume of region bounded above by sphere $x^2+y^2+z^2=9$, below by $z=0$, and sides by $x^2+y^2=4$



$x^2+y^2+z^2=9$

$r^2+z^2=9$

$x^2+y^2=4$

$r^2=4$

$r=2$

$$V=\iiint_R dV$$

$$=\int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$$=\frac{2\pi}{3}\left(27-5^{3/2}\right)$$

§ 12.7: Spherical Integrals

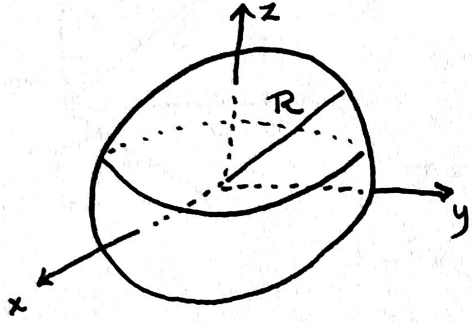
Topics

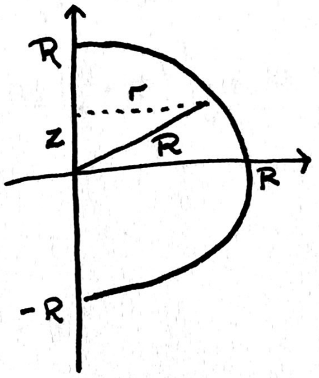
- Spherical Coordinates
- Spherical Integrals

You Should Be Able to....

- 'Derive' spherical coordinates
- Convert between Cartesian, Cylindrical, and spherical coordinates.
- Sketch surfaces in spherical form.
- Convert surfaces to/from spherical form.
- State the Jacobian for spherical integrals.
- Integrate by changing to spherical coordinates.
- Compute volumes by using spherical coordinates.

We have seen how to exploit the symmetry of regions that have a circular (radial) symmetry about an axis, ie cylindrical coordinates. But this not sufficient for many problems. For example, try the 'easy' problem of finding the volume of a sphere of radius R.

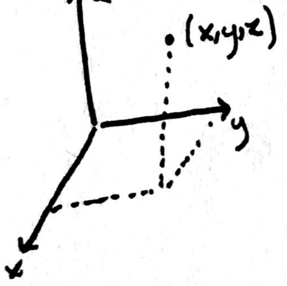


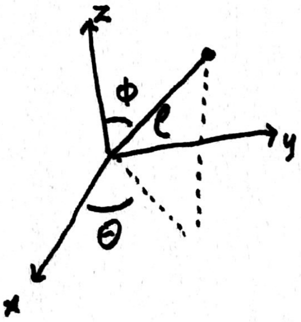


$$R^2 = z^2 + r^2$$
$$z^2 = R^2 - r^2$$
$$z = \pm\sqrt{R^2 - r^2}$$

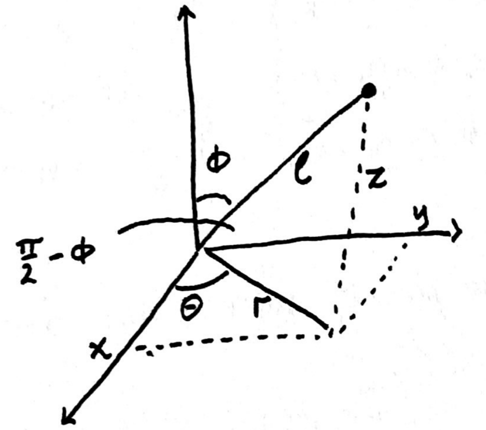
$$V = \iiint dV$$
$$= \int_0^{2\pi}\int_0^{R}\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta$$
$$= \frac{4}{3}\pi R^3$$

But this does not make full use of all the symmetry of the sphere. The sphere not only has the property that every 'slice' is a circle, but that every point has the same distance to its center, ie spherical symmetry. Instead of locating a point using (x,y,z), we now use a coordinate system that exploits this symmetry - (ρ, θ, ϕ), ie locating a point using a radius ρ, an angle left|right θ, and an angle from the vertical ϕ.





First, we need to find the conversions between our different coordinate systems. [We assume $0 < \Phi < \frac{\pi}{2}$. If $\Phi > \frac{\pi}{2}$, everything works similarly and we obtain the same formulas.]



$$x = r\cos\theta = \left(\rho\cos\left(\tfrac{\pi}{2}-\Phi\right)\right)\cos\theta = \rho\sin\Phi\cos\theta$$

$$y = r\sin\theta = \left(\rho\cos\left(\tfrac{\pi}{2}-\Phi\right)\right)\sin\theta = \rho\sin\Phi\sin\theta$$

$$z = \rho\cos\Phi$$

$$r^2 = x^2+y^2$$

$$\rho^2 = r^2+z^2 = x^2+y^2+z^2$$

$$\tan\Phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}$$

$$\theta = \theta$$

$$\tan\theta = y/x$$

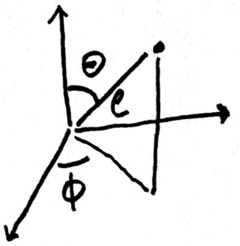
* ρ is the Greek letter 'rho'.

* By definition: $\rho \geq 0$; $0 \leq \theta \leq 2\pi$; $0 \leq \Phi \leq \pi$

* Φ is called the azimuthal angle.

* Spherical coordinates is a 'longitude/latitude' method of locating points.

* Mathematicians (sensibly) use θ for the planar angle – as it has been and always shall be – but physicists switch the roles of θ and Φ. So physicists use....



So we have conversions....

Cart. → Sph.

$(x,y,z) \mapsto (\rho, \theta, \phi)$

$$\begin{cases} \rho^2 = x^2+y^2+z^2 \\ \tan\theta = y/x \\ \tan\phi = \frac{\sqrt{x^2+y^2}}{z} \end{cases}$$

Sph. → Cart.

$(\rho,\theta,\phi) \mapsto (x,y,z)$

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

Cyl. → Sph.

$(r,\theta,z) \mapsto (\rho,\theta,\phi)$

$$\begin{cases} \rho^2 = r^2+z^2 \\ \theta = \theta \\ \tan\phi = r/z \end{cases}$$

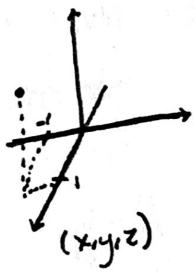
Sph. → Cyl.

$(\rho,\theta,\phi) \mapsto (r,\theta,z)$

$$\begin{cases} r = \rho \sin\phi \\ \theta = \theta \\ z = \rho\cos\phi \end{cases}$$

We now practice plotting and converting between these coordinates.

<u>Ex</u>: $(x,y,z) = (1,-1,\sqrt{6})$



$\rho^2 = 1^2 + (-1)^2 + (\sqrt{6})^2$

$\rho^2 = 1+1+6$

$\rho^2 = 8$

$\rho = \sqrt{8}$

$\rho = 2\sqrt{2}$

$\tan\theta = \frac{-1}{1}$

$\tan\theta = -1$

$\theta = -\frac{\pi}{4}$

$\wr$

$\theta = \frac{7\pi}{4}$

$\tan\phi = \frac{\sqrt{1^2+(-1)^2}}{\sqrt{6}}$

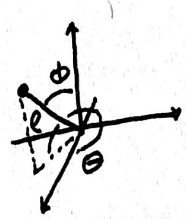
$\tan\phi = \frac{\sqrt{1+1}}{\sqrt{6}}$

$\tan\phi = \frac{\sqrt{2}}{\sqrt{6}}$

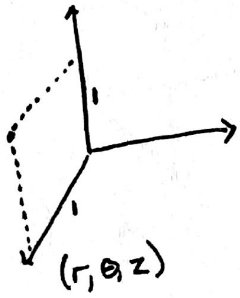
$\tan\phi = 1/\sqrt{3}$

$\phi = \frac{\pi}{6}$

$(x,y,z) = (1,-1,\sqrt{6}) \rightsquigarrow (\rho,\theta,\phi) = (2\sqrt{2}, \frac{7\pi}{4}, \frac{\pi}{6})$



<u>Ex</u>: $(r,\theta,z) = (1,0,1)$



$\rho^2 = 1^2+1^2$

$\rho^2 = 1+1$

$\rho^2 = 2$

$\rho = \sqrt{2}$

$\theta = 0$

$\tan\phi = 1/1$

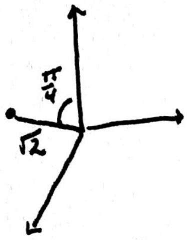
$\tan\phi = 1$

$\phi = \frac{\pi}{4}$

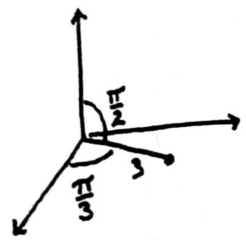
$(r,\theta,z) = (1,0,1)$

$\wr$

$(\rho,\theta,\phi) = (\sqrt{2}, 0, \frac{\pi}{4})$



Ex: $(\rho, \theta, \phi) = (3, \frac{\pi}{3}, \frac{\pi}{2})$



(ρ, θ, ϕ)

$x = 3 \sin\frac{\pi}{2} \cos\frac{\pi}{3} = 3/2$

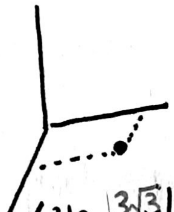
$y = 3 \sin\frac{\pi}{2} \sin\frac{\pi}{3} = 3\sqrt{3}/2$

$z = 3 \cos\frac{\pi}{2} = 0$

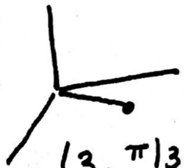
$r = 3 \sin\frac{\pi}{2} = 3$

$\theta = \frac{\pi}{3}$

$z = 3 \cos\frac{\pi}{2} = 0$



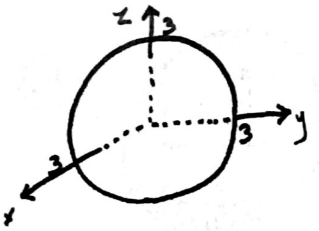
$(3/2, 3\sqrt{3}/2, 0)$
x y z



$(3, \pi/3, 0)$
r θ z

We should also be able to describe and plot surfaces in these coordinates.

Ex: $\rho = 3$, ie $\{(\rho, \theta, \phi) : \rho = 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$



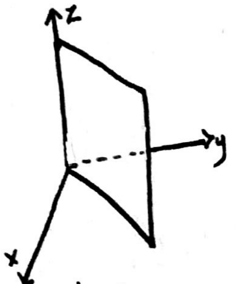
Sphere radius 3

$\rho = 3$

$\rho^2 = 9$

$x^2 + y^2 + z^2 = 9$

Ex: $\theta = \frac{\pi}{4}$, i.e. $\{(\rho, \theta, \phi) : \rho \text{ arbitrary}, \theta = \frac{\pi}{4}, 0 \le \phi \le \pi\}$



'half' plane

$\theta = \frac{\pi}{4}$

$\tan\theta = \tan\frac{\pi}{4}$

$\tan\theta = 1$

$\frac{y}{x} = 1$

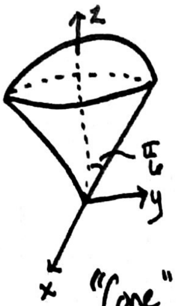
$y = x$

$\underbrace{y - x = 0}$

z-arb., plane

But $x, y \ge 0$.

Ex: $\{(\rho,\theta,\phi) : 0\le\rho\le3,\ 0\le\theta\le2\pi,\ 0\le\phi\le\frac{\pi}{6}\}$



$0\le\rho\le3$

$0\le\rho^2\le9$

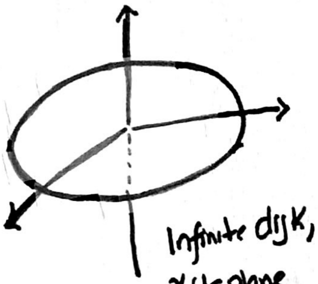
$0\le x^2+y^2+z^2\le9$

$0\le\phi\le\frac{\pi}{6}$

∡ with z-axis between 0° and 30°.

"Cone" ← Not quite, 'capped' cone.

Ex: $\phi=\frac{\pi}{2}$, ie $\{(\rho,\theta,\phi) :$ ρ arbitrary, $0\le\theta\le2\pi,\ \phi=\frac{\pi}{2}\}$



Infinite disk, ie xy-plane

$\phi=\frac{\pi}{2}$

$\tan\phi=\tan\frac{\pi}{2}$

$\underline{\tan\phi}=$ undef.

$\frac{\sqrt{x^2+y^2}}{z}=$ undef.

$z=0$

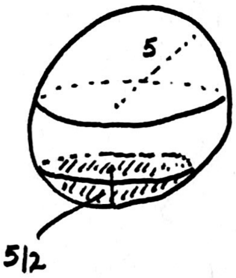
or

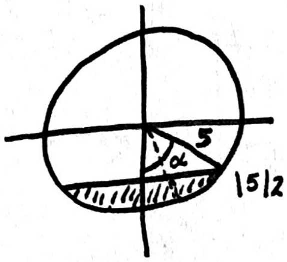
$z=\rho\cos\phi$

$z=\rho\cos\frac{\pi}{2}$

$z=0$

Ex:



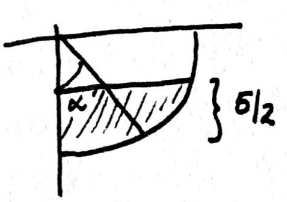


$\cos\alpha=\frac{5/2}{5}=1/2$

$\cos\alpha=1/2$

'$\alpha=\pi/3$'

$\alpha=\frac{2\pi}{3}$



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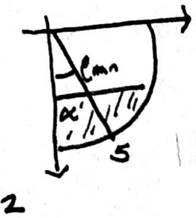
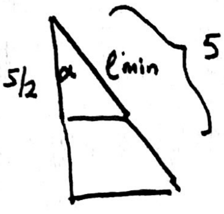


$\rho_{max}=5$

$\rho_{min}=?$



$\cos\alpha=\frac{5/2}{\rho_{min}}$

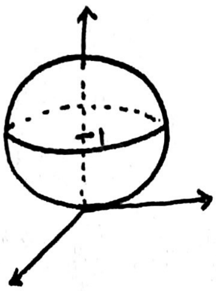
$\rho_{min}=\frac{5}{2}\frac{1}{\cos\alpha}$

$\rho_{min}=5/2\sec\alpha=\frac{5}{2}\sec\phi\rightsquigarrow\frac{-5}{2}\sec\phi$

$$\left\{(\rho,\theta,\phi):\begin{array}{l}\frac{-5}{2}\sec\phi\le\rho\le5\\ \frac{2\pi}{3}\le\phi\le\pi\\ 0\le\theta\le2\pi\end{array}\right\}$$

Ex: $\rho = 2\cos\phi$

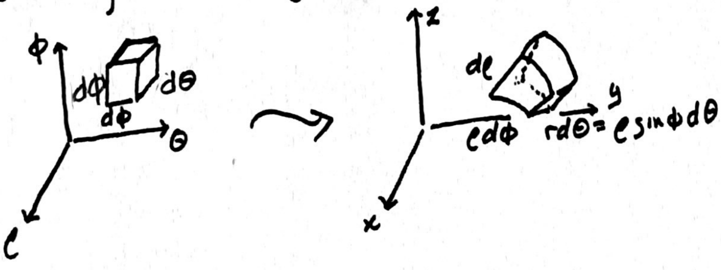
$$\rho^2 = 2\rho\cos\phi$$
$$x^2+y^2+z^2 = 2z$$
$$x^2+y^2+(z^2-2z)=0$$
$$x^2+y^2+(z^2-2z+1)=1$$
$$x^2+y^2+(z-1)^2=1$$



We want to use spherical coordinates to simplify previously difficult or impossible integrals. We use the substitution....

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

But we need to find the Jacobian to find the change in volume differentials between Cartesian and spherical coordinates.



So we have Jacobian....

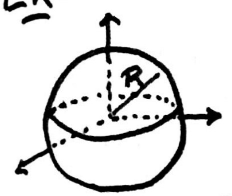
$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \det \begin{array}{c|ccc} & \rho & \theta & \phi \\ \hline x & \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ y & \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ z & \cos\phi & 0 & -\rho\sin\phi \end{array}$$

$$= \cos\phi\left(\rho^2\cos^2\theta\sin\phi\cos\phi + \rho^2\sin^2\theta\sin\phi\cos\phi\right) + \rho\sin\phi\left(\rho\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi\right)$$
$$= \rho^2\cos\phi(\sin\phi\cos\phi) + \rho^2\sin^3\phi$$
$$= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi)$$
$$= \rho^2\sin\phi$$

$$dV = \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

Now let's retry computing the volume of a sphere of radius R.

Ex:



$$V = \iiint dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3} \sin\phi \Big|_{\rho=0}^{\rho=R} d\phi \, d\theta$$

Every 'slice' in θ, ϕ,
ρ goes from 0 to R.
$0 \le \theta \le 2\pi$ } possible θ
$0 \le \phi \le \pi$ } & ϕ slices

$$= \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3} \sin\phi \, d\phi \, d\theta$$

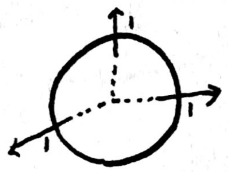
$$= \int_0^{2\pi} \frac{R^3}{3} \cdot -\cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta$$

$$= \frac{R^3}{3}\int_0^{2\pi} (-(-1) - -1) \, d\theta$$

$$= \frac{2R^3}{3}\int_0^{2\pi} d\theta$$

$$= \frac{2R^3}{3} \cdot 2\pi = \frac{4\pi}{3} R^3$$

Ex: $\iiint_R e^{(x^2+y^2+z^2)^{3/2}} dx\,dy\,dz$; where $R = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$



In spherical coordinates

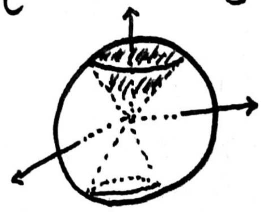
$0 \le \rho \le 1$
$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

$$\iiint_R e^{(x^2+y^2+z^2)^{3/2}} dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{(\rho^2)^{3/2}} \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 e^{\rho^3} \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{4\pi}{3}(e-1)$$

Ex: Find the volume of the region bound by the 'upper' half of the cone $z^2 = x^2+y^2$ and the sphere $x^2+y^2+z^2=9$.



$\rho^2 = x^2+y^2+z^2 = 9$

$\rho^2 = 9$

$\rho = 3$

$\underbrace{x^2+y^2}_{z^2} + z^2 = 9$

$z^2 + z^2 = 9$

$2z^2 = 9$

$z^2 = 9/2$

$z = \pm 3/\sqrt{2}$

$z = 3/\sqrt{2}$

$z = \rho\cos\phi$

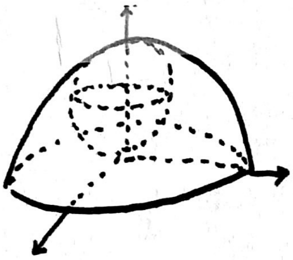
$\frac{3}{\sqrt{2}} = 3\cos\phi$

$\cos\phi = \frac{1}{\sqrt{2}}$

$\phi = \pi/4$

$$V = \iiint dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 9\pi(2-\sqrt{2})$$

Ex: Find the volume outside $x^2+y^2+(z-1)^2=1$ but inside the upper half of $x^2+y^2+z^2=4$.



$x^2+y^2+z^2=4$

$\rho^2 = 4$

$\rho = 2$

$x^2+y^2+(z-1)^2=1$

$x^2+y^2+(z^2-2z+1)=1$

$x^2+y^2+z^2 = 2z$

$\rho^2 = 2\rho\cos\phi$

$\rho = 2\cos\phi$

$$V = \iiint dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 4\pi$$

OR

$V = \frac{1}{2} V_{\text{Big sphere}} - V_{\text{Small sphere}}$

$= \frac{1}{2}\cdot\frac{4\pi}{3}\cdot 2^3 - \frac{4\pi}{3}\cdot 1^3$

$= \frac{4\pi}{3}\left(\frac{8}{2} - 1\right)$

$= \frac{4\pi}{3}(4-1)$

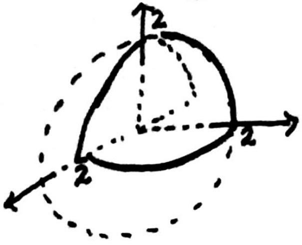
$= 4\pi$

Ex: $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

$z=0 \quad z=\sqrt{4-x^2-y^2}$ $\quad y=0 \quad y=\sqrt{4-x^2}$ $\quad x=-2 \quad x=2$

$z^2=4-x^2-y^2$ $\quad y^2=4-x^2$

$x^2+y^2+z^2=4$ $\quad x^2+y^2=4$



$y\sqrt{x^2+y^2+z^2} = \rho\sin\phi\sin\theta\sqrt{\rho^2} = \rho^2\sin\phi\sin\theta$

$0\le\theta\le 2\pi$

$0\le\phi\le\pi/2$

$0\le\rho\le 2$

$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx = \int_{0}^{\pi}\int_{0}^{\frac{\pi}{2}}\int_{0}^{2} \rho^2\sin\phi\sin\theta\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

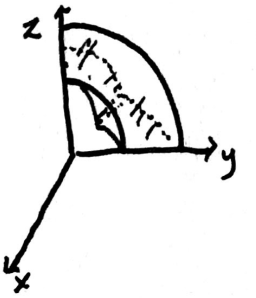
$= \frac{16\pi}{5}$

Ex: $\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{\frac{\pi}{2}}\int_{1}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ computng the volume of a region. Describe the region.

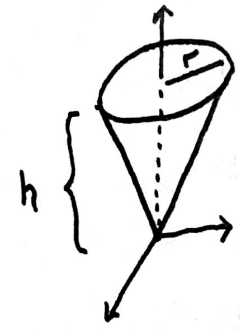
$1\le\rho\le 2$ $\quad \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = dV$

$0\le\phi\le\frac{\pi}{2}$ $\quad \int_{\frac{\pi}{2}}^{\pi}\int_{0}^{\frac{\pi}{2}}\int_{1}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \iiint_R dV$

$\frac{\pi}{2}\le\theta\le\pi$

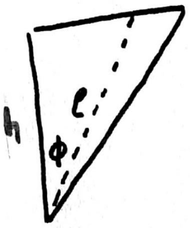


Ex: Find the volume of a right circular cone of base radius r and height h using spherical coordinates.



Fix a ϕ and θ, see how ρ varies.

θ slice:



$\cos\phi = \frac{h}{\rho} \rightarrow \rho = \frac{h}{\cos\phi} \rightarrow \rho = h\sec\phi$

Largest ϕ ~~ϕ~~:

$\tan\phi = \frac{r}{h} \rightarrow \phi = \arctan(r/h)$

$0 \leq \theta \leq 2\pi$

The region is... $\{(\rho, \theta, \phi) : 0 \leq \rho \leq h\sec\phi,\ 0 \leq \phi \leq \arctan(r/h),\ 0 \leq \theta \leq 2\pi\}$

$$V = \iiint_R dV = \int_0^{2\pi}\int_0^{\arctan(r/h)}\int_0^{h\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{3}\pi r^2 h$$

§12.4: Applications of Integration

Topics

- Average Value
- 'Totals'
- Center of Mass
- Moment of Inertia
- Probability Distributions

You Should Be Able to.....

- 'Derive' the average value.
- Compute average values of functions.
- Compute totals, eg mass and charge
- 'Derive' the center of mass
- Compute center of mass for regions.
- Compute moments of inertia.
- Check if a function is a probability function
- Find probabilities
- Compute expected values.

We have seen how to integrate in many variables. But what are these integrals good for? We will now examine several uses for these integrals.

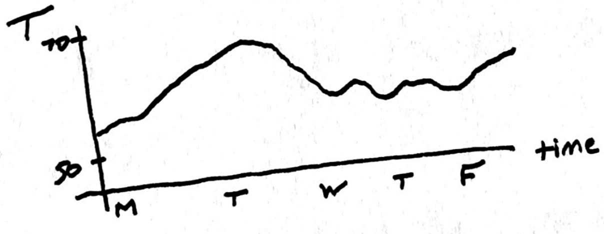
Average Value

For a discrete set of numbers, the average value is $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$.
For example, say we had temperatures for Monday - Friday.

M	T	W	T	F
56	65	61	61	63

$$X_{avg} = Temp_{avg} = \frac{56+65+61+61+63}{5} = \frac{306}{5} = 61.2$$

But temperatures vary every second. These 5 temperatures, on for each day, do not capture the true temperature variability.



We need continuous versions for the numerator & denominator of average value.

$$\bar{x} = X_{avg} = \frac{\text{sum of values}}{\text{total \# (of inputs)}}$$

The numerator is simple - the integral is the continuous version of Σ, (sums). So we might guess

$$\text{Avg Temp} = \frac{1}{5}\int_{0 \,\nwarrow \text{day 0 - Monday}}^{5 \,\leftarrow \text{day 5 - Friday}} T(x)\,dx$$

Now $\frac{1}{5}\int_0^5 T(x)\,dx = \dfrac{\int_0^5 T(x)\,dx}{5} \begin{matrix}\leftarrow \text{sum values} \\ \leftarrow \text{'Total' values}\end{matrix}$. But notice for the denominator, we have...

$$5 = \int_0^5 dx$$

But now we have a continuous version of average value.

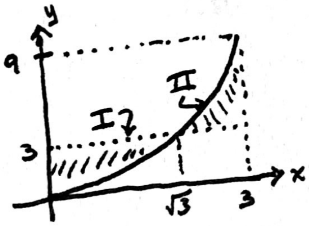
$$\underset{\text{on } [a,b]}{\nearrow}\; f_{avg} := \frac{\int_a^b f(x)\,dx}{\int_a^b dx}$$

<u>Ex</u>: Find the average value of $f(x) = x^2$ on $[0,3]$

$$\int_0^3 x^2\,dx = 9 \quad ; \quad \int_0^3 dx = 3$$

$$f_{avg} := \frac{\int_0^3 x^2\,dx}{\int_0^3 dx} = \frac{9}{3} = 3$$

*



Avg value is defined so area I = area II, ie so $\int_0^3 x^2 - 3\,dx = 0$.

Notice $\int_0^3 dx = 3$ is the length of the interval $[0,3]$. As another example...

$[-1,6]$ has length 7. But...

$$\int_{-1}^{6} dx = x\Big|_{-1}^{6} = 6-(-1) = 7$$

This immediately allows us to generalize to higher dimensions by generalizing length.

Average Values:

I) $f: [a,b] \to \mathbb{R}$

Avg. of f on $[a,b]$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{\int_a^b f(x)\,dx}{\int_a^b dx} = \frac{\int_a^b f(x)\,dx}{\text{length of } [a,b]}$$

II) $f: R \subset \mathbb{R}^2 \to \mathbb{R}$

Avg of f on R

$$f_{avg} = \frac{\iint_R f(x,y)\,dA}{\iint_R dA} = \frac{\iint_R f(x,y)\,dA}{\text{Area of } R}$$

III) $f: R \subset \mathbb{R}^3 \to \mathbb{R}$

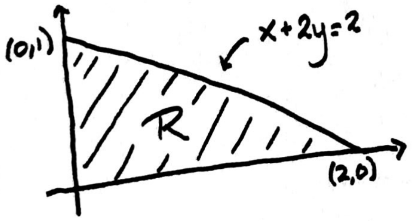
Avg of f on R

$$f_{avg} = \frac{\iiint_R f(x,y,z)\,dV}{\iiint_R dV} = \frac{\iiint_R f(x,y,z)\,dV}{\text{Volume of } R}$$

⋮

Etc.

Ex: The thickness of a metal plate at (x,y) is given by $f(x,y) = xy+1$. The shape of the plate is given by R. Find the avg. thickness.



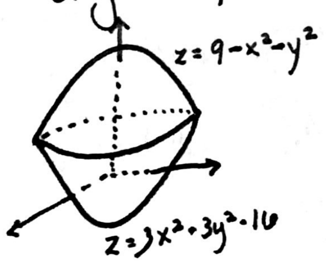
$$\text{Avg. Thickness} = f_{avg} = \frac{\iint_R f(x,y)\,dA}{\iint_R dA} = \frac{7/6}{1} = 7/6 \text{ in.}$$

$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^{2-2y} (xy+1)\,dx\,dy = 7/6$$

$$\iint_R dA = \int_0^1 \int_0^{2-2y} dx\,dy = 1$$

↳ or $A = \frac{1}{2}bh = \frac{1}{2}(2)1 = 1$

Ex: The temperature in a region bounded by $z=9-x^2-y^2$ and $z=3x^2+3y^2-16$ is given by $T(x,y)=z(x^2+y^2)$. Find the avg. temp. in this region.

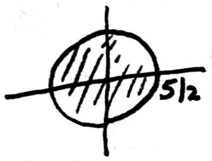


Avg Temp $= T_{avg} = \dfrac{\iiint_R T(x,y,z)\,dV}{\iiint_R dV}$

Notice cylindrical coordinates appropriate.

$z=9-x^2-y^2 \rightsquigarrow z=9-r^2$

$z=3x^2+3y^2-16=3(x^2+y^2)-16 \rightsquigarrow z=3r^2-16$



"shadow" in xy-plane

Now... $9-r^2=3r^2-16$

$4r^2=25$

$r^2=25/4$

$r^2=\pm 5/4$

$r=5/4$

$T(x,y,z)=z(x^2+y^2)$

$\Downarrow$

$T(r,\theta,z)=zr^2$

$$\iiint_R T(x,y,z)\,dV = \int_0^{2\pi}\int_0^{5/2}\int_{3r^2-16}^{9-r^2} zr^2\cdot r\,dz\,dr\,d\theta = -\frac{15625}{256}\pi$$

$$\iiint_R dV = \int_0^{2\pi}\int_0^{5/2}\int_{3r^2-16}^{9-r^2} r\,dz\,dr\,d\theta = \frac{625}{8}\pi$$

$$T_{avg} = \frac{\iiint_R T(x,y,z)\,dV}{\iiint_R dV} = \frac{-15625/256\cdot\pi}{625/8\cdot\pi} = -\frac{25}{32}$$

This is much better than the Cartesian alternative...

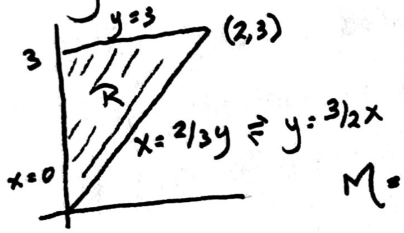
$$T_{avg} = \frac{\int_{-5/2}^{5/2}\int_{-\sqrt{25/4-x^2}}^{\sqrt{25/4-x^2}}\int_{3x^2+3y^2-16}^{9-x^2-y^2} z(x^2+y^2)\,dz\,dy\,dx}{\int_{-5/2}^{5/2}\int_{-\sqrt{25/4-x^2}}^{\sqrt{25/4-x^2}}\int_{3x^2+3y^2-16}^{9-x^2-y^2} dz\,dy\,dx}$$

But they both take work to evaluate.

'Totals'

Recall from Calculus I that if $v(t)$ was the velocity at time t, $\int_{t_0}^{t_1} v(t)\,dt$ computed the total distance traveled between time t_0 and t_1. Double and triple integrals can compute similar totals.

Ex: The density of a triangular joint at a point (x,y) is given by $\rho(x,y) = 2x+y$. Find the mass of the plate.



The integral will sum up the infinitesimal masses (density pieces) to find the total mass.

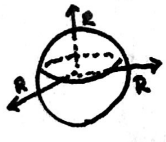
$$M = \iint_R \rho(x,y)\,dA = \int_0^2 \int_{3/2x}^{3} (2x+y)\,dy\,dx = 10g.$$

* The same idea works in 3D, ie $M = \iiint_R \rho(x,y,z)\,dV$

* This comes from the idea for constant density: $M = \rho A = \rho \iint_R dA = \iint_R \rho\,dA$

Of course, you don't have to just use a density function for mass, many types of functions apply.

Ex: In a sphere of radius R, the charge density is given by $C(x,y,z) = \sigma_0 x^2 y^2$. Find the total charge.



Obviously, spherical coordinates will be appropriate.

$$C(x,y,z) = \sigma_0 x^2 y^2 \rightsquigarrow \sigma_0 (\rho \sin\phi \cos\theta)^2 (\rho \sin\phi \sin\theta)^2 = \sigma_0 \rho^4 \sin^4\phi \cos^2\theta \sin^2\theta$$

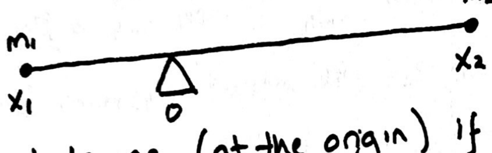
$$C = \iiint_R C(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \sigma_0 \rho^4 \sin^4\phi \cos^2\theta \sin^2\theta \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{4\pi}{105} \sigma_0 R^7$$

Center of Mass

We have seen how to find total mass. How do we find the center of mass? But what do we mean by center of mass? Let's examine the discrete case.

Ex: Take two masses m_1, m_2, located at x_1, x_2, respectively, with $x_1 < 0 < x_2$.



This 'seesaw' balances (at the origin) if $m_1 x_1 + m_2 x_2 = 0$. This is the balance point and is a natural location for our definition of center of mass. But we need general language.

Call $m_i x_i$ the i^{th} 'moment' and the sum $\sum_i x_i m_i$ the total moment. We define the center of mass as the point that if all mass were located there, the total moment would be the same.

Physically, imagine being blindfolded and a metal bar placed on your finger at its balance point. You can't tell the difference between this and a point mass, ie ball, of the same weight placed on your finger.

So we want $M\bar{x} = m_1 x_1 + m_2 x_2$, where $\bar{x}$ is the average x ← ie center of mass and M is the total mass. Then

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{M} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

So generally, the center of mass would be...

$$\bar{x} = \frac{\text{total moment}}{\text{total mass}} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

Then in three dimensions, at least in the discrete case, if we have masses m_i at (x_i, y_i, z_i), the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i} \; ; \quad \bar{y} = \frac{\sum_i m_i y_i}{\sum_i m_i} \; ; \quad \bar{z} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

Of course, we need a continuous version. Take the one dimensional case. If the density is given by $\delta(x)$, we have total mass....

$$M = \sum_i m_i \approx \sum_i \delta(x_i) \Delta x_i$$

Then the total moment (with respect to the origin) is approx...

$$\sum_i \underbrace{x_i}_{\substack{\text{approx} \\ \text{position}}} \underbrace{\delta(x_i) \Delta x_i}_{\substack{\text{approx} \\ \text{mass}}}$$

Generalizing, we then have...

$$\text{Total Mass} = \int_a^b \delta(x)\, dx$$

$$\text{Total Moment} = \int_a^b x\, \delta(x)\, dx$$

$$\text{Center mass} := \bar{x} := \frac{\text{Total Moment}}{\text{Total Mass}} = \frac{\int_a^b x\, \delta(x)\, dx}{\int_a^b \delta(x)\, dx}$$

Then generally a solid R whose density is given by $\delta(x,y,z)$, the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is...

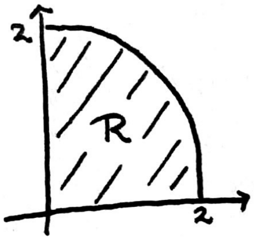
$$\bar{x} = \frac{\iiint_R x\, \delta(x,y,z)\, dV}{M}$$

$$\bar{y} = \frac{\iiint_R y\, \delta(x,y,z)\, dV}{M}$$

$$\bar{z} = \frac{\iiint_R z\, \delta(x,y,z)\, dV}{M}$$

where $M = \text{total mass} = \iiint_R \delta(x,y,z)\, dV$

Ex: Find the center of mass of the solid given by the quad. I portion of $x^2+y^2=4$, whose density at (x,y) is given by $\delta(x,y)=\sqrt{x^2+y^2}$.



Clearly, polar coordinates are the way to go here.

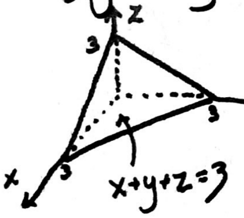
$$\text{Mass} = \iint_R \delta(x,y)\,dA = \int_0^{\frac{\pi}{2}}\int_0^2 \sqrt{r^2}\cdot r\,dr\,d\theta = \frac{4\pi}{3}$$

$$\text{X-moment}: \iint_R x\,\delta(x,y)\,dA = \int_0^{\frac{\pi}{2}}\int_0^2 r\cos\theta\sqrt{r^2}\cdot r\,dr\,d\theta = 4$$

$$\text{y-moment}: \iint_R y\,\delta(x,y)\,dA = \int_0^{\frac{\pi}{2}}\int_0^2 r\sin\theta\sqrt{r^2}\cdot r\,dr\,d\theta = 4$$

$$(\bar{x},\bar{y}) = \left(\frac{4}{4\pi/3}, \frac{4}{4\pi/3}\right) = \left(\frac{3}{\pi}, \frac{3}{\pi}\right) \approx (0.955, 0.955)$$

Ex: The mass density of a tetrahedron given by $(0,0,3)$, $(3,0,0)$, $(0,3,0)$, is given by $\delta(x,y,z) = x+y+z+1$. Find the center of mass.



Notice that the region is 'symmetric' in x,y,z. Also $\delta(x,y,z)$ is 'symmetric' in x,y,z. So we must have $\bar{x}=\bar{y}=\bar{z}$. We need only compute one of these.

$$M = \iiint_R \rho(x,y,z)\,dV = \int_0^3\int_0^{3-x}\int_0^{3-x-y}(x+y+z+1)\,dz\,dy\,dx = \frac{117}{8}$$

$$\text{X-moment} = \iiint_R x\,\rho(x,y,z)\,dV = \int_0^3\int_0^{3-x}\int_0^{3-x-y} x\,(x+y+z+1)\,dz\,dy\,dx = \frac{459}{40}$$

$$\bar{x}=\bar{y}=\bar{z}=\frac{459/40}{117/8} = \frac{51}{65} \approx 0.785$$

$$(\bar{x},\bar{y},\bar{z}) \approx (0.785, 0.785, 0.785)$$

Moment of Inertia

Mass can be considered as a measurement of matter's resistance to straight line motion. Consider the numerators involved in the computation of center of mass.

$$\bar{x}: \iiint_R \underbrace{x}_{\substack{\text{distance}\\ \text{to } yz\text{-plane}}} \underbrace{\rho(x,y,z)dV}_{\text{mass}} \; ; \; \bar{y}: \iiint_R \underbrace{y}_{\substack{\text{distance}\\ \text{to } xz\text{-plane}}} \underbrace{\rho(x,y,z)dV}_{\text{mass}} \; ; \; \bar{z}: \iiint_R \underbrace{z}_{\substack{\text{distance}\\ \text{to } xy\text{-plane}}} \underbrace{\rho(x,y,z)dV}_{\text{mass}}$$

These integrals are called moments - the product of a distance and a mass. In fact, these are the first moments. We will examine another type of moment, the second moment or moment of inertia. What if the object R is spinning? The moments of inertia about various axes a measurements of how 'difficult' it is to make R spin about that axis. They follow the 'same rule' as above.

$$I = \iiint_R \underbrace{d^2}_{\substack{\text{square of distance to}\\ \text{an axis}}} \underbrace{\rho(x,y,z)\,dV}_{\text{mass.}}$$

This is an objects resistance to rotational change about an axis. We define....

Moment of interia about x-axis $:= I_x := \iiint_R (y^2+z^2)\,\delta(x,y,z)\,dV$

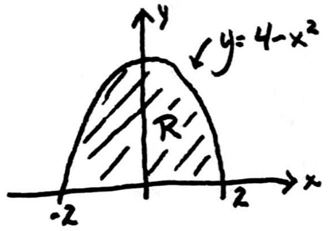
Moment of interia about y-axis $:= I_y := \iiint_R (x^2+z^2)\,\delta(x,y,z)\,dV$

Moment of inertia about z-axis $:= I_z := \iiint_R (x^2+y^2)\,\delta(x,y,z)\,dV$

For example,

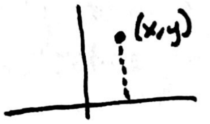
$$I_x := \iiint_R \underbrace{(y^2+z^2)}_{\substack{\text{square of distance to}\\ x\text{-axis}}} \underbrace{\delta(x,y,z)\,dV}_{\text{mass}}$$

Ex: The density at a point (x,y) in a region R is proportional to the distance between (x,y) and the x-axis. Find I_x.



Let K be the constant of proportionality.

Distance from (x,y) to x-axis: y

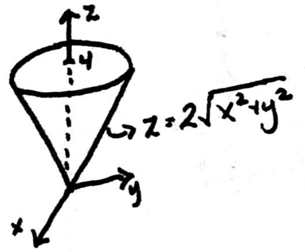


Then $\delta(x,y) = Ky$

$$I_x = \iint_R \underbrace{y^2}_{\text{square dist to x-axis}} \underbrace{\delta(x,y)\, dA}_{\text{mass}} = \int_{-2}^{2}\int_0^{4-x^2} y^2 (Ky)\, dy dx = \frac{32768}{315} K$$

Ex: Let R be the region bounded by $z^2 = 4x^2 + 4y^2$ and $z = 4$. The density varies as $\delta(x,y,z) = 5 - z$. Find the moment of inertia about the z-axis.

Cylindrical coordinates is appropriate.



$$I_z = \iiint_R \underbrace{(x^2+y^2)}_{\text{sq. of dist. to z-axis}} \underbrace{\delta(x,y,z)\, dV}_{\text{mass}}$$

$$= \int_0^{2\pi}\int_0^{2}\int_{2r}^{4} r^2 (5-z) \cdot r dz dr d\theta$$

$$= \frac{32\pi}{3}$$

Just as we had a center of mass, you can define a radius of gyration with respect to an axis, which is the distance r from that axis that we could locate a point mass M so that it has the same moment of inertia I as the original does. This is defined as...

$$r^2 M = I \Rightarrow r = \sqrt{\frac{I}{M}}$$

For example, $r_z = \sqrt{\frac{I_z}{M}}$. For the example above, $r_z = 1$.

There are many more applications to physical bodies, like in hydraulics, but we focus on one more common use of these integrals.

Probability and Statistics

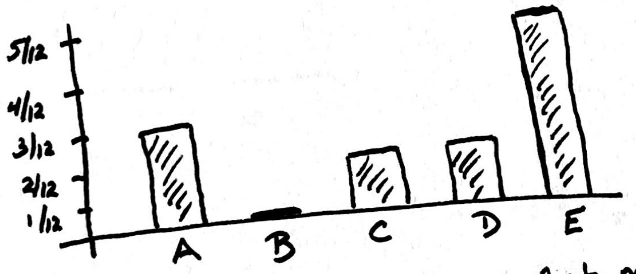
The probability of an event is the likelihood that it occurs. Say you can write the letters A, B, C, D, E on slips of paper and put them in a bag. You write A, A, A, C, C, D, D, E, E, E, E, E and put them in the bag. You randomly draw a slip of paper. What is the chance of drawing a letter. We define some terms....

Sample Space : Set of possible events/outcomes = $\{A, A, A, C, C, D, D, E, E, E, E, E\}$

Probability: $P(x)$ = probability of something in 'x' happening

Ex: $P(A) = \frac{3}{12}$; $P(B) = \frac{0}{12}$; $P(A \text{ or } C) = \frac{5}{12}$

We can represent this graphically....



This is called a probability distribution. What makes a prob. dist. though?

1) $P(x) \geq 0$: The probability must be 0 or bigger.

2) $\sum_i P(x_i) = 1$: The sum of probabilities is 1, ie 100%, because something has to happen

Notice the above satisfies this: $P(A) = \frac{3}{12}$, $P(B) = \frac{0}{12}$, $P(C) = \frac{2}{12}$, $P(D) = \frac{2}{12}$ $P(E) = \frac{5}{12}$

1) $P(A), P(B), P(C), P(D), P(E) \geq 0$

2) $P(A) + P(B) + P(C) + P(D) + P(E) = \frac{3}{12} + \frac{0}{12} + \frac{2}{12} + \frac{2}{12} + \frac{5}{12} = \frac{12}{12} = 1$

Notice also $P(A) = \frac{3}{12}$

Area Box A = $wh = 1(3/12) = 3/12$

This was a discrete probability because there were only finitely many (discrete) events. But we want a continuous version of probability. For this, we turn sums into integrals. We say a (continuous) function $f(x)$ is a probability density function (rep. a prob. distribution) if....

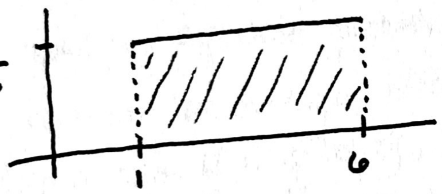
1) $f(x) \geq 0$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \leftarrow$ integral over the sample space.

If so, then we say that the probability of 'events' x between a & b, ie $x \in [a,b]$ is $\int_a^b f(x)\,dx$. Notice this the area under the curve‽ This generalizes what we had before.

Ex:

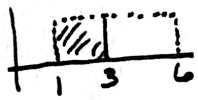
$$f(x) = \begin{cases} 1/6, & x \in [1,6] \\ 0, & \text{otherwise} \end{cases}$$



1) $f(x) \geq 0$ ✓

2) $\int_{-\infty}^{\infty} f(x)\,dx = \int_1^6 \frac{1}{6}\,dx = \frac{1}{6}\int_1^6 dx = \frac{1}{6} \cdot 6 = 1$ ✓

$P(1 \leq x \leq 3) = \int_1^3 f(x)\,dx = \int_1^3 \frac{1}{6}\,dx =$ Area under curve between 1 & 3 $= \frac{1}{6}(3-1) = \frac{2}{6} = 1/3$



$P(x \geq 4) = \int_4^{\infty} f(x)\,dx = \int_4^6 \frac{1}{6}\,dx = \frac{1}{6}(6-4) = \frac{2}{6} = 1/3$

$P(x=2) = \int_2^2 f(x)\,dx = 0$

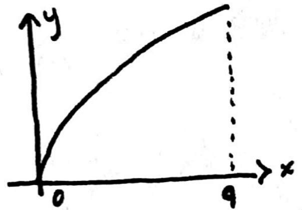
↑ For every cont. distribution $P(\text{one value}) = 0$. There are infinitely many values‽ Say you pick a # between 1 & 6. There are so many; 1, 2.5, 1.7437893....., 6, etc. The chance of me guessing any single # is 0 (notice prob. 0 doesn't mean impossible). But there is a nonzero chance of your # being between values, eg between 1 & 3 (prob. 1/3).

Ex: Find c so that $f(x) = c\sqrt{x}$ is a probability function on $[0,9]$. Then find $P(0 \le x \le 1)$ and $P(x \ge 5)$.

If $c > 0$, then $f \ge 0$ on $[0,9]$. We only need....

$$1 = \int_0^9 c\sqrt{x}\,dx = c\int_0^9 \sqrt{x}\,dx = c \cdot 18 \rightarrow c = 1/18$$

So $f(x) = \frac{1}{18}\sqrt{x}$.



$P(0 \le x \le 1)$ = Prob that x is between 0 and 1

= area under $f(x)$ between 0 & 1

$$= \int_0^1 \frac{1}{18}\sqrt{x}\,dx = \frac{1}{27} \approx 0.037$$

3.7% chance

$$P(x \ge 5) = \int_5^\infty f(x)\,dx = \int_5^9 \frac{1}{18}\sqrt{x}\,dx = 1 - \frac{5\sqrt{5}}{27}$$

≈ 0.586

58.6% chance

We now can associate probability to values. But what happens 'on average'. What do we expect? Say you flip a coin. If it is heads, you pay \$5, and if it is tails you win \$10. What do you win 'on average'? Most would arrive at \$5. But how? How do we compute the expected value?

$$E(X) = \underbrace{\tfrac{1}{2}}_{\text{prob}}\underbrace{(-5)}_{\text{value}} + \underbrace{\tfrac{1}{2}}_{\text{prob}}\underbrace{(10)}_{\text{value}} = \$2.50$$

Generally, we have $E(X) = \sum_i p(x_i)\, x_i$. The continuous version is then clearly $\int x p(x)\,dx$. Generalizing to 2D, we have....

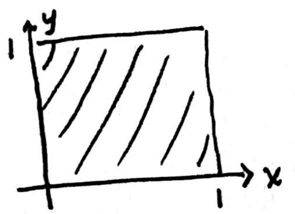
$f(x,y)$ is a prob. dist. if $f(x,y) \ge 0$ and $\iint_R f(x,y)\,dA = 1$. Then $P(R) = \iint_R f(x,y)\,dA$. The average value of x, y are...

$$E(X) = \iint_R x\, f(x,y)\,dA$$

$$E(Y) = \iint_R y\, f(x,y)\,dA$$

Ex: The joint probability distribution function (PDF) of x, y is…

$$f(x,y)=\begin{cases} x+cy^2, & 0\le x\le 1,\ 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$$



a) Find c

b) Find $P(0\le x\le 1/2,\ 0\le y\le 1/2)$

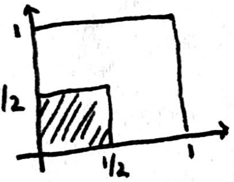
c) Find $P(x\le y)$

d) Find EX & EY.

a) $1=\iint_R f(x,y)\,dA=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dA=\int_0^1\int_0^1 (x+cy^2)\,dx\,dy=\frac{1}{2}+\frac{1}{3}c$

So $c=3/2$. *Note: $f(x,y)\ge 0$ for $(x,y)\in\mathbb{R}^2$!

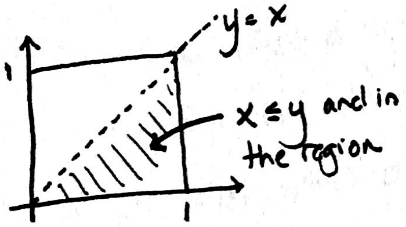
b)



$P(0\le x\le 1/2,\ 0\le y\le 1/2)=\iint_R f(x,y)\,dA$

$=\int_0^{1/2}\int_0^{1/2}\left(x+\frac{3}{2}y^2\right)dx\,dy = 3/32 \approx 9.38\%$ chance

c)



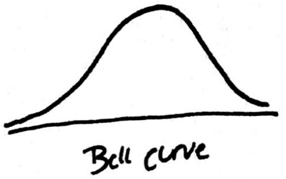
$P(x\le y)=\int_0^1\int_0^x\left(x+\frac{3}{2}y^2\right)dy\,dx=\frac{11}{24}\approx 45.8\%$ chance

d) $E(X)=\iint_R x f(x,y)\,dA=\int_0^1\int_0^1 x\left(x+\frac{3}{2}y^2\right)dx\,dy = 7/12$

$E(Y)=\iint_R y f(x,y)\,dA=\int_0^1\int_0^1 y\left(x+\frac{3}{2}y^2\right)dx\,dy = 5/8$

There are many more concepts to explore, eg standard deviation, covariances, etc.
But note we have met one famous distribution before: e^{-x^2}

$\text{Prob}(a\le x\le b)=\int_a^b e^{-x^2}\,dx$



Bell curve

Generally, if we have a normal dist. with mean μ & S.D. σ

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

§13.1 & 13.5: Vector Fields, Curl, and Divergence

Topics

- Vector fields
- Divergence
- Curl
- Vector formulas
- Applications

You Should Be Able to....

- Define a vector field
- Sketch vector fields
- Draw flow/stream lines
- Know what a gradient/conservative field is.
- Compute the divergence of a vector field
- Explain what divergence measures.
- Compute the curl of a vector field.
- Explain what curl measures
- Explain what irrotational & incompressible/solenoidal mean
- Summarize where these topics appear in the sciences.

We have studied several types of functions:

Multivariable Functions: $f: R \subseteq \mathbb{R}^n \to \mathbb{R}$

$f(x) = x^2 + 1$

$f(x,y) = x \sin y$

$f(x,y,z) = x^2 + xy - e^{x \sin z}$

Vector Valued | Parametric Functions: $f: R \subset \mathbb{R} \to \mathbb{R}^n$

$f(t) = (\cos t, \sin t)$ 

$f(t) = (t, t^2)$ 

$f(t) = \langle t, 1, \sin t \rangle$

We want to study another type of function: Vector fields

A vector field (also called a force field or velocity field) is a function $F: R \subset \mathbb{R}^n \to \mathbb{R}^m$. For us, this will almost always be $F: R \subset \mathbb{R}^2 \to \mathbb{R}^2$ or $F: R \subset \mathbb{R}^3 \to \mathbb{R}^3$, which will take a point (in $\mathbb{R}^2$ or $\mathbb{R}^3$) and assign it a vector (in $\mathbb{R}^2$ or $\mathbb{R}^3$), hence the name.

Ex: Let $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\vec{F}(x,y) = x\hat{\imath} + y\hat{\jmath} = \langle x, y \rangle$. Plot this vector field.

$\vec{F}(0,0) = \langle 0,0 \rangle$

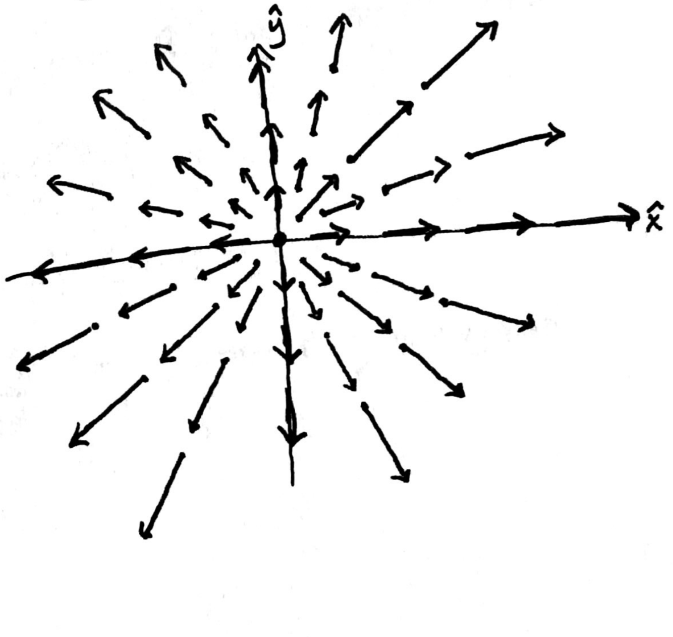
$\vec{F}(1,0) = \langle 1,0 \rangle = \hat{\imath}$

$\vec{F}(0,1) = \langle 0,1 \rangle = \hat{\jmath}$

$\vec{F}(1,1) = \langle 1,1 \rangle$

$\vec{F}(-1,2) = \langle -1,2 \rangle$

⋮



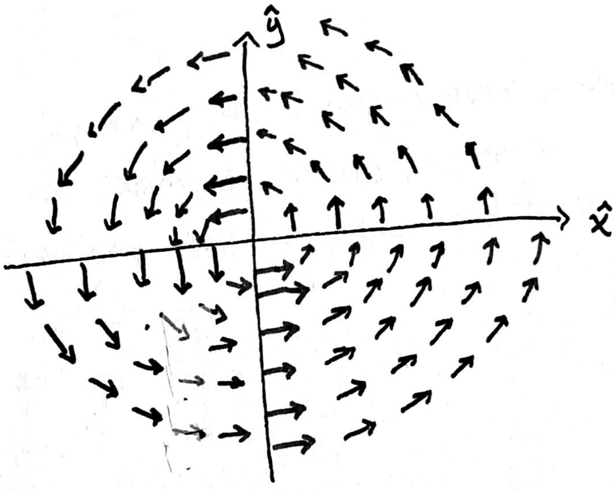
Ex: $\vec{F}(x,y) = \frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{\sqrt{x^2+y^2}} = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$ ← unit vector!

$\vec{F}(0,1) = \langle -1, 0 \rangle = -\hat{\imath}$

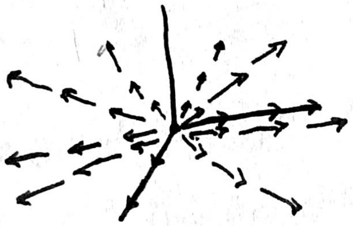
$\vec{F}(1,0) = \langle 0, 1 \rangle = \hat{\jmath}$

$\vec{F}(1,1) = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

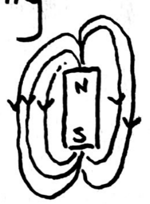
$\vec{F}(1,2) = \left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$



Ex: $\vec{F}(x,y,z) = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k} = \langle x, y, z \rangle$



But what are vector fields used for?



1) Velocity Fields: You may have seen wind maps - at each point giving the strength and direction of wind. They can also be used for water/fluid flows, magnetic/electric fields, etc.

2) Gravitational Fields: G univ. grav. constant, m_1, m_2 masses. Then $\vec{F}(x,y,z) = \frac{-Gm_1m_2}{x^2+y^2+z^2}\vec{u}$, where $\vec{u}$ is the vector from the origin to (x,y,z).

Equivalently, $\vec{F} = \frac{-Gm_1m_2}{|\vec{r}|^2}\hat{u}$

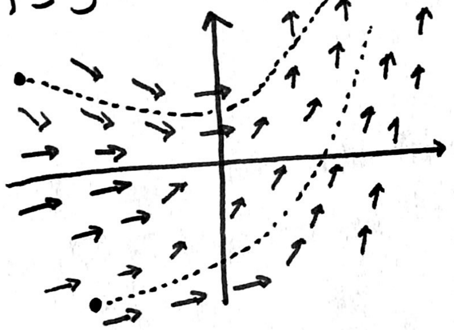
3) Electric Fields: (Coulomb's Law) Charges q_1, q_2

$\vec{F}(x,y,z) = \frac{C q_1 q_2}{|\vec{r}|^2}\vec{u}$, where C is a constant.

The last two are known as inverse square laws:

$$\vec{F}(x,y,z) = \frac{K}{|F|^2}\vec{u}$$

Vector fields can be used to find the path of motion even when it cannot be solved for exactly. For example, suppose the vector field represents wind velocity. Drop the leaf at a point and track where the wind pushes it.



These lines are called flow/stream lines.

Flow/Stream Line: A flow line of a vector field $\vec{F}: X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable path $\vec{x}: I \to \mathbb{R}^n$ such that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$.

* We say $\vec{F}$ is continuous if each of its components are continuous. Sim. for diff.

There will be a special kind of vector field we will examine in more depth later.

Gradient/Conservative Field: We say that a vector field $\vec{F}$ is a gradient field (or a conservative field) if there is a function f (called the potential function) such that $\vec{F} = \nabla f$.

Ex: $\vec{F}(x,y) = \langle 2x, y\rangle$ is conservative because if $f(x,y) = x^2 + \frac{1}{2}y^2$ then $F = \nabla f$.

* The terminology comes from Physics, ie conserved quantities.

In fact, every inverse square law $\vec{F} = \frac{K}{|F|^2}\vec{u}$ is conservative with potential function $f(x,y,z) = \frac{-K}{\sqrt{x^2+y^2+z^2}}$. Not all vector fields are gradient fields. But we will give a way of telling which functions are gradient fields, as well as their properties, later. Recall the del operator $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$. We now examine its relationship to vector fields.

Divergence

Let $\vec{F}: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a diff. vector field. The divergence of $\vec{F}$, written div $\vec{F}$ or $\nabla \cdot F$ (read 'del dot $\vec{F}$') is the scalar field

$$\text{div } F = \nabla \cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \dots + \frac{\partial F_n}{\partial x_n} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

Ex: $\vec{F} = \langle x, y \rangle$

$$\text{div } F = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2$$

Ex: $\vec{F} = \langle x^2+y, x-y^2 \rangle$

$$\text{div } F = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2+y) + \frac{\partial}{\partial y}(x-y^2) = 2x + (-2y) = 2x - 2y$$

Ex: $\vec{F} = x^2y\,\hat{i} + xz\,\hat{j} + xyz\,\hat{k}$

$$\text{div } F = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2y, xz, xyz \rangle = 2xy + 0 + xy = 3xy$$

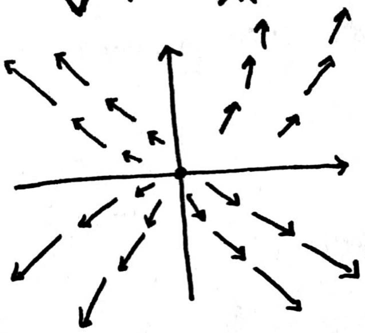
The divergence of a vector field (at a point) measures the net flow or 'flux' in/out of that point.

div $F > 0$: flow away/out of a point ← source

div $F < 0$: flow toward/into a point ← sink

Ex: $\vec{F} = \langle 2x, 3y \rangle$

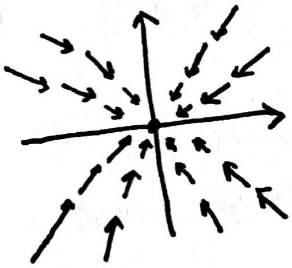
$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) = 2 + 3 = 5$ ← so all points a source.



$\nabla \cdot \vec{F} = 5$

← Greater flow 'away' from each point than into it. So $\vec{F}$ 'diverging' at every point. Each point is a source.

If we had $\vec{F} = \langle -2x, -3y \rangle$, then $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-3y) = -5$



$\nabla \cdot \vec{F} = -5$

← Greater flow 'towards' each point than out of it. So $\vec{F}$ 'contracting' at every point. Each point is a sink.

Ex: $\vec{F} = \langle x^2 + xy, y^2 - x^2 y \rangle$

div F $= \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 + xy) + \frac{\partial}{\partial y}(y^2 - x^2 y) = (2x + y) + (2y - x^2) = 3y + 2x - x^2$

$(\text{div } F)(1,1) = (\nabla \cdot \vec{F})(1,1) = (3y + 2x - x^2)\Big|_{(1,1)} = 3 + 2 - 1 = 4$ ← source pt.

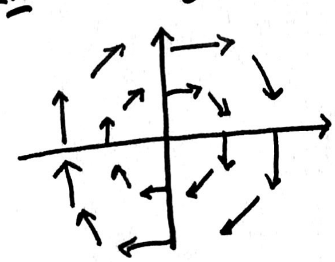
$(\text{div } F)(3,1) = (\nabla \cdot \vec{F})(3,1) = (3y + 2x - x^2)\Big|_{(3,1)} = 3 + 6 - 9 = 0$ ← incompressible pt.

$(\text{div } F)(2,-2) = (\nabla \cdot \vec{F})(2,-2) = (3y + 2x - x^2)\Big|_{(2,-2)} = -6 + 4 - 4 = -6$ ← sink pt.

Not all vector fields have any divergence.

Ex: $\vec{F} = \langle y, -x \rangle$

div F: $\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \cdot \langle y, -x \rangle = 0 + 0 = 0$



$\vec{F}$ is like a rotating fluid – without source or sink.

✱ A vector field with $\nabla \cdot \vec{F} = 0$ at every point is called incompressible or solenoidal.

Our last vector field example did not 'flow out' or 'flow in' at any point but rather just swirled. Can we find something that measures this rotation?

Curl

Let $\vec{F}: \mathcal{U} \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a diff. vector field (on $\mathbb{R}^3$). The curl of $\vec{F}$, denoted curl $\vec{F}$ or $\nabla \times \vec{F}$, is the vector field

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

The curl measures the tendency for a vector field to rotate about a point. $\|\text{curl } \vec{F}\| = \|\nabla \times \vec{F}\|$ measures the speed of this rotation.

Ex: $\vec{F}(x,y,z) = xy\,\hat{\imath} - z\hat{\jmath} + (xy - z^2)\hat{k} = \langle xy, -z, xy - z^2 \rangle$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \times \langle xy, -z, xy - z^2 \rangle$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -z & xy - z^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy - z^2) - \frac{\partial}{\partial z}(-z)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}(xy - z^2) - \frac{\partial}{\partial z}(xy)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(xy)\right]\hat{k}$$

$$= (x - (-1))\hat{\imath} - (y - 0)\hat{\jmath} + (0 - x)\hat{k}$$

$$= (x+1)\hat{\imath} - y\hat{\jmath} - x\hat{k}$$

$$= \langle x+1, -y, -x \rangle$$

$$(\text{curl } \vec{F})(1,0,1) = (\nabla \times \vec{F})(1,0,1) = \langle x+1, -y, -x \rangle \Big|_{(1,0,1)} = \langle 2, 0, -1 \rangle$$

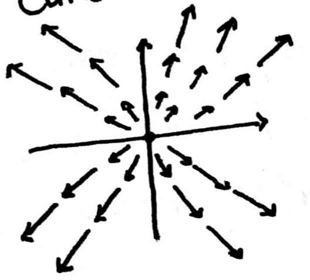
If the vector field is two dimensional, embed it in three dimensions.

Ex: $\vec{F} = x\hat{i} + y\hat{j} = \langle x, y \rangle$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle x, y, 0 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix}$$

$$= \langle 0, 0, 0 \rangle$$

$$= \vec{0}$$

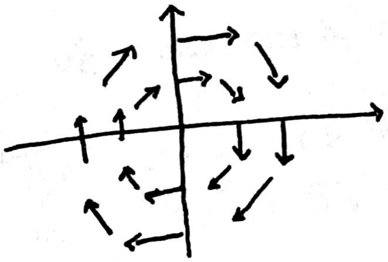


Only flow in/out (really just out) of each point. There is no rotation. So curl free or irrotational.

* Vector fields $\vec{F}$ with curl $\vec{F} = \vec{\nabla} \times \vec{F} = \vec{0}$ at every point are called irrotational.

Ex: $\vec{F} = y\hat{i} - x\hat{j} = \langle y, -x \rangle$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix} = \langle 0, 0, -2 \rangle$$



No flow in/out at any point but there is rotation at every point.

Ex: $\vec{F} = e^x \cos y \, \hat{i} + e^x \sin y \, \hat{j} + z\hat{k} = \langle e^x \cos y, e^x \sin y, z \rangle$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \cos y & e^x \sin y & z \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(e^x \sin y)\right)$$

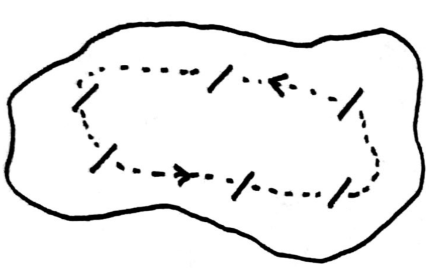
$$-\hat{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(e^x \cos y)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(e^x \sin y) - \frac{\partial}{\partial y}(e^x \cos y)\right)$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(e^x \sin y + e^x \sin y)$$

$$= 2e^x \sin y \, \hat{k}$$

$$= \langle 0, 0, 2e^x \sin y \rangle$$



Imagine a stick floating on the water in a pond. In the left, curl $\vec{F} = \vec{0}$ because the stick does not rotate as it moves while on the right, curl $\vec{F} \neq \vec{0}$ because the stick rotates.

As a brief summary,

- Div $\vec{F} = \vec{\nabla} \cdot \vec{F}$ and measures the flow in/out of points.
- Curl $\vec{F} = \vec{\nabla} \times \vec{F}$ and measures the rotation of a vector field. It is ⊥ to the motion and is direction of axis of rotation.
- Div $\vec{F}$ gives a scalar (field).
- Curl $\vec{F}$ gives a vector (field).

We make note of a few important theorems:

Thm: Let $f: \mathcal{U} \subseteq \mathbb{R}^3 \to \mathbb{R}$ be of class C^2, then curl (grad f) = $\vec{0}$, ie $\nabla \times \nabla f = \vec{0}$; that is, gradient fields are irrotational.

Thm: Let $\vec{F}: \mathcal{U} \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of class C^2, then div (cur $\vec{F}$) = 0, ie $\nabla \cdot \nabla \times \vec{F} = 0$; that is, curl $\vec{F}$ is an incompressible vector field.

So we summarize these in a few identities:

1) $\nabla \times \nabla f = \vec{0}$

2) $\nabla \cdot \nabla \times \vec{F} = 0$

3) $\nabla \cdot (f\vec{F}) = f(\nabla \cdot \vec{F}) + (\nabla f)\,\vec{F}$

4) $\nabla \times (f\vec{F}) = f(\nabla \times \vec{F}) + (\nabla f) \times \vec{F}$

5) $\nabla \cdot (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\nabla \times \vec{G})$

6) $\nabla \times (\vec{F} \times \vec{G}) = (\nabla \cdot \vec{G})\vec{F} - (\nabla \cdot \vec{F})\vec{G} + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}$

7) $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$

8) $\nabla \times (\vec{F} + \vec{G}) = \nabla \times \vec{F} + \nabla \times \vec{G}$

There is plenty more that could be said. For example, we could introduce the Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, ie

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

or more generally $\nabla^2 f = \sum \frac{\partial^2 f}{\partial x_i^2}$. This is also denoted by Δf. This appears in electric & grav. potentials, heat/fluid flow, etc. Instead, we make a few final remarks about the applications of this to the sciences (all this is really the start of Vector Analysis).

Maxwell's Equation

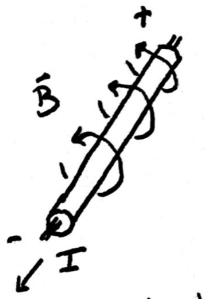
Let $\vec{E}$ denote an electric field, $\vec{B}$ a magnetic field, ϵ_0 the permittivity of free space, ρ charge density, $\vec{J}$ the current density field, and μ_0 be the permeability of free space. Then

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ Gauss' Law ← There are sources for electric fields

$\vec{\nabla} \cdot \vec{B} = 0$ No magnetic monopoles ← Mag. fields are solenoidal

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's Law

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$ Ampère's Law



$\vec{\nabla} \times \vec{B} = 0$ outside cable
$\vec{\nabla} \times \vec{B} \neq \vec{0}$ inside cable

Heat Transfer/Fluid Flow

Let ρ be the density, heat capacity c, temp. dist. $T(x,y,z,t)$, and heat flux vector $\vec{q}$.

Conservation of Heat: $\frac{\partial}{\partial t}(\rho c T) + \nabla \cdot \vec{q} = 0$

Fick's Law: $\vec{q} = -K \nabla T$

Heat Equation: $\frac{\partial T}{\partial t} = K \nabla^2 T$

Similarly in Fluid flow, if ρ is fluid dens. & $\vec{v}$ is fluid velocity

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \rho = 0$$