Calculus III Problem Book

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Contents

Preface

One can only learn Mathematics by doing Mathematics. It is then be necessary to solve problems—lots of them! There are 799 problems in this text. The new student to Multivariable Calculus should try to solve as many as possible. However, solving problems is not enough. Trying every problem 'type' in Calculus could be a lifetime journey. Treat the problems as small lights in the dark, illuminating the paths connecting different concepts. Whenever possible, the student should have in mind the connection between the calculus being performed and the underlying geometry. The problems throughout this text—even the subject itself—cannot be separated from underlying geometrical concepts. There is space before each problem section for brief topic notes for reference.

As for texts, the author strongly suggests *Vector Calculus* by Colley or *Calculus* by Larson and Edwards. These were a common reference when considering what problem types to integrate into the text. The problems themselves were written and compiled by the author from lecture notes of previous iterations of the course. Accordingly, these notes could have been taken or supplemented by sources the author has since forgotten. If the author has seemingly missed a reference or has committed any other error, please email him at cgmcwhor@syr.edu so that he may rectify his error!

Chapter 1

Spatial Geometry & Vectors

1.1 Basic n–Euclidean Geometry

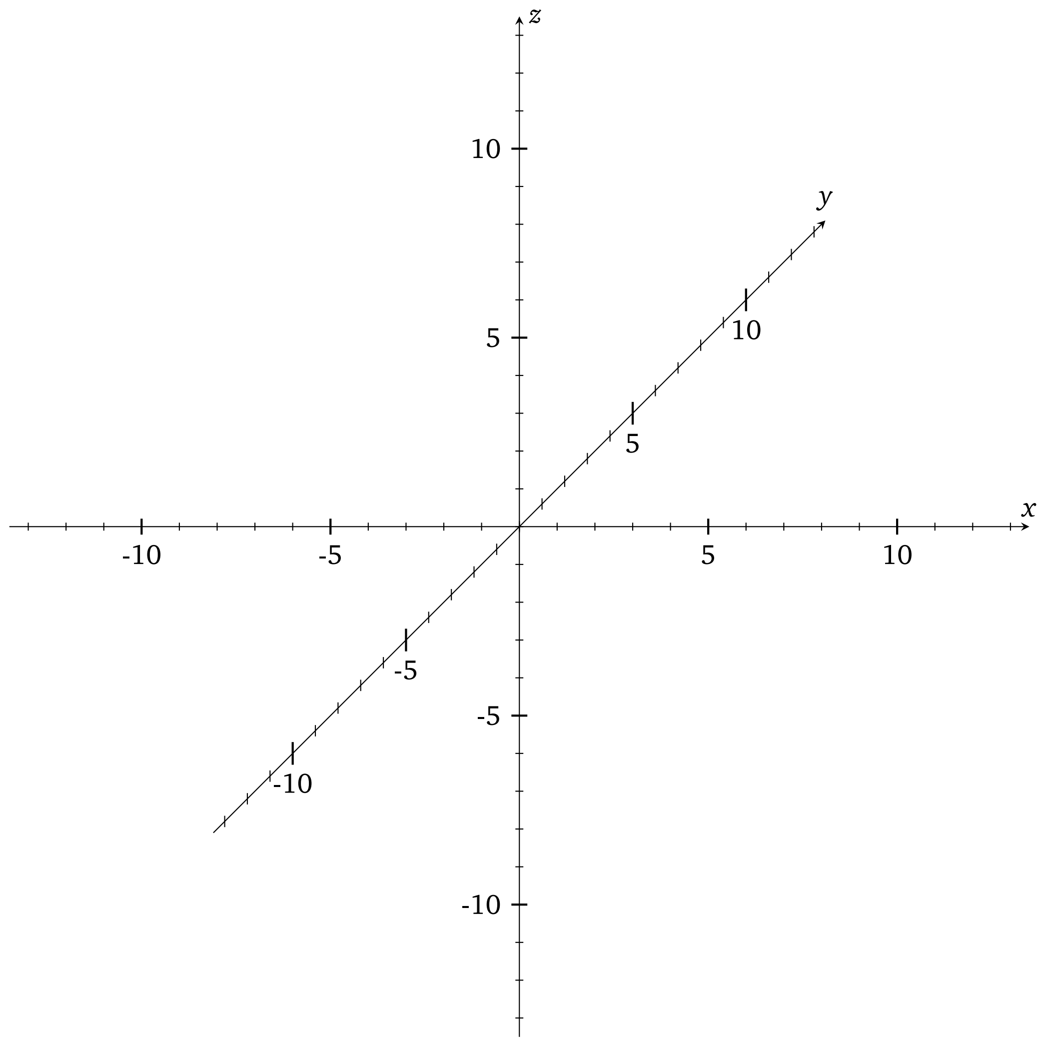
Euclidean n–space:

Coordinate Axes:

Distance:

1.1 | Exercises

1. One the set of axes below, sketch the points $(1,2,3)$, $(6,-7,2)$, and $(-3,0,8)$. Choose a point and carefully draw dotted lines connecting the chosen point perpendicularly to the coordinate planes.



2. Draw an appropriate set of coordinate axes – labeled – and plot the points $(3,6,-2)$, $(-5,-5,5)$, and $(0,0,-7)$. Choose one of the first two points and carefully draw dotted lines connecting the chosen point perpendicularly to the coordinate planes.

3. Find the distance between the points $(2,-1,-3)$ and $(4,3,-1)$. Which point is closer to the xy–plane? Which point is closer to the yz–plane?

4. Find the distance between the points $(4,5,-2)$ and $(3,1,-1)$. Which points is closer to the xz–plane? Which points is closer to the yz–plane?

5. Consider a triangle formed by the points $A(1,0,-1)$, $B(1,-2,-1)$ and $C(1,-2,-3)$. Sketch this triangle in 3–space. Determine if the triangle is an isosceles triangle. Determine if the triangle is a right triangle. Determine if the triangle is an equilateral triangle.

6. Consider a triangle formed by the points $M(-1,2,-1)$, $N(-1,2,-3)$, and $P(-1,6,-2)$. Sketch this triangle in 3–space. Determine the the triangle is an isosceles triangle. Determine if the triangle is a right triangle. Determine if the triangle is an equilateral triangle.

7. For the point $(3,5,4)$, determine the following:

(a) The distance to the xy–plane.

(b) The distance to the yz–plane.

(c) The distance to the xz–plane.

(d) The distance to the x–axis.

(e) The distance to the y–axis.

(f) The distance to the z–axis.

8. For the point $(-1,4,2)$, determine the following:

(a) The distance to the xy–plane.

(b) The distance to the yz–plane.

(c) The distance to the xz–plane.

(d) The distance to the x–axis.

(e) The distance to the y–axis.

(f) The distance to the z–axis.

9. Determine if the following three points lie along a straight line: $A(-5,7,-4)$, $B(1,1,5)$, and $C(-1,3,2)$.

10. Determine if the following three points lie along a straight line: $M(3,-4,2)$, $N(0,-1,8)$, and $P(2,-3,4)$.

11. Find at least 6 points that have distance 3 from the point $(1,-2,6)$. What shape does the set of all points having distance 3 from the points $(1,-2,6)$ make? Sketch the shape, the points found, and the given point in the same plot.

12. Find at least 6 points that have distance 4 from the point $(2,0,-5)$. What shape does the set of all points having distance 4 from the points $(2,0,-5)$ make? Sketch the shape, the points found, and the given point in the same plot.

13. Show that the midpoint of the line segment connecting the points $P_1(x_1,y_1,z_1)$ and $P(x_2,y_2,z_2)$ is given by

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

1.2 Introduction to Vectors

Vector:

Displacement Vector:

Triangle/Parallelogram Law:

Scalar Multiple:

Parallel/Equal Vectors:

Length:

Unit Vector:

Standard Basis Vectors:

1.2 | Exercises

1. Show that $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$.

2. Show that if $\mathbf{v}$ is a nonzero vector then $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

3. Given the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ below, find

(a) $\mathbf{u}+\mathbf{v}$

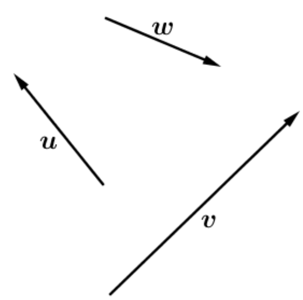
(b) $\mathbf{u}+\mathbf{w}$

(c) $\mathbf{u}-\mathbf{v}$

(d) $2\mathbf{v}$

(e) $-\frac{1}{2}\mathbf{w}$

(f) $\mathbf{u}+\mathbf{v}-\mathbf{w}$



4. Given the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ below, find

(a) $\mathbf{u}-\mathbf{v}$

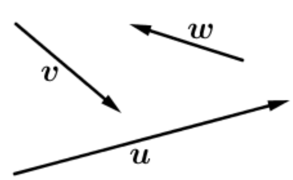
(b) $\mathbf{u}+\mathbf{v}$

(c) $\mathbf{u}-\mathbf{w}$

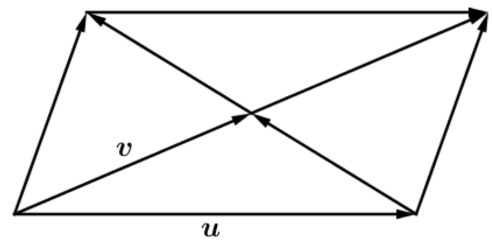
(d) $-2\mathbf{v}$

(e) $\frac{1}{2}\mathbf{w}$

(f) $\mathbf{v}+\mathbf{u}-\mathbf{w}$



5. Given the partially labeled parallelogram below, label all other vectors in the parallelogram.



6. Find and sketch the displacement vector between the two given points:

(a) $P(-1,1)$, $Q(3,-1)$

(b) $M(2,1)$, $N(3,5)$

(c) $A(4,-1)$, $B(0,4)$

7. Find and sketch the displacement vector between the two given points:

(a) $P(2,1,1)$, $Q(3,0,-1)$

(b) $M(4,2,1)$, $N(0,3,5)$

(c) $A(4,-1,1)$, $B(-1,4,-1)$

8. Given $\mathbf{u} = \langle 1,-2 \rangle$ and $\mathbf{v} = \langle 3,2 \rangle$, find the following:

(a) $3\mathbf{u}$

(b) $\mathbf{u}+\mathbf{v}$

(c) $\mathbf{u}-\mathbf{v}$

(d) $2\mathbf{u}-3\mathbf{v}$

(e) $||\mathbf{u}||$

(f) $||\mathbf{u}+\mathbf{v}||$

9. Given $\mathbf{u} = \langle 2,-1,1 \rangle$ and $\mathbf{v} = \langle 3,0,1 \rangle$, find the following:

(a) $-2\mathbf{u}$

(b) $\mathbf{u}-\mathbf{v}$

(c) $2\mathbf{u}+\mathbf{v}$

(d) $3\mathbf{u}-\mathbf{v}$

(e) $||\mathbf{u}||$

(f) $||\mathbf{u}-\mathbf{v}||$

10. Describe geometrically the collection of points $r\langle 1,3 \rangle + s\langle 2,1 \rangle$, where r and s are integers.

11. Find a unit vector in the same direction as $3\mathbf{i}-4\mathbf{j}$.

12. Find a unit vector in the same direction as $-5\mathbf{i}+12\mathbf{j}$.

13. Find a unit vector in the same direction as $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$.

14. Find a unit vector that points in the 'opposite' direction as $2\mathbf{i}-3\mathbf{j}$.

15. Find a unit vector in the 'opposite direction' as $-2\mathbf{i}+5\mathbf{k}$.

16. Find the angle between the given vector and the x–axis and the y–axis: $2\mathbf{i}-2\sqrt{3}\mathbf{j}$.

17. Find the angle between the given vector and the x–axis and the y–axis: $\dfrac{9\mathbf{i}+9\mathbf{j}}{\sqrt{2}}$.

18. Find the angle between the given vector and the x–axis and the y–axis: $2\mathbf{i}+5\mathbf{j}$.

19. If a vector in the plane has length 3 and makes angle $\frac{\pi}{3}$ with the positive x–axis, find the vector.

20. If a vector in the plane has length 5 and makes angle $\frac{\pi}{3}$ with the negative y–axis, find the vector.

21. A rocket launches from a launch pad traveling 6000 mph east, 10000 mph north, and 4000 mph vertically. In 30 minutes, how high will the rocket be off the ground? How far East will it be? How far from the launch pad will it be? How far would you have to drive from the launch pad to look up and see the rocket straight above you?

22. Complete the following parts:

(a) Write the chemical equation $CO + H_2O = H_2 + CO_2$ as an equation in ordered triples. Represent it as a vector in space.

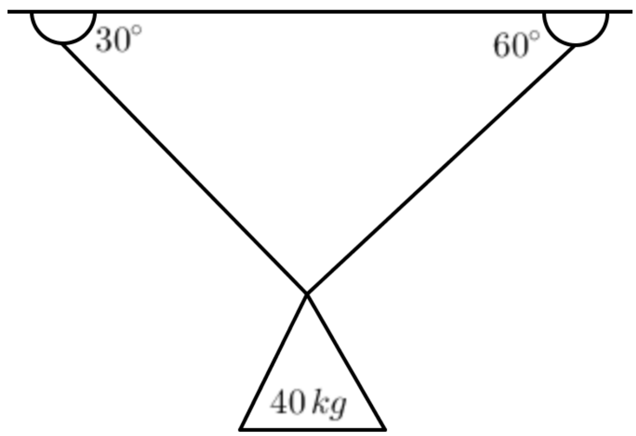
(b) Write the chemical equation $pC_3H_4O_3 + qO_2 = rCO_2 + sH_2O$ as an equation in ordered triples with unknown coefficients.

(c) Find the smallest possible integer solution for p, q, r, and s.

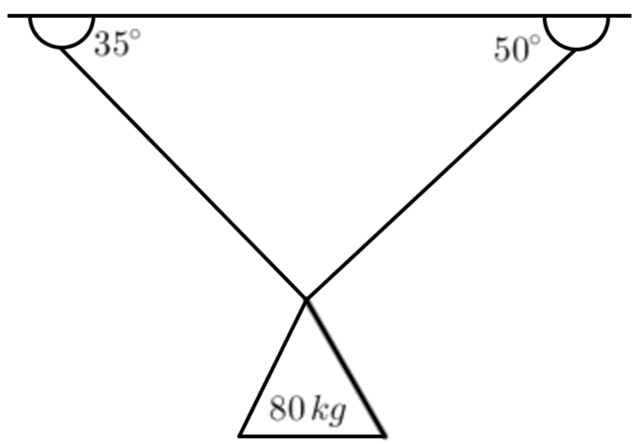
(d) Demonstrate the solution by plotting it in space. What are the other possible solutions? How do they relate geometrically to the vector solution you found?

23. If $\mathbf{u}$ and $\mathbf{v}$ are vectors, describe the set of points inside the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. What if the vectors originate at the point (p_1, p_2, p_3)?

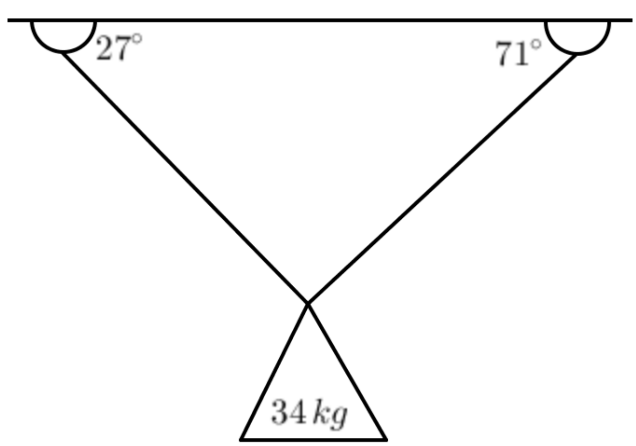
24. Find the tension in each wire in the diagram below.



25. Find the tension in each wire in the diagram below.



26. Find the tension in each wire in the diagram below.



1.3 Dot Product

Dot Product:

Angle between Vectors:

Test for Orthogonality:

Projection:

1.3 | Exercises

1. Determine which of the following are meaningful expressions:

(a) $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$

(b) $(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})$

(c) $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}$

(d) $(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$

(e) $|\mathbf{v}|\,(\mathbf{u}\cdot\mathbf{w})$

(f) $|\mathbf{u}|\cdot(\mathbf{v}\cdot\mathbf{w})$

2. Given $\mathbf{u}=\langle 2,-1,3\rangle$ and $\mathbf{v}=\langle 1,0,-2\rangle$, find

(a) $\mathbf{u}\cdot\mathbf{u}$

(b) $|\mathbf{u}|$

(c) $\mathbf{u}\cdot\mathbf{v}$

(d) $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$

3. Given $\mathbf{u}=\langle 3,-5,1\rangle$ and $\mathbf{v}=\langle 2,-2,1\rangle$, find

(a) $\mathbf{v}\cdot\mathbf{v}$

(b) $|\mathbf{u}|$

(c) $\operatorname{proj}_{\mathbf{u}}\mathbf{v}$

(d) $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$

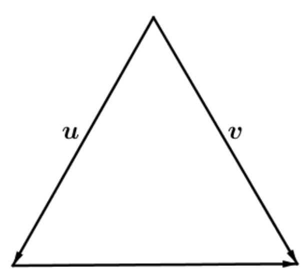
4. If $\mathbf{u}=\langle 1,2,3\rangle$ and $\mathbf{v}=\langle 3,4,5\rangle$, what is the angle between $\mathbf{u},\mathbf{v}$? Sketch these vectors and the angle between them.

5. If $\mathbf{u}=\langle -1,2,0\rangle$ and $\mathbf{v}=\langle 2,0,-3\rangle$, what is the angle between $\mathbf{u},\mathbf{v}$? Sketch these vectors and the angle between them.

6. If $|\mathbf{u}|=2$, $|\mathbf{v}|=3$, and the angle between them is $\pi/6$, what is $\mathbf{u}\cdot\mathbf{v}$?

7. If $|\mathbf{u}|=4$, $|\mathbf{v}|=\sqrt{2}$, and the angle between them is $4\pi/3$, what is $\mathbf{u}\cdot\mathbf{v}$?

8. Given the equilateral triangle below (each side is length 3), place an appropriate vector to label the other side and find $\mathbf{u}\cdot\mathbf{v}$.



9. Recall given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can form a right triangle using the projection $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$. Show that $\mathbf{u}-\operatorname{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$.

10. If $\mathbf{u} = \langle 3, 0, 4 \rangle$, find a vector $\mathbf{v}$ such that $|\text{proj}_{\mathbf{u}}\, \mathbf{v}| = \frac{1}{5}$.

11. A truck drags a wood pallet across the ground. The rope attaching the pallet to the truck makes an angle of $\pi/6$ with the ground and the tension in the rope is 1000 N. How much work does the truck do pulling the pallet 3 km?

12. A person pulls a sled along the ground. The tension in the rope is 10 N and the rope makes an angle of $\pi/4$ with the ground. What is the work done pulling the sled 20 m?

13. Find the acute angles between the curves $y = x^2 - 3x - 1$ and $y = 4x - 11$. [The angle is defined to be the angle between their tangents at the point.]

14. Find the acute angles between the curves $y = x^2 - 8x + 21$ and $y = 5$ at their points of intersection. [The angle is defined to be the angle between their tangents at the point.]

15. Find the acute angles between the curves $y = x^3 + 3$ and $y = x^2 + 4x - 1$t their points of intersection. [The angle is defined to be the angle between their tangents at the point.]

16. Find the acute angle between the curves $y = \sin\theta$ and $y = \cos\theta$ at the smallest positive θ value of intersection. [The angle is defined to be the angle between their tangents at the point.]

17. Find the angle between the diagonal and an adjacent edge in a cube.

18. Find the work done by a force given by $\mathbf{F} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ moving an object at the point $(1, 0, -2)$ in a straight line to the point $(5, 6, 7)$?

19. Find the work done by a force given by $\mathbf{F} = 3\mathbf{i} - 5\mathbf{k}$ moving an object at the point $(2, 2, 0)$ in a straight line to the point $(2, -3, 4)$?

20. Show that if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then $|\mathbf{u}| = |\mathbf{v}|$.

21. Is it possible for $\text{proj}_{\mathbf{b}}\, \mathbf{a} = \text{proj}_{\mathbf{a}}\, \mathbf{b}$? If so, under what conditions is it true?

22. Suppose an object starts at point P and is pushed to point Q with constant force $\mathbf{F}$. If θ is the angle between the displacement vector, $\mathbf{d}$ and the force vector, show that the work is $\mathbf{F} \cdot \mathbf{d}$.

1.4 Cross Product

Cross Product:

Determinants:

Cross Product Formula:

Test for 'Parallelity':

Parallelogram Area:

Volume of Parallelepiped:

Torque:

1.4 | Exercises

1. Calculate the determinant
$$\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix}$$

2. Calculate the determinant
$$\begin{vmatrix} -4 & 9 \\ 6 & 1 \end{vmatrix}$$

3. Calculate the determinant
$$\begin{vmatrix} 3 & 6 & -1 \\ 4 & 0 & 3 \\ 5 & 5 & 1 \end{vmatrix}$$

4. Calculate the determinant
$$\begin{vmatrix} -1 & -1 & -1 \\ 2 & 2 & -5 \\ 4 & 6 & 4 \end{vmatrix}$$

5. Calculate the determinant
$$\begin{vmatrix} 1 & -3 & 4 & 1 & 0 \\ 0 & 4 & -2 & -2 & 6 \\ 7 & 1 & 3 & 1 & 1 \\ -2 & 0 & 4 & 5 & 4 \\ 3 & 4 & 0 & -1 & -4 \end{vmatrix}$$

6. Given $\mathbf{u} = \langle 1, 3, 0\rangle$ and $\mathbf{v} = \langle -2, 5, 0\rangle$, find $\mathbf{u} \times \mathbf{v}$.

7. Given $\mathbf{u} = -5\mathbf{i} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{j} + 4\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.

8. Given $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 3, 4, 5\rangle$, find $\mathbf{u} \times \mathbf{v}$.

9. Given $\mathbf{u} = \mathbf{k} - 6\,\mathbf{i}$ and $\mathbf{v} = \mathbf{k} - 2\mathbf{i} - 2\mathbf{j}$, find $\mathbf{u} \times \mathbf{v}$.

10. Without using the determinant, calculate $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$ and $\mathbf{k} \times (\mathbf{j} \times \mathbf{j})$.

11. Without using the determinant, calculate $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$.

12. If $\mathbf{u} \times \mathbf{v} = 3\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$, find $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.

13. Find two unit vectors perpendicular to both $\langle 1, 0, -2\rangle$ and $\langle 3, 3, 1\rangle$.

14. Find two unit vectors perpendicular to both $\langle 5, 1, 2\rangle$ and $\langle -2, -2, 6\rangle$.

15. Find the area of the parallelogram spanned by the vectors $\langle 2, 3\rangle$ and $\langle -3, 5\rangle$.

16. Find the area of the parallelogram spanned by the vectors $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $5\mathbf{k} - 3\mathbf{i}$.

17. Calculate the area of the parallelogram having vertices $(1, 1)$, $(3, 2)$, $(1, 3)$, and $(-1, 2)$.

18. Calculate the area of the parallelogram having vertices $(1,2,3)$, $(4,-2,1)$, $(-3,1,0)$, and $(0,-3,-2)$.

19. Find the volume of the parallelepiped determined by $\mathbf{u}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{v}=2\mathbf{j}-3\mathbf{k}$, and $\mathbf{w}=\mathbf{i}+\mathbf{k}$.

20. Find the volume of the parallelepiped having vertices $(3,0,-1)$, $(4,2,-1)$, $(-1,1,0)$, $(3,1,5)$, $(0,3,0)$, $(4,3,5)$, $(-1,2,6)$, and $(0,4,6)$.

21. Determine if the vectors $\mathbf{i}+5\mathbf{j}-2\mathbf{k}$, $3\mathbf{i}-\mathbf{j}$, and $5\mathbf{i}+9\mathbf{j}-4\mathbf{k}$ are coplanar.

22. Find the area of the triangle having vertices $(0,1,2)$, $(3,4,5)$, and $(-1,-1,0)$.

23. Find the area of the triangle having vertices $(-1,-1,0)$, $(2,3,4)$, and $(5,6,1)$.

24. Use the cross product of $\langle\cos\theta,\sin\theta\rangle$ and $\langle\cos\phi,\sin\phi\rangle$ to show that

$$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$$

25. If $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=0$, what is the geometric relationship between $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

26. Show that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=0$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

27. Show that

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

28. Show that a triangle with vertices $P(x_1,y_1)$, $Q(x_2,y_2)$, and $R(x_3,y_3)$ is given by absolute value of

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

29. Show that

$$\begin{vmatrix} x^2 & y^2 & z^2 \\ 2x & 2y & 2z \\ 2 & 2 & 2 \end{vmatrix}\neq 0$$

This is a simple example of a Wronskian, which can determine if a collection of functions is linearly independent or not.

30. If the vertices of a parallelogram are (listed in order) $(1,0,2)$, $(1,4,3)$, $(2,1,4)$, and $(2,-3,3)$, find the area of the parallelogram. Find the area of the parallelogram projected to the xy–plane, to the xz–plane?

31. Show that

$$(\mathbf{u}-\mathbf{v})\times(\mathbf{u}+\mathbf{v})=2(\mathbf{u}\times\mathbf{v})$$

32. Show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}$$

33. Prove the *Jacobi Identity*:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}$$

34. Show that

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2\,|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

35. Show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

36. Find a formula for the surface area of the tetrahedron formed by **a**, **b**, and **c**.

37. Let $\mathbf{r}_1, \ldots, \mathbf{r}_n$ be the vectors connecting the origin to masses $m_1, \ldots, m_n$. The center of mass of the collection of these masses is

$$\mathbf{c} = \frac{\sum_{i=1}^n m_i \mathbf{r}_i}{M},$$

where $M = \sum_{i=1}^n m_i$. Show that for any vector $\mathbf{r}$,

$$\sum_{i=1}^n m_i \|\mathbf{r} - \mathbf{r}_i\|^2 = \sum_{i=1}^n m_i \|\mathbf{r}_i - \mathbf{c}\|^2 + M\|\mathbf{r} - \mathbf{c}\|^2$$

1.5 Functions in n–Space

Function:

Domain:

Partial Function:

Parametric Equations:

1.5 | Exercises

Functions on n–Space

1. Given the function $f(x,y) = 2x - y$, find $f(3,4)$.

2. Given the function $f(x,y) = \dfrac{x+y}{xy}$, find $f(-1,3)$.

3. Given the function $f(x,y,z) = 2x + 3y - z$, find $f(5,-1,4)$.

4. Given the function $f(x,y,z) = \dfrac{2x-y+z}{x^2+y^2+z^2}$, find $f(1,1,1)$.

5. Find the domain of the function $f(x,y) = x + y$. Plot this region.

6. Find the domain of the function $f(x,y) = \dfrac{x+y}{x-y}$. Plot this region.

7. Find the domain of the function $f(x,y) = \dfrac{2}{xy}$. Plot this region.

8. Find the domain of the function $f(x,y) = \sqrt{2x-3y}$. Plot this region.

9. Find the domain of the function $f(x,y) = \ln(x^2 - y)$. Plot this region.

10. Simplify the function $f(x,y) = \dfrac{x^2-y^2}{x-y}$.

11. Simplify the function $f(x,y) = \dfrac{x^3+y^3}{x+y}$.

12. Simplify the function $f(x,y) = \dfrac{x^3y+2y}{y^2+xy}$.

13. Simplify the function $f(x,y) = \ln(x^2y^2) - \ln(xy)$.

Parametric Equations

14. Give a parametric function for the line segment connecting the points $(1,-3)$ and $(5,2)$. Find $x(t)$ and $y(t)$. Sketch this function.

15. Give a parametric function for the line segment connecting the points $(5,5)$ and $(-1,0)$. Find $x(t)$ and $y(t)$. Sketch this function.

16. Give a parametric function for the line segment connecting the points $(-1,0,4)$ and $(4,4,-3)$. Find $x(t)$, $y(t)$, and $z(t)$. Sketch this function.

17. Give a parametric function for the line segment connecting the points $(0,1,-2)$ and $(5,-1,3)$. Find $x(t)$, $y(t)$, and $z(t)$. Sketch this function.

18. Give a parametric function for a circle of radius 3 centered at the origin. Find $x(t)$ and $y(t)$. Sketch this function.

19. Give a parametric function for a circle of radius 4 centered at the point $(-1,3)$. Find $x(t)$ and $y(t)$. Sketch this function.

20. Give a parametric function for a circle of radius 1 centered at the point $(4,4)$. Find $x(t)$ and $y(t)$. Sketch this function.

21. Give a parametric function for an ellipse centered at the origin with semimajor axis 3 and semiminor axis 2. Find $x(t)$ and $y(t)$. Sketch this function.

22. Give a parametric function for an ellipse centered at the point $(-4,3)$ with semimajor axis 5 and semiminor axis 1. Find $x(t)$ and $y(t)$. Sketch this function.

23. Give a parametric function for an ellipse centered at the point $(2,-1)$ with semimajor axis 6 and semiminor axis 4. Find $x(t)$ and $y(t)$. Sketch this function.

Vector Valued Functions

24. If $\mathbf{x}(t) = \langle t^2+1, 1-t, 4\rangle$, find $\mathbf{x}(0)$ and $\mathbf{x}(1)$.

25. If $\mathbf{x}(t) = \cos \pi t\, \mathbf{i} + (t^3 - t + 1)\mathbf{j} + \ln t\, \mathbf{k}$, find $\mathbf{x}(1)$.

26. If $\mathbf{x}(t) = \langle \arctan 2t, t\cos t, \sqrt{t}\rangle$, find $\mathbf{x}'(t)$.

27. If $\mathbf{x}(t) = (1-t^4)\,\mathbf{i} + te^{2t}\,\mathbf{j} + \dfrac{1}{\sqrt{t^3}}\,\mathbf{k}$, find $\mathbf{x}(t)$.

28. If $\mathbf{a}(t) = \langle 6t, 0\rangle$, $\mathbf{v}(0) = \langle 0, -1\rangle$, and $\mathbf{x}(0) = \langle 4, 1\rangle$, find $\mathbf{x}(t)$.

29. If $\mathbf{a}(t) = \langle -\pi^2 \sin \pi t, 6t, \dfrac{-1}{t^2}\rangle$, $\mathbf{v}(1) = \langle -\pi.2, 1\rangle$, and $\mathbf{x}(1) = \langle 0,1,0\rangle$, find $\mathbf{x}(t)$.

30. If $\mathbf{a}(t) = 2\mathbf{i} + (6t-4)\mathbf{j}$, $\mathbf{v}(0) = \mathbf{0}$, and $\mathbf{x}(0) = \mathbf{i} + 3\mathbf{k}$, find $\mathbf{x}(t)$.

31. If $\mathbf{a}(t) = \langle -\sin t, 0, -\cos t\rangle$, $\mathbf{v}(0) = \langle 1,1,0\rangle$, and $\mathbf{x}(0) = \langle 0,0,1\rangle$, find $\mathbf{x}(t)$.

32. If $\mathbf{a}(t) = 4e^{2t-2}\,\mathbf{i} - \dfrac{1}{t^2}\,\mathbf{j} - 24t\,\mathbf{k}$, $\mathbf{v}(1) = \mathbf{i} + \mathbf{j} - 13\,\mathbf{k}$, and $\mathbf{x}(1) = -4\,\mathbf{k}$, find $\mathbf{x}(t)$.

1.6 Lines, Planes, & Surfaces

Line:

Plane:

Sphere:

Cylinder:

Level Curve/Surface:

1.6 | Exercises

Spheres

1. Find the equation of a sphere with radius 4 and center $(1,0,-5)$.

2. Find the equation of a sphere with radius 3/2 and center $(3,3,-2)$.

3. Find the center and radius of the sphere $x^2+y^2+z^2-2x+6y=-6$.

4. Find the center and radius of the sphere $x^2+y^2+z^2+10-4y+2z=-21$.

5. Find the center and radius of the sphere $x^2+y^2+z^2-x+3y-2z=-5/4$.

Cylinders

6. Plot the cylinder $(x-3)^2+y^2=4$.

7. Plot the cylinder $(z+5)^2+(y-1)^2=9$.

8. Plot the cylinder $z=x^2$.

9. Plot the cylinder $x=(z-3)^2$.

10. Plot the cylinder $x=(y+3)^2+2$.

Lines

11. Find the equation of the line through the points $(-1,0,3)$ and $(5,2,2)$.

12. Find the equation of the line through the points $(0,2,-7)$ and $(5,-6,0)$.

13. Find the equation of the line through the points $(\pi,6,e^2)$ and $(1,\sqrt{2},-\frac{1}{3})$.

14. Find the equation of the line through the point $(1,2,3)$ parallel to the vector $\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

15. Find the equation of the line through the point $(1,-1,0)$ and parallel to the vector $\langle -2,5,3\rangle$.

16. Find the equation of the line through the point $(3,-1,2)$ and perpendicular to the plane $2x-3y+5z=6$.

17. Find the equation of the line through the point $(5,0,5)$ and perpendicular to the plane $x-z=7$.

18. Find the equation of the line through $(1,1,1)$ and perpendicular to the vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}-\mathbf{j}-\mathbf{k}$.

19. Find the equation of the line through $(-7,1,1)$ and perpendicular to the vectors $2\mathbf{x}+3\mathbf{y}$ and $5\mathbf{z}-\mathbf{y}$.

20. Determine if the lines l_1: $x = 4t+1$, $y = 4-t$, $z = 3t+4$ and l_2: $x = -2-8t$, $y = 2t-8$, $z = -6t-8$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

21. Determine if the lines l_1: $x = t-2$, $y = 4-t$, $z = 2t+1$ and l_2: $x = 2t-4$, $y = 8-2t$, $z = 2t+2$ are the same skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

22. Determine if the lines $l_1(t) = (7,1,20) + t(2,1,5)$ and $l_2(s) = (4,-17,-13) + s(-1,5,6)$ are the same skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

23. Determine if the lines $l_1 : \dfrac{x}{4} = \dfrac{y-9}{-3} = \dfrac{z+1}{4}$ and $l_2 : \dfrac{x+12}{8} = \dfrac{y-18}{-6} = \dfrac{z+13}{8}$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

24. Determine if the lines $l_1(t) = (2t+1, 3t-4, 5-t)$ and $l_2(s) = (2s-1, 3s+1, 5-s)$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

25. Determine if the lines l_1: $x = t$, $y = 1-t$, $z = 1-t$ and l_2: $x = 6t$, $y = -t-1$, $z = 3t+1$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

26. Determine if the lines $l_1(t) = (1-t, 5t-3, t)$ and $l_2(s) = (9-2s, 3-10s, 2s)$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

27. Determine if the lines $l_1(t)$: $x = 2t+5$, $y = 2t+1$, $z = 3-t$ and $l_2(s)$: $x = 11s-1$, $y = 4-s$, $z = 12s$ are perpendicular.

28. Determine if the lines $l_1(t) = (-1,3,6) + t(-2,0,1)$ and $l_2(t) = (11,-1,15) + t(-3,2,-6)$ are perpendicular.

29. Determine if the lines $l_1(t) = (t+6, t+7, t+9)$ and $l_2(t) = (13-4t, 3t-7, t+1)$ are perpendicular.

30. Determine if the lines $l_1(t)$: $x = 3t+1$, $y = 2t+5$, $z = t-1$ and l_2: $x = 7t$, $y = 7-14t$, $z = 6t+5$ are perpendicular.

Planes

31. Find a normal vector to the plane $2x - 3y + z = 6$ and three points on the plane.

32. Find a normal vector to the plane $x + y - 4z = 3$ and three points on the plane.

33. Find a normal vector to the plane $2x - 3z = 5$ and three points on the plane.

34. Find a normal vector to the plane $3y + 2z - x = 7$ and three points on the plane.

35. Find a normal vector to the plane $2z - 5y = -1$ and three points on the plane.

36. Find the equation of the plane containing the point $(1,2,3)$ with normal vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

37. Find the equation of the plane containing the point $(0,-3,5)$ with normal vector $\langle -2,1,-3\rangle$.

38. Find the equation of the plane containing the point $(7,-2,4)$ with normal vector $\mathbf{i}+\mathbf{j}-\mathbf{k}$.

39. Find the equation of the plane containing the point $(1/2,3,0)$ with normal vector $\langle 2,-3,5\rangle$.

40. Find the equation of the plane containing the point $(0,5,0)$ with normal vector $-\frac{1}{5}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{1}{4}\mathbf{k}$.

41. Find the equation of the plane containing the point $(2,2,-1)$ with normal vector $\langle 30,60,-60\rangle$.

42. Find the equation of the plane containing the points $(1,1,1)$, $(0,1,3)$, and $(-1,2,-1)$.

43. Find the equation of the plane containing the points $(2,1,3)$, $(2,2,-5)$, and $(0,0,1)$.

44. Find the equation of the plane containing the points $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$.

45. Find the equation of the plane containing the points $(0,0,2)$, $(5,2,-4)$, and $(-1,-2,4)$.

46. Find the equation of the plane with x–intercept a, y–intercept b, and z–intercept c.

47. Find the equation of the plane containing the line $x=3t$, $y=2-2t$, $z=t+1$ and parallel to the plane $x+2y+z=2$.

48. Find the equation of the plane containing the line $l(t)=(4,8,20)t+(1,1,1)$ and parallel to the plane $5y-2z=10$.

49. Find the equation of the plane containing the line $\dfrac{x}{-12}=\dfrac{y-1}{3}=\dfrac{z+1}{-10}$ and parallel to the plane $-2x+2y+3z=7$.

50. Find the equation of the plane containing the line $x=6-28t$, $y=4t+4$, $z=5-24t$ and parallel to the plane $-x+5y+2z=16$.

51. Find the equation of the plane containing the line $l(t)=(3-t,2t-1,3t+2)$ and parallel to the plane $x-y+z=-1$.

52. Find the equation of the plane perpendicular to the planes $x-2y+3z=13$ and $6+4x-2z=0$ and containing the point $(1,0,1)$.

53. Find the equation of the plane perpendicular to the planes $2y-z=9$ and $x+2y=14$ and containing the point $(7,3,9)$.

54. Find the equation of the plane perpendicular to the planes $2x-3y+z=25$ and $x+y-4z=3$ and containing the point $(0,6,0)$.

55. Find the equation of the plane perpendicular to the planes $-3x+5y+4z=2$ and $2x+2y+2z=14$ and containing the point $(1,2,3)$.

56. Find the equation of the plane perpendicular to the planes $2x + y - z = 7$ and $-x + 3y + 2z = 17$ and containing the point $(1, 0, -1)$.

57. Find the equation for the line of intersection of the planes $x + 2y - z = 7$ and $x + y - z = 0$.

58. Find the equation for the line of intersection of the planes $x + 2y + 2z = 16$ and $3x + z = 6$.

59. Find the equation for the line of intersection of the planes $x + y - 3z = 1$ and $5x - y + 2z + 1 = 0$.

60. Find the equation for the line of intersection of the planes $3x + 3y + 2z = 0$ and $2x + y + 2z = 12$.

61. Find the equation for the line of intersection of the planes $x + 4y - z = 4$ and $3x + 2y - z = 2$.

62. Determine if the planes $x + y + z = 5$ and $2y + 2y = 10 - 2x$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

63. Determine if the planes $x + y - 2z + 1 = 0$ and $2x + z = 2y + 8$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

64. Determine if the planes $2x + y + 3z + 4 = 0$ and $4x + 2y + 6z = 10$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

65. Determine if the planes $x - 2y + z + 4 = 0$ and $3z - 3x = 0$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

66. Determine if the planes $2x + y - z = 13$ and $3x + y + 4z = 0$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

67. Determine if the planes $2x - 4y + 2z + 8 = 0$ and $-5x + 10y - 5z = 20$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

68. Determine if the planes $2x + y + 3z = 9$ and $-14x - 8y + 12z + 2 = 0$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

69. Determine if the planes $4x + 4y + z = 6$ and $5x + 3y + z = 13$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

70. Determine if the planes $3x - y + 2z = 2$ and $3y - 9x = 6z - 15$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

71. Determine if the line $l(t) = (2, -1, 2)t + (1, 1, 1)$ and the plane $2x - y + 2z = 9$ are perpendicular, parallel, or neither.

72. Determine if the line $l(t) = (t + 2, 5 - 3t, 5t)$ and the plane $3x + y + 6z + 7 = 0$ are perpendicular, parallel, or neither.

73. Determine if the line $x = 3t + 4$, $y = 7 - t$, $z = t$ and the plane $x + 5y + 2z = 10$ are perpendicular, parallel, or neither.

74. Find the equation of the line perpendicular to the plane $5x-7y+4z=12$ containing the point $(-1,4,8)$.

75. Find the equation of the plane perpendicular to the line $x=4-t$, $y=2t+1$, $z=5t+6$ and containing the line $l(t)=(3-8t,11t+2,4-6t)$.

(Quadratic) Surfaces

76. Use appropriate level curves to sketch the surface given by the equation $x+y+z=7$.

77. Use appropriate level curves to sketch the surface given by the equation $(x-2)^2+y^2+(z+1)^2=9$ and describe the surface.

78. Use appropriate level curves to sketch the surface given by the equation $4\frac{x}{2}+(y-1)^2+z^2=4$ and describe the surface.

79. Use appropriate level curves to sketch the surface given by the equation $z=4x^2+9y^2$ and describe the surface.

80. Use appropriate level curves to sketch the surface given by the equation $y^2=x$ and describe the surface.

81. Use appropriate level curves to sketch the surface given by the equation $z^2-x^2-y^2=1$ and describe the surface.

82. Use appropriate level curves to sketch the surface given by the equation $z^2=4x^2+9y^2$ and describe the surface.

83. Use appropriate level curves to sketch the surface given by the equation $z=4y^2-x^2$ and describe the surface.

84. Use appropriate level curves to sketch the surface given by the equation $x^2+y^2-z^2=1$ and describe the surface.

85. Use appropriate level curves to sketch the surface given by the equation $2x+x^2+36y^2+4z^2=144y-141$ and describe the surface.

86. Use appropriate level curves to sketch the surface given by the equation $-y=(x-1)^2+z^2$ and describe the surface.

87. Use appropriate level curves to sketch the surface given by the equation $2x-3y-z=6$.

88. Use appropriate level curves to sketch the surface given by the equation $y=z^2-4z+7$ and describe the surface.

89. Use appropriate level curves to sketch the surface given by the equation $\frac{x^2}{4} + \frac{z^2}{9} - 1 = y^2$ and describe the surface.

90. Use appropriate level curves to sketch the surface given by the equation $x^2 + (y+5)^2 + (z+1)^2 = 4$ and describe the surface.

91. Use appropriate level curves to sketch the surface given by the equation $x^2 = z^2 + y^2 - 2y + 1$ and describe the surface.

92. Use appropriate level curves to sketch the surface given by the equation $x + 3 = z^2 - 9y^2$ and describe the surface.

93. Use appropriate level curves to sketch the surface given by the equation $y^2 - 4x^2 - 4(z+1)^2 = 4$ and describe the surface.

94. Use appropriate level curves to sketch the surface given by the equation $y - 2z = 4$.

95. Use appropriate level curves to sketch the surface given by the equation $x^2 - y^2 - \frac{z^2}{9} = 1$ and describe the surface.

96. Use appropriate level curves to sketch the surface given by the equation $x^2 + 36z^2 + 9y^2 = 288z - 576$ and describe the surface.

97. Use appropriate level curves to sketch the surface given by the equation $x + 1 = y^2 + (z-1)^2$ and describe the surface.

98. Use appropriate level curves to sketch the surface given by $x^2 + y^2 + z^2 + 2 = 2(x + y + z)$ and describe the surface.

99. Use appropriate level curves to sketch the surface given by the equation $(y+1)^2 + (z-2)^2 - \frac{x^2}{2} = 1$ and describe the surface.

100. Use appropriate level curves to sketch the surface given by the equation $y^2 - x^2 = z^2 - 6z - 2y + 8$ and describe the surface.

101. Use appropriate level curves to sketch the surface given by the equation $y - 2 = x^2 - z^2 + 2x + 3$ and describe the surface.

102. Use appropriate level curves to sketch the surface given by the equation $x = 4 - z^2$ and describe the surface.

103. Use appropriate level curves to sketch the surface given by the equation $9x^2 + 9y^2 + z^2 = 54x + 36y - 108$ and describe the surface.

104. Use appropriate level curves to sketch the surface given by the equation $x^2 - y^2 - z^2 = 4x - 4$ and describe the surface.

105. Use appropriate level curves to sketch the surface given by the equation $x + 4y = 3y - z + 1$.

106. Use appropriate level curves to sketch the surface given by the equation $z = x^2 + 3$ and describe the surface.

107. Use appropriate level curves to sketch the surface given by $4x^2 + 4y^2 + 4z^2 + 16z = 8z - 19$ and describe the surface.

108. Use appropriate level curves to sketch the surface given by the equation $y^2 - 2 = x^2 + z^2 + 2(x + y + z)$ and describe the surface.

109. Use appropriate level curves to sketch the surface given by the equation $(z+2)^2 - y^2 - (x-3)^2 = 9$ and describe the surface.

110. Use appropriate level curves to sketch the surface given by the equation $x = \dfrac{y^2}{4} + z^2$.

111. Use appropriate level curves to sketch the surface given by the equation $x = z^2 - \dfrac{y^2}{9}$ and describe the surface.

112. Use appropriate level curves to sketch the surface given by $x^2 + y^2 +^2 + 8x + 16 = 16$ and describe the surface.

113. Use appropriate level curves to sketch the surface given by the equation $z^2 - x^2 - y^2 = 2y + 2z - 5x + 25$ and describe the surface.

114. Use appropriate level curves to sketch the surface given by the equation $x + 2z = 4$.

115. Use appropriate level curves to sketch the surface given by the equation $x^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z+3}{2}\right)^2 = 1$ and describe the surface.

116. Use appropriate level curves to sketch the surface given by the equation $z^2 + (y-3)^2 = x^2 + 9$ and describe the surface.

117. Use appropriate level curves to sketch the surface given by the equation $\dfrac{y^2}{4} - 3x^2 - z^2 = 1$ and describe the surface.

118. Use appropriate level curves to sketch the surface given by the equation $y = 1 - 6z - z^2$ and describe the surface.

119. Use appropriate level curves to sketch the surface given by the equation $z + 3 = (x-1)^2 - (y+2)^2$ and describe the surface.

120. Use appropriate level curves to sketch the surface given by the equation $z + 1 = (y-3)^2 + (x+1)^2$ and describe the surface.

1.7 Computing Distances

Distance Point–Point:

Distance Point–Line:

Distance Point–Plane:

Distance Line–Line:

Distance Plane–Plane:

Distance Surface–Surface:

1.7 | Exercises

Distance between Points

1. Find the distance between the points $(-1,3,7)$ and $(9,-2,3)$.

2. Find the distance between the points $(0,4,-10)$ and $(3,1,3)$.

3. Find the distance between the points $(12,1,5)$ and $(7,4,6)$.

Distance between Point & Line

4. Find the distance from the point $(-2,4,-2)$ to the line $l(t)=(-1,2,0)+t(4,5,3)$.

5. Find the distance from the point $(4,9,4)$ to the line $l: x=2-5t,\ y=t+5, z=6t+3$.

6. Find the distance from the point $(-1,2,-2)$ to the line $l: \dfrac{x+2}{5}=\dfrac{y+2}{3}=\dfrac{z-2}{4}$.

7. Find the distance from the point $(2,9,4)$ to the line $l: x=2-5t,\ y=t+5,\ z=3-4t$.

8. Find the distance from the point $(2,7,-3)$ to the line $l(t)=(t+4,1-3t-6t)$.

9. Find the distance from the point $(-4,6,-1)$ to the line $l: \dfrac{x}{4}=\dfrac{y-4}{5}=\dfrac{z}{5}$.

10. Find the distance from the point $(-3,-1,5)$ to the line $l(t)=(-2,1,7)+t(4,4,-6)$.

11. Find the distance from the point $(1,9,4)$ to the line $l: y=4, x-2=3-z$.

12. Find the distance from the point $(5,7,3)$ to the line $l(t)=(6t,4-t,-4t-3)$.

Distance between Points & Plane

13. Find the distance from the point $(1,1,-3)$ to the plane $x+2y+2z+4=0$.

14. Find the distance from the point $(2,2,0)$ to the plane $2x+5y+4z=1$.

15. Find the distance from the point $(-4,0,2)$ to the plane $2x-4y+4z=3$.

16. Find the distance from the point $(1,3,1)$ to the plane $2y-x+z=0$.

17. Find the distance from the point $(0,0,7)$ to the plane $3x-4z=1$.

18. Find the distance from the point $(3,0,3)$ to the plane $4x-2y+5z=3$.

19. Find the distance from the point $(5,5,3)$ to the plane $2x+y-2z+5=0$.

20. Find the distance from the point $(1,1,-1)$ to the plane $x-2y+z+4=0$.

21. Find the distance from the point $(-2,1,2)$ to the plane $4x+3z+2=0$.

Distance between Skew Lines

22. Find the distance between the lines $l_1(t)=(3,3,1)+t(0,-1,5)$ and $l_2(t)=(-2,1,6)+t(2,2,1)$.

23. Find the distance between the lines $l_1\colon\ x+5=\dfrac{y}{2}=\dfrac{z+4}{-2}$ and $l_2\colon\ z=5, \dfrac{x+2}{4}=\dfrac{y-6}{3}$.

24. Find the distance between the lines $l_1\colon\ x=-2t,\ y=-4t-1, z=3$ and $x=4t-2,\ y=1-3t, z=1$.

25. Find the distance between the lines $l_1(t)=(5t+5,1,-5t-2)$ and $l_2(t)=(2t+6,4-2t,-3)$.

26. Find the distance between the lines $l_1\colon\ \dfrac{x-2}{4}=\dfrac{y-1}{2}=\dfrac{z+2}{5}$ and $l_2\colon\ y=1, \dfrac{x+3}{-4}=\dfrac{z-2}{-4}$.

27. Find the distance between the lines $l_1\colon\ x=2t,\ y=-t,\ z=t+5$ and $l_2\colon\ x=7-t,\ y=3t+9,$ $z=4-3t$.

28. Find the distance between the lines $l_1(t)=(1,-3t-1,4)$ and $l_2(t)=(-4,3-t,3-t)$.

29. Find the distance between the lines $l_1\colon\ x=4t-2,\ y=t+3, z=5t-1$ and $l_2\colon\ x=8-3t,\ y=4t,$ $z=t-3$.

30. Find the distance between the lines $l_1(t)=(1,1,2)+t(4,0,-4)$ and $l_2(t)=(3,0,4)+t(1,0,2)$.

Distance between Parallel Planes

31. Find the distance between the planes $-7x-6y+6z=4$ and $7x+6y=6z+7$.

32. Find the distance between the planes $9x+4y+z=12$ and $9x+4y+z=-2$.

33. Find the distance between the planes $8x+4y+z=-11$ and $8x+4y+z=7$.

34. Find the distance between the planes $11x+5y-z=9$ and $11x+5y-z=-12$.

35. Find the distance between the planes $2x+y+2z+25=0$ and $2x+y+2z+16=0$.

36. Find the distance between the planes $x+y+z+11=0$ and $x+y+z+23=0$.

37. Find the distance between the planes $-6x+2y-3z=12$ and $-6x+2y-3z=5$.

38. Find the distance between the planes $5x+4y+2z+10=0$ and $5x+4y+2z+25=0$.

39. Find the distance between the planes $x+2y+2z+13=0$ and $x+2y+2z+25=0$.

Distance Formulas

40. Show that the distance from a point p and a line l with direction vector $\mathbf{u}$ containing a point q is given by

$$d = \frac{\|\vec{pq} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

41. Show that the distance from a point (x_0, y_0, z_0) to a plane $Ax + By + Cz + D = 0$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

42. If $l_1(t) = \mathbf{b}_1 + t\mathbf{a}$ and $l_2(t) = \mathbf{b}_2 + t\mathbf{a}$ are parallel lines in $\mathbb{R}^3$, show the distance between them is given by

$$d = \frac{\|\mathbf{a} \times (\mathbf{b}_2 - \mathbf{b}_1)\|}{\|\mathbf{a}\|}$$

43. If $l_1(t) = \mathbf{b}_1 + t\mathbf{a}_1$ and $l_2(t) = \mathbf{b}_2 + t\mathbf{a}_1$ are skew lines in $\mathbb{R}^3$, show that the distance between them is given by

$$d = \frac{|(\mathbf{a}_1 \times \mathbf{a}_2) \cdot (\mathbf{b}_2 - \mathbf{b}_1)|}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}$$

44. Show that the distance between parallel planes with normal vector $\mathbf{n}$ is given by

$$d = \frac{|\mathbf{n} \cdot (\mathbf{x}_2 - \mathbf{x}_1)|}{\|\mathbf{n}\|}$$

where $\mathbf{x}_i$ is the position vector on the ith plane.

45. Show that the distance between parallel planes $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$ is

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

1.8 Cylindrical & Spherical Coordinates

Polar Coordinates:

Cylindrical Coordinates:

Spherical Coordinates:

1.8 | Exercises

1. Convert the following Cartesian coordinates to polar coordinates. Plot these points in both Cartesian and polar coordinates.

(a) $(-\sqrt{2}, -\sqrt{2})$

(b) $(-2, 2\sqrt{3})$

(c) $(3, 3\sqrt{3})$

2. Convert the following polar coordinates to Cartesian coordinates. Plot these points in both Cartesian and polar coordinates.

(a) $(4\sqrt{2}, 7\pi/4)$

(b) $(4, 4\pi/3)$

(c) $(2\sqrt{2}, 3\pi/4)$

3. Convert the following Cartesian coordinates to cylindrical coordinates. Plot these points in both Cartesian and cylindrical coordinates.

(a) $(-1, \sqrt{3}, 2)$

(b) $(\sqrt{3}, -3, 1)$

(c) $(-5, -5, -4)$

4. Convert the following cylindrical coordinates to Cartesian coordinates. Plot these points in both cylindrical coordinates and Cartesian coordinates.

(a) $(4, 5\pi/6, -2)$

(b) $(10, -\pi/4, -6)$

(c) $(1, 4\pi/3, 3)$

5. Convert the following Cartesian coordinates to spherical coordinates. Plot these points in both spherical and Cartesian space.

(a) $(1, -1, \sqrt{6})$

(b) $(0, \sqrt{3}, 1)$

(c) $(\sqrt{2}, -\sqrt{2}, 2)$

6. Convert the following spherical coordinates to Cartesian coordinates. Plot these points both in spherical and Cartesian space.

(a) $(4, \pi/3, \pi/6)$

(b) $(16, 4\pi/3, 3\pi/4)$

(c) $(20, -\pi/3, \pi/4)$

7. Convert the following spherical coordinates to cylindrical coordinates. Plot these points both in spherical and cylindrical coordinates.

(a) $(2, \pi/3, \pi/6)$

(b) $(1, 4\pi/3, \pi/2)$

(c) $(4, \pi/2, 5\pi/6)$

8. Convert the following cylindrical coordinates to spherical coordinates. Plot these points in both cylindrical coordinates and spherical coordinates.

(a) $(4, \pi/2, 0)$

(b) $(5\sqrt{3}, 5\pi/6, 5)$

(c) $(1, -\pi/4, -\sqrt{3})$

9. Describe the curve given by $r = 3$ in polar coordinates. Sketch this.

10. Describe the curve given by $\theta = 3\pi/4$ in polar coordinates. Sketch this.

11. Describe the curve given by $|\,\theta\,| = \pi/4$ in polar coordinates. Sketch this.

12. Describe the surface given by $r = 2$ in cylindrical coordinates. Sketch this.

13. Describe the surface given by $\theta = \pi/3$ in cylindrical coordinates. Sketch this.

14. Describe the surface given by $\rho = 5$ in spherical coordinates. Sketch this.

15. Describe the surface given by $\phi = \pi/4$ in spherical coordinates. Sketch this.

16. Describe the surface given by $\theta = \pi$ in spherical coordinates. Sketch this.

17. Describe the curve given by $r^2 = 6r\cos\theta$ in polar coordinates. Sketch this. What if this were in cylindrical coordinates?

18. Describe the surface given by $z = 2r$ in cylindrical coordinates. Sketch this.

19. Describe the surface given by $\rho\cos\phi = 2\rho\sin\phi$ in spherical coordinates. Sketch this.

20. Describe the surface given by $\rho = 2a\cos\phi$ in spherical coordinates. Sketch this.

Chapter 2

Partial Derivatives & their Applications

2.1 Limits

Limit:

Squeeze Theorem:

‘Scholastic Approach’:

2.1 | Exercises

For Exercises 1–18, find the limit or explain why the limit does not exist.

1. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} x^2+4xy-yz^4+y^2-3z^2+4$

2. $\displaystyle\lim_{(x,y,z)\to(1,0,1)} x^2+4xy-z^2+3\sin(xyz)+5$

3. $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{e^{x-z}}{x+y+1}$

4. $\displaystyle\lim_{(x,y,z)\to(3,2,6)} \ln\left(2x-3y+e^{z-x}\right)$

5. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{2x+2y}{x+y}$

6. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{x^2+2xy+y^2}{x+y}$

7. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{y^2+2x^2}$

8. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

9. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x-1}{(x-1)^2+y^2}$

10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

11. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x^3-y^3}{x-y}$

12. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{x^2-y^2}{x+y}$

13. $\displaystyle\lim_{(x,y)\to(2,0)} \frac{x^2-y^2-4x+4}{x^2+y^2-4x+4}$

14. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4}$

15. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y}{\sqrt{x}-\sqrt{y}}$

16. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

17. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{3x^2+4y^2+5z^2}{x^2+y^2+z^2}$

18. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy-xz-yz}{x^2+y^2+z^2}$

19. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^4}{(x^2+y^4)^3}$ exists along every straight line through the origin but that the limit does not exist. [Hint: For the second part, you may want to try a polynomial path.]

20. Evaluate the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

(b) $\lim\limits_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \dfrac{\sin(x+y)}{x+y}$

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{xy}$

Evaluate the following limits. [Note: It in some cases it may be easier to change coordinates]

21. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$

22. $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

23. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+xy+y^2}{x^2+y^2}$

24. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$

25. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}$

26. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2}$

27. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xz}{x^2+y^2+z^2}$

28. Use the definition of the limit to show that $\lim\limits_{(x,y,z)\to(2,-1,3)} (2x+5y-z) = -4$.

29. Use the definition of the limit to show that $\lim\limits_{(x,y,z)\to(1,1,2)} (x-y+3z) = 6$.

30. Use the definition of the limit to show that $\lim\limits_{(x,y,z)\to(0,-1,3)} (7x-4y-z) = 1$.

31. Use the definition of the limit to show that $\lim\limits_{(x,y,z)\to(3,-2,5)} (x+y+z) = 0$.

32. Show that $\lim\limits_{(x,y,)\to(0,0} \dfrac{x^3+y^3}{x^2+y^2} = 0$ by completing the following steps:

(a) Show $|x| \le \|(x,y)-(0,0)\|$ and $|y| \le \|(x,y)-(0,0)\|$.

(b) Show that $|x^3+y^3| \le 2(x^2+y^2)^{1/3}$. [Hint: The Triangle Inequality.]

(c) Show that $\left|\dfrac{x^3+y^3}{x^2+y^2}\right| < 2\delta$ if $\|(x,y)-(0,0)\| < \delta$.

(d) Use the proceeding parts to show that the limit is 0.

2.2 Partial Derivatives

Ordinary Derivative:

Partial Derivative:

Clairaut's Theorem:

Higher Derivatives:

Chain Rule:

2.2 | Exercises

Partial Derivatives

In Exercises 1–10, given the function f, find as many of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ as possible.

1. $f(x,y) = x^2y^3 + x - y + 1$

2. $f(x,y) = x\sqrt{y} - y\sqrt{x}$

3. $f(x,y) = \frac{x}{y}$

4. $f(x,y) = x \ln y$

5. $f(x,y) = x \ln(xy)$

6. $f(x,y) = \arctan(xy)$

7. $f(x,y) = ye^{xy}$

8. $f(x,y) = x^y$

9. $f(x,y) = \frac{y}{1 - xy}$

10. $f(x,y) = e^{2y} \sin(\pi x)$

In Exercises 11–20, given the function f, find as many of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial x \partial z}$ as possible.

11. $f(x,y,z) = x^4yz^2 - x^3z + yx^2 + 4$

12. $f(x,y,z) = \frac{x\sqrt{y}}{z}$

13. $f(x,y,z) = \frac{x^z}{y}$

14. $f(x,y,z) = \frac{\cos x \ln y}{\sqrt{z}}$

15. $f(x,y,z) = e^{xyz}$

16. $f(x,y,z) = z^{y/x}$

17. $f(x,y,z) = \frac{x + y}{y + z}$

18. $f(x,y,z) = \frac{x + e^{2y} - z}{xy}$

19. $f(x,y,z) = y\arctan(xyz)$

20. $f(x,y,z) = \ln y^4 e^{\pi x}\sin(3z)$

Chain Rule

21. If $T(x,y) = x^2e^y - xy^3$ and $x = \cos t$ and $y = \sin t$, find $\dfrac{dT}{dt}$.

22. If $u(x,y,z) = xe^{yz}$, where $x(t) = e^t$, $y(t) = t$, and $z(t) = \sin t$, find $\dfrac{du}{dt}$.

23. If $z = f(x,y)$, where $x = g(t)$ and $y = h(t)$, find a formula for $\dfrac{dz}{dt}$ using $g(t)$ and $h(t)$.

24. If $z = x^2y^3 + y\cos x$, where $x = \ln t^2$ and $y = \sin(4t)$, find $\dfrac{dz}{dt}$.

25. If $z = e^{2t}\sin(3\theta)$, where $r(s,t) = st - t^2$ and $\theta(s,t) = \sqrt{s^2+t^2}$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

26. If $z(x,y) = x^2y - y^2$ and $x(t) = t^2$ and $y(t) = 2t$, then find $\dfrac{dz}{dt}$.

27. If $z = ye^{x^2}$, where $x(u,v) = \sqrt{uv}$ and $y(u,v) = \dfrac{1}{v}$, find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

28. If a curve is given by $F(x,y) = 0$, show that the slope of the tangent line is given by

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

29. If a curve is given by $F(x,y) = 0$ and $g(x,y)$ is defined along the curve, find $\dfrac{dg}{dx}$.

30. If a ship is traveling along a path given by $\mathbf{x}(t) = (t, \cos t, \sin t)$, describe the path and find the temperature of the ship, $H(x,y,z)$, as a function of time.

2.3 Arclength, Tangent Planes, & Differential Geometry

Arclength:

Unit Tangent:

Curvature:

(Principal) Unit Normal Vector:

Binormal Vector:

Tangent Plane:

Total Differential:

2.3 | Exercises

Derivatives & Tangents

1. If $x(t) = 2t+1$ and $y(t) = 3-t$, what is $\dfrac{dy}{dx}$?

2. If $x(t) = t^2-t+1$ and $y(t) = 4-t^2$, what is $\dfrac{dy}{dx}$?

3. If $x(t) = \cos t$ and $y(t) = t\sin t$, what is $\dfrac{dy}{dx}$?

4. If $x(t) = \ln t$ and $y(t) = \tan t$, what is $\dfrac{dy}{dx}$?

5. If $x(t) = \sqrt{t^3}$ and $y(t) = \dfrac{e^t}{t}$, what is $\dfrac{dy}{dx}$?

6. Find $\dfrac{d^2y}{dx^2}$ if $x(t) = 4t^2+3$ and $y(t) = 1-t$.

7. Find $\dfrac{d^2y}{dx^2}$ if $x(t) = \sec t$ and $y(t) = \tan t$.

8. Find $\dfrac{d^2y}{dx^2}$ if $x(t) = \ln t$ and $y(t) = t^{3/2}$.

9. Find $\dfrac{d^2y}{dt^2}$ if $x(t) = e^t$ and $y(t) = t\cos^2 t$.

10. Find $\dfrac{d^2y}{dt^2}$ if $x(t) = t - \dfrac{1}{t}$ and $y(t) = \arctan t$.

11. Find the tangent line to the curve $x(t) = t^2+3t+1$, $y(t) = 4t+7$ when $t = 2$.

12. Find the tangent line to the curve $x(t) = \cos t$, $y(t) = \sin t$ when $t = \dfrac{\pi}{4}$.

13. Find the tangent line to the curve $C(t) = (\ln(t+1), \arctan(t-1))$ when $t = 0$.

14. Find the tangent line to the curve $x(t) = \dfrac{3t}{1+t^3}$, $y(t) = \dfrac{3t^2}{1+t^3}$ when $t = 1$.

15. Find the tangent line to the curve $x(t) = \sec t$, $y(t) = \tan t$ when $t = \dfrac{\pi}{3}$.

Arclength

16. Find the length of the curve $l(t) = (1,2)$, where $0 \le t \le \sqrt{\pi}$.

17. Find the length of the curve $x(t) = 4t-1$, $y(t) = 5-3t$, where $1 \le t \le 3$.

18. Find the length of the curve $x(t) = \dfrac{2t^{3/2}}{3}$, $y(t) = 1-t$, where $0 \le t \le 1$.

19. Find the length of the curve $x(t) = \sin t$, $y(t) = \cos t$, where $0 \le t \le \pi$.

20. Find the length of the curve $x(t) = \arctan t,\ y(t) = t^2$, where $-1 \le t \le 3$.

21. Find the length of the curve $x(t) = \sqrt{2}t,\ y(t) = \frac{t^2}{2},\ z(t) = \ln t$, where $1 \le t \le 4$.

22. Find the length of the curve $\mathbf{x}(t) = (r\cos t, r\sin t, st)$, where $0 \le t \le 2\pi$.

23. Find the length of the curve $\mathbf{x}(t) = \mathbf{i} + \frac{t^2}{2}\mathbf{j} + t\mathbf{k}$, where $0 \le t \le 1$.

24. Find the length of the straight line path connecting (x_0, x_1) and (y_0, y_1) two different ways.

25. For a curve $y = f(x)$ such that $f'(x)$ exists and is continuous for $x \in [a, b]$, show that the length of the curve between $(a, f(a))$ and $(b, f(b))$ is exactly

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Tangent Plane

26. Find the tangent plane for the function $f(x, y) = x^2y + x + y + 1$ at the point $(1, -2, 0)$.

27. Find the tangent plane for the function $f(x, y) = \frac{x}{y}$ at the point $(-2, 3, -\frac{2}{3})$.

28. Find the tangent plane for the function $f(x, y) = \sin x \cos y$ at the point $(\pi/2, \pi/2, 0)$.

29. Find the tangent plane for the function $f(x, y) = \frac{x + y}{x + 2}$ at the point $(1, 0, 1)$.

30. Find the tangent plane for the function $f(x, y) = x \arctan(xy)$ at the point $(1, 0, 0)$.

31. Find the tangent plane for the function $f(x, y) = x\sqrt{y} - \frac{1}{\sqrt{x^3}}$ at the point $(1, 1, 0)$.

32. Find the tangent plane for the function $f(x, y) = e^{x-1}\ln(xy + 2)$ at the point $(1, -1, 3)$.

33. Find the tangent plane for the function $f(x, y) = y\sin(\pi xy) + \frac{1}{2}$ at the point $(1, 1/2, 1)$.

34. Find the tangent plane for the function $f(x, y) = \sqrt{xy}$ at the point $(2, 18, 6)$.

35. Find the tangent plane for the function $f(x, y) = y2^{xy} - 2$ at the point $(0, 1, -1)$.

Total Differentials

36. Find the total differential for the function $f(x, y) = xy$.

37. Find the total differential for the function $f(x, y) = x^3y^2 + 2$.

38. Find the total differential for the function $f(x, y) = x + y + \frac{x}{y}$.

39. Find the total differential for the function $f(x, y) = x \arctan y$.

40. Find the total differential for the function $f(x, y) = ye^{xy}$.

41. Find the total differential for the function $f(x, y, z) = x^2y + y^2z + 4$.

42. Find the total differential for the function $f(x, y, z) = \frac{xy^2}{\sqrt{z^3}}$.

43. Find the total differential for the function $f(x, y, z) = e^x \sin y + \cos z$.

44. Find the total differential for the function $f(x, y, z) = x \arctan(yz^2)$.

45. Find the total differential for the function $f(x, y, z) = \frac{2 + \ln xz}{y}$.

Approximations

46. Approximate $(10.2)^3(7.9)^2$.

47. Approximate $\frac{3.2^3}{\sqrt{4.1 \cdot 8.8}}$.

48. Approximate $\sin(\pi - 0.2)\cos(0.1)$.

49. Approximate $7.9\cos(2.2(\pi - 0.1))$.

Differential Geometry

50. Compute $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ for the curve $\mathbf{x}(t) = 5\cos 3t\,\mathbf{i} + 6t\,\mathbf{j} + 5\sin 3t\,\mathbf{k}$.

51. Compute $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ for the curve $\mathbf{x}(t) = \left(t, \frac{(t+1)^{3/2}}{3}, \frac{(1-t)^{3/2}}{3}\right)$, $-1 < t < 1$.

52. Compute $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ for the curve $\mathbf{x}(t) = (e^{2t}\sin t, e^{2t}\cos t, 1)$.

53. Show that a circle has constant curvature.

54. Show that a helix has constant curvature.

2.4 Gradients & Directional Derivatives

Vector Field:

Del Operator:

Gradient:

Divergence:

Curl:

Normal Line:

2.4 | Exercises

Vector Fields

1. Plot the vector field $\langle x, y\rangle$. Find the curl of this vector field.

2. Plot the vector field $y\mathbf{i}+x\mathbf{j}$. Find the curl of this vector field.

3. Plot the vector field $\langle x, -y\rangle$. Find the curl of this vector field.

4. Plot the vector field $\langle -y, x\rangle$. Find the curl of this vector field.

5. Plot the vector field $(y-x)\mathbf{i}+(x-y)\mathbf{j}$. Find the curl of this vector field.

6. Plot the vector field $\langle x+y, y-x\rangle$. Find the curl of this vector field.

7. Plot the vector field $x\mathbf{i}+\mathbf{j}$. Find the curl of this vector field.

8. Plot the vector field $\langle y, y+1\rangle$. Find the curl of this vector field.

9. Plot the vector field $\langle x, y, z\rangle$. Find the curl of this vector field.

10. Plot the vector field $\langle x, 1, z\rangle$. Find the curl of this vector field.

Directional Derivatives

Find the directional derivative of the function f at the given point in the direction given by the angle indicated by the angle θ.

11. $f(x,y)=x^2y$, $(1,1)$, $\theta=0$

12. $f(x,y)=\dfrac{x+1}{2-y}$, $(2,-2)$, $\theta=\frac{\pi}{2}$

13. $f(x,y)=x\arctan y$, $(1,\sqrt{3})$, $\theta=\dfrac{2\pi}{3}$

14. $f(x,y)=\ln(xy)$, $(1,2)$, $\theta=\dfrac{5\pi}{4}$

15. $f(x,y)=xye^y$, $(-1,0)$, $\theta=\dfrac{11\pi}{6}$

In Exercises 16–25, find the directional derivative of the given function f at the point $\mathbf{x}$ in a direction parallel to the vector $\mathbf{u}$.

16. $f(x,y)=x^2+xy+y^2$, $\mathbf{x}=(1,-1)$, $\mathbf{u}=\dfrac{2\mathbf{i}-\mathbf{j}}{\sqrt{5}}$

17. $f(x,y)=x\sin y$, $\mathbf{x}=(1,\frac{\pi}{2})$, $\mathbf{u}=2\mathbf{i}+3\mathbf{j}$

18. $f(x,y) = \dfrac{xy\sin x}{y}$, $\mathbf{x} = (\pi, 10)$, $\mathbf{u} = \langle -1, 1\rangle$

19. $f(x,y) = e^x + \dfrac{x}{y}$, $\mathbf{x} = (1,2)$, $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$

20. $f(x,y) = \arctan(xy)$, $\mathbf{x} = (\frac{1}{2}, 2)$, $\mathbf{u} = \langle 1, 0\rangle$

21. $f(x,y,z) = xe^y + x^2e^z + y^3e^z$, $\mathbf{x} = (0,1,0)$, $\mathbf{u} = \dfrac{\mathbf{k}-\mathbf{i}}{\sqrt{2}}$

22. $f(x,y,z) = xyz\sin(yz)$, $\mathbf{x} = (1, \frac{\pi}{2}, 2)$, $\mathbf{u} = \langle -1, 0, -1\rangle$.

23. $f(x,y,z) = xz^2 + \sqrt{x^2+y^2}$, $\mathbf{x} = (3,4,1)$, $\mathbf{u} = 3\mathbf{u} + 4\mathbf{j} - \mathbf{k}$

24. $f(x,y,z) = e^{x^2+y^2+z^2}$, $\mathbf{x} = (1,-1,1)$, $\mathbf{u} = \langle 1, -1, 1\rangle$

25. $f(x,y,z) = \dfrac{\sqrt{2}\,x\,e^y}{z+1}$, $\mathbf{x} = (2,0,1)$, $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$

Tangent Planes

In Exercises 26–35, find the tangent plane to the given function at the given point.

26. $f(x,y) = x^2y + x + y - 2$, $\mathbf{x} = (2,3)$

27. $f(x,y) = xe^y$, $\mathbf{x} = (5,0)$

28. $f(x, y0 = \tan^{-1}(xy)$, $\mathbf{x} = (1,-1)$

29. $f(x,y) = \dfrac{x^2}{y^3}$, $\mathbf{x} = (3,-1)$

30. $f(x,y) = \dfrac{x\sin y}{x+1}$, $\mathbf{x} = (1, \frac{\pi}{4})$

31. $f(x,y,z) = xy + xz + yz$, $\mathbf{x} = (1,2,3)$

32. $f(x,y,z) = \dfrac{3x+y}{x+2z}$, $\mathbf{x} = (1,-1,1)$

33. $f(x,y,z) = \dfrac{xe^z}{y+2}$, $\mathbf{x} = (-3,1,0)$

34. $f(x,y,z) = \sqrt{\dfrac{xy}{z^3}}$, $\mathbf{x} = (4,1,1)$

35. $f(x,y,z) = \dfrac{\ln(xz)}{y}$, $\mathbf{x} = (1,1,1)$

Normal Lines

36. Find the normal line to the curve given by $y = 2x - 3$ at the point $(-1, 2)$.

37. Find the normal line to the curve given by $x^2 - y^3 = 1$ at the point $(3, 2)$.

38. Find the normal line to the curve given by $x^3 - x^2 - x + 2 = y^2$ at the point $(2, 2)$.

39. Find the normal line to the curve given by $x^2 - xy^2 + y^3 = 1$ at the point $(2, -1)$.

40. Find the normal line to the curve given by $x^2y = y^2x + 6$ at the point $(2, -1)$.

41. Find the normal line to the surface given by $x - 3y + z = 4$ at the point $(1, 0, 3)$.

42. Find the normal line to the surface given by $x^2 + y^2 + z^2 = 9$ at the point $(3, 0, 0)$.

43. Find the normal line to the surface given by $z = x^2 + y^2$ at the point $(-1, 3, 10)$.

44. Find the normal line to the surface given by $x^2 + y^2 = z^2 + 1$ at the point $(1, 1, 1)$.

45. Find the normal line to the surface given by $z = y^2 - x^2$ at the point $(4, 1, -15)$.

Gradients

46. Find the maximum rate of change of the function $f(x, y) = e^x \cos y$ at the point $(0, \frac{\pi}{6})$ and the direction which it occurs.

47. Find the maximum rate of change of the function $f(x, y) = \sin(x + y)$ at the point $(1, -1)$ and the direction which it occurs.

48. Find the minimum rate of change of the function $f(x, y) = \dfrac{x^2y - y^2x}{3}$ at the point $(2, -2)$ and the direction which it occurs,

49. Find the maximum rate of change of the function $f(x, y) = \dfrac{\ln(xy)}{x}$ at the point $(1, 1)$ and the direction which it occurs.

50. Find the maximum rate of change of the function $f(x, y) = \dfrac{x + y}{y - x}$ at the point $(2, 1)$ and the direction which it occurs.

51. Find the minimum rate of change of the function $f(x, y, z) = xy \sin(xz)$ at the point $(2, 1, \pi)$ and the direction which it occurs.

52. Find the minimum rate of change of the function $f(x, y, z) = xz \ln(yz)$ at the point $(2, 1, 1)$ and the direction which it occurs.

53. Find the maximum rate of change of the function $f(x, y, z) = \dfrac{x^2yz^3 + 1}{z + 2}$ at the point $(-1, -1, -1)$ and the direction which it occurs.

54. Find the maximum rate of change of the function $f(x,y,z)=x^2y-yz^2+xz^3$ at the point $(-1,-1,2)$ and the direction which it occurs.

55. Find the minimum rate of change of the function $f(x,y,z)=\dfrac{xy}{\ln(xz)}$ at the point $(-e,1,-1)$ and the direction which it occurs.

'Applied' Gradients

56. Find the points on the surface $f(x,y)=x^3+2xy+y+5$ that are parallel to the plane $5x+3y-z=0$.

57. Find the points on the hyperboloid $9x^2-45y^2+5z^2-45$ where the tangent plane to the surface is parallel to the plane $x+5y-2z=7$.

58. Find the points on the paraboloid $z=x^2+y^2$ where the tangent plane to the surface is parallel to the plane $x+y+z=1$. Find the equation of the tangent planes at these points.

59. Find the point(s) where the tangent plane to the surface $z=x^2-6x+y^3$ is parallel to the plane $4x-12y+z=7$.

60. Find the points where the tangent plane to $x^2+y^2+z^2=9$ is parallel to the plane $2x+2y+z=1$.

61. Describe the set of points on $x^2+3y^2+z^2+xz=6$ where the tangent plane is parallel to the z–axis.

Other

62. Show that the sum of the x, y, and z intercepts of any tangent plane to the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{c}$ is constant.

63. Let a,b be constants and f,g be functions. Use the definition of the gradient to show that it has the following properties:

(a) $\nabla(af+bg)=a\nabla f+b\nabla g$

(b) $\nabla(fg)=f\nabla g+g\nabla f$

(c) $\nabla\left(\dfrac{f}{g}\right)=\dfrac{g\nabla f-f\nabla g}{g^2}$

(d) $\nabla f^n=nf^{n-1}\nabla f$

64. Let $f(x,y)$ be the function given by

$$f(x,y)=\begin{cases}\dfrac{x|y|}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & \text{otherwise}\end{cases}$$

(a) Use a computer system to graph $f(x,y)$.

(b) Use the definition of the derivative to calculate $f_x(0,0)$ and $f_y(0,0)$.

(c) Use the definition of the directional derivative to determine for which unit vectors $\mathbf{u} = \langle a, b\rangle$ $D_{\mathbf{u}} f(0,0)$ exists.

(d) Is $f(x,y)$ differentiable at the origin? Put your result in the context of the previous parts.

65. Show that the equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at the point (x_0, y_0, z_0) can be expressed as

$$x\frac{x_0}{a^2}+y\frac{y_0}{b^2}+z\frac{z_0}{c^2}=1$$

2.5 Max/Min & Lagrange Multipliers

Max/Mins:

Lagrange Multipliers:

Compact Regions:

2.5 | Exercises

Critical Points

1. Find and classify the critical points for the function $f(x,y) = x^2 + xy + y^2$.

2. Find and classify the critical points for the function $f(x,y) = x^2 + xy + y^2 + 2x - 2y + 5$.

3. Find and classify the critical points for the function $f(x,y) = \ln(x^2 + y^2 + \pi)$.

4. Find and classify the critical points for the function $f(x,y) = x^2 - xy^2 + y^3$.

5. Find and classify the critical points for the function $f(x,y) = (x+y)(1-xy)$.

6. Find and classify the critical points for the function $f(x,y) = e^x \sin y$.

7. Find and classify the critical points for the function $f(x,y) = \cos x \sin y$.

8. Find and classify the critical points for the function $f(x,y) = x + y + \ln(xy)$.

9. Find and classify the critical points for the function $f(x,y) = (x^2 - y^2)e^{-x}$.

10. Find and classify the critical points for the function $f(x,y) = \dfrac{x^2y^2 - 8x + y}{xy}$.

11. Find and classify the critical points of the function $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$.

12. Find and classify the critical points for the function $f(x,y,z) = x^3 + xy^2 + x^2 + y^2 + 3z^2$.

13. Find and classify the critical points for the function $f(x,y,z) = x^2 + y^2 + 2z^2 + xz$.

14. Find and classify the critical points for the function $f(x,y,z) = x^3 + xy + yz + y^2$.

15. Find and classify the critical points for the function $f(x,y,z) = x^2 + y^2 - z^2$.

16. Find and classify the critical points for the function $f(x,y,z) = x^2 + y^2 + 7z^2 - xy - 3yz + 4$.

17. Find and classify the critical points for the function $f(x,y,z) = xy + xz + 2yz + \dfrac{1}{x}$.

18. Find and classify the critical points for the function $f(x,y,z) = e^z(z^2 - y^2 - 2x^2)$.

19. Show that the function $f(x,y) = x^3y^3$ has a critical point at the origin. Show that the Hessian fails to give any information about $f(x,y)$ at the point $(0,0)$. Use other methods to determine the behavior of the function $f(x,y)$ at $(0,0)$.

20. Show that the function $f(x,y) = x^2y^2$ has a critical point at the origin. Show that the Hessian fails to give any information about the behavior $f(x,y)$ at the point $(0,0)$. Use other methods to determine the behavior of the function $f(x,y)$ at $(0,0)$.

21. Show that the function $f(x,y) = e^{-(x^2+y^2)}$ has a critical point at the origin. Show that the Hessian fails to give any information about the function $f(x,y)$ at the point $(0,0)$. Use other methods to determine the behavior of the function $f(x,y)$ at $(0,0)$.

22. Show that the function $f(x,y,z) = x^2y^4z^3$ has a critical point at the origin. Show that the Hessian fails to give any information about the function $f(x,y)$ at the point $(0,0,0)$. Use other methods to determine the behavior of the function $f(x,y)$ at $(0,0,0)$.

Lagrange Multipliers

23. If x and y are such that $x+2y=4$, find the maximum and minimum values of $f(x,y) = x^2+y^2-2x-2y$.

24. Find the maximum and minimum values of $f(x,y) = xy$ along the curve $3x^2+y^2=6$.

25. Find the critical values of $f(x,y) = x^2+y^2-2x-2y$ along a circle of radius 2 centered at the origin.

26. A electrons orbit about a nucleus is given by $x^2+y^2=1$. The energy of the particle is given by $f(x,y) = 3xy+2$. Find the maximum and minimum energy states of the electron.

27. Find the point(s) on the curve $x^2+xy=1$ to the origin.

28. Find the point(s) on the curve $x^2+4xy-5x+2y^2-3y=0$ closest to the point $(2,1)$.

29. Find the points on $z^2 = xy+1$ closest to the origin.

30. An observatory is being constructed. The base will consist of a right circular cylinder of height h with a half sphere of radius r sitting atop it. If the material for the half sphere top costs $20/m^2, the siding costs $8/m^2, and the bottom costs $5/m^2, what ratio of height to diameter minimizes costs if the total volume of the structure must be 200π cubic meters.

31. Find the maximum volume of a rectangular box that is contained in the ellipsoid $x^2+9y^2+4z^2=9$, assuming the edges of the box are parallel to the coordinate axes.

32. A rectangular box with the top of the box removed is made from 12 square ft of cardboard. What is the maximum possible the box can be constructed to have?

33. Find the critical points of $f(x,y,z) = x+y+2z$ if $x^2+y^2+z^2=3$.

34. On the surface $x^2+2y^2+3z^2=1$, find the maximum and minimum values of $f(x,y,z) = x^2-y^2$.

35. Let S be the surface created by intersecting $z^2 = x^2+y^2$ and $z = x+y+2$. Find the points on S nearest and farthest from the origin.

Compact Regions

36. Find the absolute maximum and minimum of the function $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ over the region $D = \{(x,y)\colon -1 \le x \le 1, -1 \le y \le 1\}$.

37. Find the absolute maximum and minimum of the function $f(x,y) = 2x^2 - y^2 + 6y$ over $D = \{(x,y)\colon x^2 + y^2 \le 16\}$.

38. Find the absolute maximum and minimum of the function $f(x,y) = 5 - 3x + 4y$ on the triangular region with vertices $(0,0)$, $(4,0)$, and $(4,5)$.

39. Find the absolute maximum and minimum of the function $f(x,y) = x^3 - xy + y^2 - x$ on $D = \{(x,y)\colon x, y \ge 0, x + y \le 2\}$.

40. Find the absolute maximum and minimum of the function $f(x,y) = x^2 + y^2 - x$ over the square with vertices $(\pm 1, \pm 1)$.

41. Find the absolute maximum and minimum of the function $f(x,y) = e^{xy}$ over $D = \{(x,y)\colon 2x^2 + y^2 \le 1\}$.

42. Find the absolute maximum and minimum of the function $f(x,y) = xy^3$ over $D = \{(x,y)\colon x, y \ge 0, x^2 + y^2 \le 1\}$.

Chapter 3

Multiple Integration

3.1 Double Integrals

Double Integral:

3.1 | Exercises

Rectangular Regions

1. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2 \int_0^3 x + y \; dx \; dy$.

2. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-1}^3 \int_2^5 x - y \; dy \; dx$.

3. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_{-1}^1 3x^2 - 2y \; dy \; dx$.

4. Sketch the region of integration and evaluate the integral $\displaystyle\int_1^2 \int_3^6 x^2 + y^2 \; dx \; dy$.

5. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-1}^1 \int_0^1 \int_0^\pi x \sin y \; dy \; dx$.

6. Sketch the region of integration and evaluate the integral $\displaystyle\int_{\pi/4}^{\pi/2} \int_0^\pi \cos x \sin y \; dy \; dx$.

7. Sketch the region of integral and evaluate the integral $\displaystyle\int_0^3 \int_0^1 x^2 e^y \; dy \; dx$.

8. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-1}^1 \int_{-\pi}^\pi y + y^2 \cos x \; dx \; dy$.

9. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_0^1 \sqrt[3]{xy} \; dx \; dy$.

10. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_0^1 15\sqrt{x+y} \; dx \; dy$.

11. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_0^1 x\, e^{xy} \; dy \; dx$.

12. Sketch the region of integration and evaluate the integral $\displaystyle\int_1^2 \int_1^e \frac{\ln y}{xy} \; dy \; dx$.

13. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_0^1 xy\sqrt{x^2+y^2} \; dy \; dx$.

14. Sketch the region of integration and evaluate the integral $\displaystyle\int_1^2 \int_1^2 \frac{x}{y} + \frac{y}{x} \; dy \; dx$.

‘Irregular’ Regions

15. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2 \int_0^{x^2} y \; dy \; dx$.

16. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-2}^0 \int_{x^2}^4 y \; dy \; dx$.

17. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1\int_{2x}^{8x} y+1\,dy\,dx.$

18. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-1}^3\int_0^x 3y\,dy\,dx.$

19. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1\int_x^{2x} xy\,dy\,dx.$

20. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^3\int_0^{3+2x-x^2} 2x-2y+1\,dy\,dx.$

21. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-3}^3\int_{x^2-9}^{x^2} \frac{1}{9}\,dy\,dx.$

22. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-2}^7\int_0^{(y+2)/3} 3-x\,dx\,dy.$

23. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1\int_y^{\sqrt[3]{y}} x^2+1\,dx\,dy.$

24. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2\int_{(4x-10)/5}^{(15-x)/5} \frac{y+5}{7}\,dy\,dx.$

25. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2\int_{-e^x}^{e^x} 3y^2\,dy\,dx.$

26. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1\int_{2^x}^{3^x} \frac{2}{y}\,dy\,dx.$

27. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-4}^0\int_0^{\sqrt{4-x^2}} 2y\,dy\,dx.$

28. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-3}^3\int_{-\sqrt{9-y^2}}^0 5x^3\,dx\,dy.$

29. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx.$

30. Sketch the region of integration and evaluate the integral $\displaystyle\int_{\pi/4}^{5\pi/4}\int_{\cos x}^{\sin x} dy\,dx.$

31. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2\int_{y^2}^4 dx\,dy.$

32. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta.$

33. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^\pi\int_0^{\sin x} (1+\cos x)\,dy\,dx.$

34. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1\int_0^x \sqrt{1-x^2}\,dy\,dx.$

35. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2\int_x^2 x\sqrt{1+y^3}\,dy\,dx.$

36. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{2+y^3}\, dy\, dx$.

37. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_{2x}^2 4e^{y^2}\, dy\, dx$.

38. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_y^1 \sin x^2\, dx\, dy$.

39. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\, dy\, dx$.

40. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$.

41. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^{\ln 2} \int_{-1}^1 \tan x \sqrt{e^y+1}\, dx\, dy$.

42. Change the order of integration in $\displaystyle\int_0^1 \int_1^{e^y} f(x,y)\, dx\, dy$.

43. Evaluate $\iint_R xy\, dA$, where R is the region enclosed by $y = \dfrac{x}{2}$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

44. Evaluate $\iint_R (2x - y^2)\, dA$, where R is the region enclosed by $x + y = 1$, $y = x + 1$, and $y =$.

45. Evaluate $\iint_R (4 - y)\, dA$, where R is the region enclosed by the circle $x^2 + y^2 = 4$.

Areas

46. Use a double integral to show that the area between the functions $f(x), g(x)$ between $x = a$ and $x = b$, where $f(x) \geq g(x)$ on $[a, b]$, is given by

$$\int_a^b f(x) - g(x)\, dx$$

47. Use a double integral to find the area of the rectangle with vertices $(3,3)$, $(7,3)$, $(7,-4)$, and $(3,-4)$.

48. Use a double integral to find the area of the region bound by $x = 0$, $y = 0$, and $y = 9 - x^2$.

49. Use a double integral to find the area bound by $y = \sqrt{x}$, $x = 0$, and $y = 2$.

50. Use a double integral to find the area bound by $y = x^3$ and $y = \sqrt{x}$.

51. Use a double integral to find the area between $y = x^3$ and $y = x^2$ in Quadrant 1.

52. Use a double integral to find the area below the parabola $y = 4x - x^2$ and above both the line $y = 6 - 3x$ and the x–axis.

53. Use a double integral to find the area bound by $x = 0$, $y = 4$, $y = -4$, and $x = y^2$.

Volumes

54. Find the volume of the solid region formed by the region above the plane $z = 4 - x - y$ and below the rectangle $\{(x, y)\colon 0 \le x \le 1, 0 \le y \le 2\}$.

55. Find the volume of the region bound by $z = 2 - x - 2y$ and the coordinate axes.

56. Find the volume of the region bound above by $z = xy^2$ and below by the region in the plane formed by $y = x^3$, $y = -x^2$, $x = 0$, and $x = 1$.

57. Find the volume of the solid bound by $z = 0$, $x = 0$, $y = 0$, $x = \sqrt[3]{y}$, $x = 2$ but below the function $f(x, y) = e^{x^4}$.

58. Find the volume of the region bound by $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ in the first octant.

59. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

60. Find the volume of the region below $z = 4 - x^2 - 2y^2$ and above $z = 0$.

61. Find the volume of the region above the plane $z = 1 - y$ and below the paraboloid $z = 1 - x^2 - y^2$.

Other Double Integral Problems

62. Evaluate the improper integral $\displaystyle\int_1^\infty \int_0^{1/x} y\,dy\,dx$.

63. Evaluate the improper integral $\displaystyle\int_0^3 \int_0^\infty \frac{x^2}{1+y^2}\,dy\,dx$.

64. Evaluate the improper integral $\displaystyle\int_0^\infty \int_0^\infty xye^{-(x^2+y^2)}\,dx\,dy$.

65. Is the following true or false, if it is true explain why and if it is false give an example to show it.

$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

66. Is the following true or false, if it is true explain why and if it is false give an example to show it.

$$\int_0^1 \int_0^{2x} f(x, y)\,dy\,dx = \int_0^2 \int_{y/2}^1 f(x, y)\,dx\,dy$$

67. Is the following true or false, if it is true explain why and if it is false give an example to show it.

$$\int_0^1 \int_0^x f(x,y)\,dy\,dx = \int_0^1 \int_0^y f(x,y)\,dx\,dy$$

3.2 Triple Integrals

Triple Integrals:

3.2 | Exercises

Rectangular Prism Regions

1. Evaluate the integral $\int_1^3 \int_{-1}^1 \int_0^1 3\,dx\,dy\,dz$.

2. Evaluate the integral $\int_{-1}^2 \int_3^4 \int_2^5 \,dy\,dz\,dx$.

3. Evaluate the integral $\iiint_R dV$, where $R = [-1,3] \times [0,1] \times [0,5]$.

4. Evaluate the integral $\iiint_R xyz\,dV$ over the region R given by $[0,1] \times [1,2] \times [2,3]$.

5. Evaluate the integral $\int_0^1 \int_0^1 \int_0^1 (y^2 + z^2)\,dz\,dy\,dx$.

6. Evaluate the integral $\int_0^1 \int_{x^2}^x \int_0^{xy} 6\,dz\,dy\,dx$.

7. Evaluate $\iiint_R (x^2 e^y + xyz)\,dV$, where R is the region $[-2,3] \times [0,1] \times [0,5]$.

Irregular Regions

8. Evaluate the integral $\iiint_R x\, dV$, where R is the region bound by the coordinate axes and $x+y+z=4$.

9. Evaluate the integral $\iint_R xy\, dV$, where R is the region enclosed by $z=x+y$, $z=0$, $y=x^2$, and $x=y^2$.

10. Evaluate the integral $\iiint_R 2x\, dV$, where R is the region lying under $2x+3y+z=6$ and in the first octant.

11. Evaluate $\iiint_R (1+xy)\, dV$, where R is the bound by the coordinate planes ant $x+y+z=1$.

12. Evaluate $\iiint_R (2x-y+z)\, dV$, where R is the region bound by the cylinder $z=y^2$, $x=0$, $x=1$, $y=-2$, $y=2$, and $z=0$.

13. Evaluate $\iiint_R y\, dV$, where R is the region bounded by $x+y+z=2$, $x^2+z^2=1$, and $y=0$.

14. Evaluate $\iiint_R 8xyz\, dV$, where R is the region bounded by $y=x^2$, $y+z=9$, and the xy–plane.

15. Evaluate $\iiint_R z\, dV$, where R is the region in the first octant bounded by $y^2+z^2=9$, $y=x$, $x=0$, and $z=0$.

16. Evaluate $\iiint_R (1-z^2)\, dV$, where R is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

17. Evaluate $\iiint_R 3x\, dV$, where R is the region in the first octant bounded by $z=x^2+y^2$, $x=0$, $y=0$, and $z=4$.

18. Evaluate $\iiint_R (x+y)\, dV$, where R is the region bounded by $z=x^2+y^2$, $x=0$, $y=0$, and $z=4$.

19. Evaluate $\iiint_R z\, dV$, where R is the region bounded by $z=0$, $x^2+4y^2=4$, and $z=x+2$.

Volume Integrals

20. Show that a circular cylinder with base $x^2+y^2=r^2$ and height h has volume $\pi r^2 h$.

21. Find the volume of a cone of height h with base radius r.

22. Find the volume of the tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,1,0)$, and $(0,0,3)$.

23. Find $\iiint_R \frac{24xy}{13}\, dV$, where R is the region where $0 \le z \le 1+x+y$ and above the region in the plane bound by $y=\sqrt{x}$, $y=0$, and $x=1$.

24. Find $\iiint_R f(x,y,z)\, dV$, where $f(x,y,z)=3x-2y$ and R is the region bounded by the coordinate planes and the plane $2x+3y+z=6$ in the first octant.

25. Compute the integral $\iiint_D f(x,y,z)\, dV$, where $f(x,y,z)=2x+z$ and D is the region bound by the surfaces $z=x^2$ and $z=2-x^2$ from $0 \le y \le 3$.

26. Evaluate the integral $\iiint_R (x^2+y^2+z^2)\, dV$, where R is the region bounded by $x+y+z=1$, $x=0$, $y=0$, and $z=0$.

27. Find the volume of the region bound by the surfaces $z=x^2+y^2$, $z=0$, $x=0$, $y=0$, and $x+y=1$.

28. Find the volume of the region bound by the planes $z=x+y$, $z=10$, $x=0$, and $y=0$.

29. Find the volume of the region beneath $z^2=xy$ but above the region in the plane bound by $x=y$, $y=0$, and $x=4$.

30. Find the volume of the solid sitting in the first octant bound by the coordinate planes and $z=x^2+y^2+9$ and $y=4-x^2$.

31. Find the volume of the solid bounded by $z=4x^2+y^2$ and the cylinder $y^2+z=2$.

3.3 Change of Variables

Change of Variables:

‘Standard Transformations’:

3.3 | Exercises

General Change of Variable Integrals

1. Describe the image of $[0,1]\times[0,1]$ under the transformation $T(u,v)=(3u,-v)$.

2. What is the image of $[0,1]\times[0,1]$ under the transformation $T(u,v)=\left(\frac{u-v}{\sqrt{2}},\frac{u+v}{\sqrt{2}}\right)$.

3. If

$$T(u,v)=\begin{pmatrix}2 & 3\\ -1 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

and D^* is the parallelogram with vertices $(0,0)$, $(1,3)$, $(-1,2)$, and $(0,5)$. What is $D=T(D^*)$?

4. Find $\iint_R y^2\,dA$, where R is the region bounded by $xy=1$, $xy=2$, $xy^2=1$, and $xy^2=2$.

5. Evaluate $\iint_R \left(\frac{x-y}{x+y+2}\right)^2\,dx\,dy$, where R is the square with vertices $(-1,0)$, $(0,-1)$, $(1,0)$, and $(0,1)$.

6. Evaluate $\iint_R (x+y)\,dx\,dy$, where R is the region $y=x$, $y=x+1$, $xy=1$, and $xy=2$.

7. Evaluate $\iint_R \sqrt{\frac{x+y}{x-2y}}\,dA$, where R is the region enclosed by $y=x/2$, $y=0$, and $x+y=1$.

8. Evaluate $\iint_R \cos(x+2y)\sin(x-y)\,dx\,dy$, where R is the region bounded by $y=0$, $y=x$, and $x+2y=8$.

9. Evaluate $\iint_R e^{\frac{y-x}{y+x}}\,dA$, where R is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,2)$.

10. Evaluate $\iint_R (x-y)^2\,dA$, where R is the parallelogram $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$.

11. Evaluate $\iint_R x^2\,dA$, where R is the ellipse $9x^2+4y^2=36$.

12. Evaluate $\iint_R \frac{y}{x}\,dx\,dy$, where R is the region bounded by $x^2-y^2=1$, $x^2-y^2=4$, $y=0$, and $y=\frac{x}{2}$.

13. Use a change of variables on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ to show that its volume is $\frac{4}{3}\pi$.

14. Find $\iint_R \cos\left(\frac{x-y}{x+y}\right)\,dx\,dy$, where R is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

15. Evaluate $\iiint_R x-y\,dV$, where R is the parallelepiped with vertices $(0,0,0)$, $(2,0,0)$, $(3,1,0)$, $(1,1,0)$, $(0,1,2)$, $(2,1,2)$, $(3,2,2)$, and $(1,2,2)$.

16. Use the substitution $u = x^2 - y^2$ and $v = \dfrac{y}{x}$ to evaluate $\displaystyle\iint_R \frac{dA}{x^2}$, where R is the region under $y = \dfrac{1}{x}$ in Quadrant 1 and to the right of $x^2 - y^2 = 1$.

17. Show that for a polar change of coordinates, $dx\ dy = dA = r\ dr\ d\theta$.

18. Show that for a cylindrical change of coordinates, $dx\ dy\ dx = dV = r\ dr\ d\theta\ dz$.

19. Show that for a spherical change of coordinates, $dx\ dy\ dz = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$.

Polar Integrals

20. Evaluate $\displaystyle\int_R dA$, where R is *any* circle of radius R.

21. Evaluate $\displaystyle\iint_R x^2 + y^2\ dA$, where R is the circle of radius 2 centered at the origin.

22. Evaluate $\displaystyle\iint_R x\ dA$, where $R = \{(r,\theta)\colon 1 \le r \le 2, 0 \le \theta \le \frac{\pi}{4}\}$.

23. Evaluate
$int_R\sqrt{1+x^2+y^2}\ dA$, where R is the part of the interior of $x^2+y^2=4$ in Quadrant I.

24. Evaluate $\displaystyle\iint_R (x^2+y^2)^{3/2}\ dA$, where R is the circle of radius 3 centered at the origin.

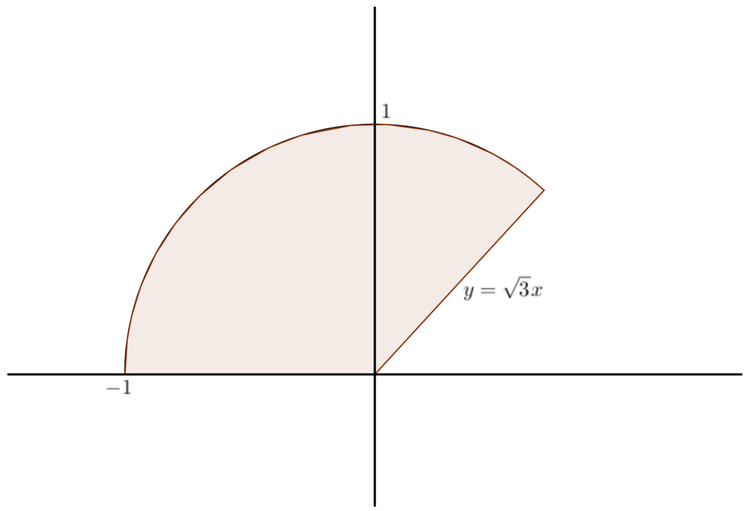
25. Evaluate $\displaystyle\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} e^{x^2+y^2}\ dx\ dy$.

26. Evaluate $\displaystyle\int_0^3\int_0^x \frac{dy\ dx}{\sqrt{x^2+y^2}}$.

27. Evaluate $\displaystyle\iint_R \frac{dA}{\sqrt{4-x^2-y^2}}$, where R is the disk of radius 1 centered at $(0,1)$.

28. Evaluate $\displaystyle\iint_R y^2\ dA$, where R is the region between the circle of radius 1 and the square with side length 2 centered at the origin.

29. Evaluate $\displaystyle\iint_R \cos(x^2+y^2)\ dA$, where R is the region below.



30. Find $\displaystyle\iint_R \frac{x}{\sqrt{x^2+y^2}}\,dA$, where R is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

31. Evaluate $\displaystyle\iint_R \sqrt{x^2+y^2}\,dA$, where R is the region give by $0 \le r \le 1+\cos\theta$ for $0 \le \theta \le 2\pi$.

32. Evaluate $\displaystyle\iint_R \sin(x^2+y^2)\,dA$, where R is the circle centered at the origin with radius 2.

Cylindrical Integrals

33. Evaluate $\displaystyle\iiint_R (x^2+y^2+z^2)\,dV$, where R is the region inside $x^2+y^2 \le 4$, bounded above by $z=5$ and below by $z=-3$.

34. Evaluate $\displaystyle\iiint_R (x^2+y^2+2z^2)\,dV$, where R is the solid cylinder defined by $x^2+y^2 \le 4$, $-1 \le z \le 2$.

35. Evaluate $\displaystyle\iiint_R z^2\sqrt{x^2+y^2}\,dV$, where R is the solid cylinder formed by $x^2+y^2 \le 4$, $z=-1$, and $z=3$.

36. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{2xy} x^2+y^2\,dz\,dy\,dx.$

37. Find the volume of the solid bounded above by the sphere $x^2+y^2+z^2=9$ and below by the xy–plane and laterally by $x^2+y^2=4$.

38. Find $\displaystyle\iiint_R y\,dV$, where R is the region below $z=x+2$, above $z=0$, and between $x^2+y^2=1$ and $x^2+y^2=4$.

39. Evaluate $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy.$

40. Find the volume under the plane $y = z$, above $z = 0$, and within $x^2 + y^2 = 1$.

Spherical Integrals

41. Evaluate $\iiint_R z\, dV$, where R is the upper half of the unit sphere.

42. Find the volume of the region bounded above by $x^2 + y^2 + z^2 = 8$ and below by $z^2 = x^2 + y^2$.

43. Evaluate $\iiint_R \dfrac{dV}{\sqrt{x^2 + y^2 + z^2 + 3}}$, where R is the sphere of radius 2 centered at the origin.

44. Evaluate $\iiint_R 4z\, dV$, where R is the upper half unit sphere.

45. Evaluate $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2\, dz\, dx\, dy$.

46. Find the volume of the solid above the cone $z^2 = x^2 + y^2$ and below $z = 1$.

47. Evaluate $\iiint_R e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where $R = \{(x, y, z) \colon x^2 + y^2 + z^2 \le 1, x, y, z \ge 0\}$.

48. Evaluate $\displaystyle\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} y\sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$.

Chapter 4

Line Integrals

4.1 Line Integrals

Line Integral:

4.1 | Exercises

1. Evaluate $\int_C \cos \pi y \; dx$, where C is the line segment from $(0,0)$ to $(1,1)$.

2. Evaluate $\int_C e^x \; dy$, where C is the curve (t^2, t^2) for $0 \le t \le 1$.

3. Evaluate $\int_C x^2 y^3 \; dx$, where C is the curve (t^2, t) for $0 \le t \le 1$.

4. Evaluate $\int_C 2xy \; dx$, where C is the curve $y = x^2 + 9$ from $(0,9)$ to $(3,18)$.

5. Calculate $\int_C f \; ds$, where $f(x,y) = 2x + y$ and C is the line segment from $(-1,1)$ to $(2,-3)$.

6. Calculate $\int_C f \; ds$, where $f(x,y) = xy - x + y$ and C is the line segment from $(3,3)$ to $(3,1)$.

7. Calculate $\int_C f \; ds$, where $f(x,y) = x\sqrt{y}$ and C is the line segment from $(-1,0)$ to $(2,3)$.

8. Calculate $\int_C f \; ds$, where $f(x,y) = xy^4$ and C is the upper half of the circle $x^2 + y^2 = 9$.

9. Evaluate $\int_C f \; ds$, where $f(x,y,z) = xyz$ and C is the path $\mathbf{x}(t) = (t, 2t, 3t)$, $0 \le t \le 2$.

10. Evaluate $\int_C f \; ds$, where $f(x,y,z) = \dfrac{x+z}{y+z}$ and C is the curve $\mathbf{x}(t) = (t, t, t^{3/2})$, where $1 \le t \le 3$.

11. Evaluate $\int_C f \; ds$, where $f(x,y,z) = x + y + z$ and C is the straight line segments from $(-1,5,0)$ to $(1,6,4)$ then to $(0,1,1)$.

12. Evaluate $\int_C (2x + 9z) \; ds$, where $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$ for $0 \le t \le 1$.

13. Evaluate $\int_C f \; ds$, where $\mathbf{x}(t) = (\cos t, \sin t, t)$, $\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3$, and $f(x,y,z) = xy + z$.

14. Evaluate $\int_C f \; ds$, where

$$\mathbf{x}(t) = \begin{cases} (2t, t), & 0 \le t \le 1 \\ (t+1, 5-4t), & 1 < t \le 3 \end{cases}$$

15. Evaluate the integral $\int_C \mathbf{F} \cdot ds$, where $\mathbf{F} = \langle z, y, -x \rangle$ and C is the path $(t, \sin t, \cos t)$, $0 \le t \le \pi$.

16. Evaluate $\int_C \mathbf{F} \cdot ds$, where C is the curve $\mathbf{x}(t, 3t^2, 2t^3)$ and $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

17. Find $\int_C \mathbf{F} \cdot ds$, where $\mathbf{F} = \langle x, y, z \rangle$ and C is the path $\mathbf{x}(t) = (2t+1, t, 3t-1)$, $0 \le t \le 1$.

18. Evaluate $\displaystyle\int_C \mathbf{F}\cdot ds$, where $\mathbf{F} = (y+2)\,\mathbf{i} + x\,\mathbf{j}$ and C is the path $\mathbf{x}(t) = (\sin t, -\cos t)$, $0 \le t \le \pi/2$.

19. Evaluate $\displaystyle\int_C \mathbf{F}\cdot ds$, where $\mathbf{F} = \langle y\cos z, x\sin z, xy\sin z^2\rangle$ and C is the path $\mathbf{x}(t) = (t, t^2, t^3)$, $0 \le t \le 1$.

20. Evaluate $\displaystyle\int_C x\,dy - y\,dx$, where C is the curve $\mathbf{x}(t) = (\cos 3t, \sin 3t)$, $0 \le t \le \pi$.

21. Evaluate $\displaystyle\int_C \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}}$, where C is curve $\mathbf{x}(t) = (e^{2t}, \cos 3t, e^{2t}\sin 3t)$, $0 \le t \le 2\pi$.

22. Evaluate $\displaystyle\int_C (x^2 - y)\,dx + (x - y^2)\,dy$, where C is the line segment from $(1,1)$ to $(3,5)$.

23. Evaluate $\displaystyle\int_C x^2y\,dx - (x+y)\,dy$, where C is the trapezoid with vertices $(0,0)$, $(3,0)$, $(3,1)$, and $(1,1)$, oriented counterclockwise.

24. Evaluate $\displaystyle\int_C x^2y\,dx - xy\,dy$, where C is the curve with $y^2 = x^3$ from $(1,-1)$ to $(1,1)$.

25. Evaluate $\displaystyle\int_C yz\,dx - xz\,dy + xy\,dz$, where C is the line segment from $(1,1,2)$ to $(5,3,1)$.

26. Evaluate $\displaystyle\int_C z\,dx + x\,dy + y\,dz$, where C is the curve obtained by intersecting $z = x^2$ and $x^2+y^2 = 4$ and oriented counterclockwise around the z–axis.

27. Find the work done by the force $\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + (x+y+z)\,\mathbf{k}$ on a particle moving along the parabola $y = 3x^2, z = 0$ from the origin to the point $(2,12,0)$.

4.2 The Fundamental Theorem for Line Integrals

Fundamental Theorem of Calculus:

Fundamental Theorem for Line Integrals:

Open Set:

Closed Set:

(Path) Connected:

Domain:

Simple Curve:

Closed Curve:

Simply Connected:

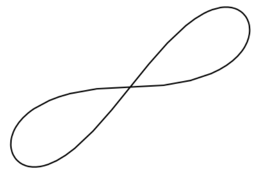
Path Independence:

Conservative:

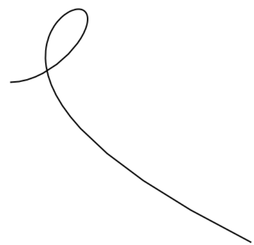
4.2 | Exercises

Curves & Regions

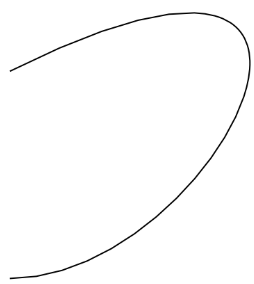
1. Is the curve below closed? Is the curve simple?



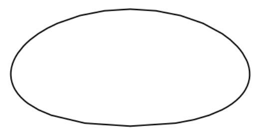
2. Is the curve below closed? Is the curve simple?



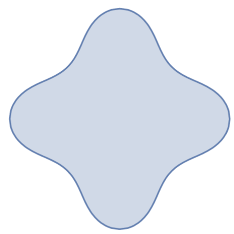
3. Is the curve below closed? Is the curve simple?



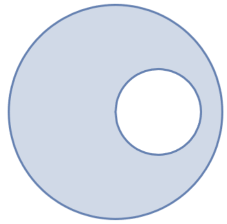
4. Is the curve below closed? Is the curve simple?



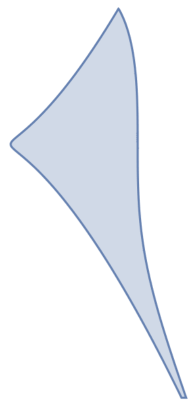
5. Is the region below simply connected? Is it open? Is it path connected? Explain why or why not.



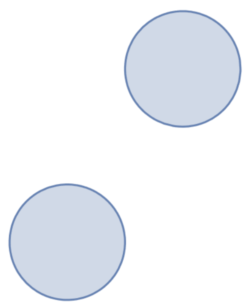
6. Is the region below simply connected? Is it open? Is it path connected? Explain why or why not.



7. Is the region below simply connected? Is it open? Is it path connected? Explain why or why not.



8. Is the region below simply connected? Is it open? Is it path connected? Explain why or why not.



Conservative Vector Fields

9. Determine if the vector field $\mathbf{F} = \mathbf{i} + \mathbf{j}$ is conservative. If it is, find a potential function for $\mathbf{F}$.

10. Determine if the vector field $\mathbf{F} = \langle 2xy - 1, x^2 + 1 \rangle$ is conservative. If it is, find a potential function for $\mathbf{F}$.

11. Determine if the vector field $\mathbf{F} = \dfrac{\mathbf{i}}{y+1} + \left(\dfrac{1}{y} - \dfrac{x+y}{y^2}\right)\mathbf{j}$ is conservative. If it is, find a potential function for $\mathbf{F}$.

12. Determine if the vector field $\mathbf{F} = \langle y - \cos x, x + \dfrac{1}{y} \rangle$ is conservative. If it is, find a potential function for $\mathbf{F}$.

13. Determine if the vector field $\mathbf{F} = (\cos xy - xy\sin xy)\mathbf{i} - (x^2 \sin xy + 1)\mathbf{j}$ is conservative. If it is, find a potential function for $\mathbf{F}$.

14. Determine if the vector field $\mathbf{F} = (y\cos xy\cos yz\mathbf{i} + (x\cos xy\cos yz - z\sin xy\sin yz + y^2)\mathbf{j} - y\sin xy\sin yz\mathbf{k}$ is conservative, if it is, find a potential function for $\mathbf{F}$.

15. Determine if the vector field $\langle 2xz, z, 1+x^2+y\rangle$ is conservative. If it is, find a potential function for $\mathbf{F}$.

16. Determine if the vector field $\langle 3x^2+\sin y, \frac{1}{z}+x\cos y, -\frac{y}{z^2}\rangle$ is conservative. If it is, find a potential function for $\mathbf{F}$.

17. Determine if the vector field $\mathbf{F} = \log y\sin z\mathbf{i} + x\cos y\ln z\mathbf{j} + \frac{x\sin y}{z}\mathbf{k}$. If it is, find a potential function for $\mathbf{F}$.

18. Determine if the vector field $\mathbf{F} = \frac{z}{2\sqrt{x}}\mathbf{i} + (e^y - \pi z\sec^2(\pi yz))\mathbf{j} + (\sqrt{x} - \pi y\sec^2(\pi yz))\mathbf{k}$. If it is, find a potential function for $\mathbf{F}$.

Evaluating Integrals

19. Show that the line integral $\int_C (3x-5y)\,dx + (7y-5x)\,dy$, where C is the line segment from $(1,3)$ to $(5,2)$ is path independent and evaluate the integral.

20. Show that the line integral $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$, where C is the semicircular arc of $x^2+y^2=4$ from $(2,0)$ to $(-2,0)$ is path independent and evaluate the integral.

21. Show that the line integral $\int_C (2y-3z)\,dx + (2x+z)\,dy + (y-3x)\,dz$, where C is the line segment from $(0,0,0)$ to $(0,1,1)$ then the line segment to $(1,2,3)$ is independent of path and evaluate the integral.

22. Let $\mathbf{F} = \langle 2x\frac{3}{2}\sqrt{y}\rangle$. Compute the integral $\int_C \mathbf{F}\cdot ds$, where $C : [0,1] \to \mathbb{R}^2$ is the upper quarter of the ellipse, oriented counterclockwise, centered at the origin with semimajor axis (along the x-axis) of length 4 and semiminor axis (the y-axis) of length 3.

23. Let $\mathbf{F} = x\mathbf{i} - y^2\mathbf{j}$. Compute the integral $\int_C \mathbf{F}\cdot ds$, where $C : [0,1] \to \mathbb{R}^2$ is the curve given by $\mathbf{x}(t) = (t, e^{t^4})$, where $0 \le t \le 1$.

24. Let $\mathbf{F} = (2xy+1)\mathbf{i} + (x^2-1)\mathbf{j}$. Compute the integral $\int_C \mathbf{F}\cdot ds$, where C is the path $C : [0,1] \to \mathbb{R}^2$ given by

$$\mathbf{x}(t) = \left(e^{t^2-t} + \sin\left(\pi\cos\left(\frac{\pi}{2}t\right)\right) - 2, \frac{1}{t^2+2t-4} - \sin(\pi t)\right)$$

25. Let $\mathbf{F} = \frac{1-2xy}{y^2-1}\mathbf{i} + \frac{x^2-2xy+x^2y^2}{(y^2-1)^2}\mathbf{j}$. Compute the integral $\int_C \mathbf{F} \cdot ds$, where $C : [0,1] \to \mathbb{R}^2$ is the curve given by

$$\mathbf{x}(t) = \left(2^t + t^4 + \sqrt{\frac{2t^2+t}{3}}, (t+1)^t + \frac{1}{t+1} + \frac{t}{2}\right)$$

26. Let $\mathbf{F} = \left\langle \sqrt[3]{y} - \frac{y}{3\sqrt[3]{x}}, \frac{x}{3\sqrt[3]{y^2}} - \sqrt[3]{x} \right\rangle$. Compute the integral $\int_C \mathbf{F} \cdot ds$, where $C : [0,1] \to \mathbb{R}^2$ is the curve given by

$$\mathbf{x}(t) = \left((t+1)^{(t+1)^t} + t^2 + \arctan^2(t^2-t) + 1, \frac{1}{t^2+1}\ln(t^2-t+1) + \sin^4(\pi t \cos^3(\pi t)) + \frac{t}{2}\right)$$

27. Let $\mathbf{F} = 2x\mathbf{i} + \cos y \cos z\mathbf{j} - \sin y \sin z\mathbf{k}$. Compute the integral $\int_C \mathbf{F} \cdot ds$, where C is the path from $(0,0,0)$ to $(1,3,1)$, $(1,3,1)$ to $(-4,5,6)$, and finally $(-4,5,6)$ to $(0,0)$.

4.3 Green's Theorem

Green's Theorem:

4.3 | Exercises

1. Let $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ and D be the region bound by the curves $y = x$ and $y = x^2$ in the plane. Verify Green's Theorem for the integral $\displaystyle\oint_{\partial D} \mathbf{F} \cdot ds$.

2. Evaluate the integral $\oint_C -y\;dx + x\;dy$, where C is a circle of radius a, oriented counterclockwise.

3. Calculate

$$\oint_C xy\;dx + x^2y^3\;dy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$, oriented counterclockwise.

4. Show that if D is any region to which Green's Theorem applies that then we have

$$\text{area } D = \frac{1}{2}\oint_{\partial D} -y\;dx + x\;dy$$

5. Use the previous exercise to find the area of an ellipse with semimajor and semiminor axes of length a, b, respectively.

6. Verify Green's Theorem for $D = \{(x,y) \colon x^2 + y^2 \leq 4\}$ and $\mathbf{F} = -x^2y\,\mathbf{i} + xy^2\,\mathbf{j}$.

7. Verify Green's Theorem for D the square centered at the origin with side length 2 and $\mathbf{F} = y\,\mathbf{i} + x^2\,\mathbf{j}$.

8. Calculate

$$\oint_C y^2\;dx + x^2\;dy$$

where C is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, oriented counterclockwise.

9. Find the work done by the vector field $\mathbf{F} = (4y - 3x)\,\mathbf{i} + (x - 4y)\,\mathbf{j}$ on a particle moving counterclockwise twice around the ellipse $x^2 + 4y^2 = 4$.

10. Evaluate

$$\oint_C y^2\;dx + x^2\;dy$$

where C is the boundary of the triangle with vertices $(0,0)$, $(1,1)$ and $(1,0)$, oriented clockwise.

11. Calculate

$$\oint_C \left(2y + \tan(\ln(x^2+1))\right)\;dx + \left(5x - e^{-y^2} + \sin^2 y^4\right)\;dy$$

where C is the circle of radius 3 centered at the origin.

12. Calculate

$$\oint_C \mathbf{F} \cdot ds$$

where $\mathbf{F} = \langle e^y, -\sin \pi x\rangle$ and C is the triangle with vertices $(1,0)$, $(0,1)$, and $(-1,0)$, oriented clockwise.

13. Calculate

$$\oint_C y^4\,dx + 2xy^3\,dy$$

where C is the ellipse $x^2 + 2y^2 = 2$.

14. If D is a region to which Green's Theorem applies and ∂D is oriented properly, show that

$$\text{area } D = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx$$

15. Show that if C is the boundary of any rectangular region in $\mathbb{R}^2$, then

$$\oint_C (x^2y^3 - 3y)\,dx + x^3y^2\,dy$$

depends only on the area of the rectangle, not on the placement of the rectangle in $\mathbb{R}^2$.

16. Show that if C is a simple closed curve forming a region D to which Green's Theorem applies, C being oriented properly, then

$$\oint_C -y^3\,dx + (x^3 + 2x + y)\,dy$$

is strictly positive.

17. Let D be a region to which Green's Theorem applies and suppose $u(x,y)$ and $v(x,y)$ are two functions of class C^2 whose domain include D. Show that

$$\iint_D \frac{\partial(u,v)}{\partial(x,y)}\,dA = \oint_C (u\nabla v)\cdot ds$$

where $C = \partial D$ is oriented as in Green's Theorem.

18. Let $f(x,y)$ be a function of class C^2 such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

, i.e. f is harmonic. Show that if C is any closed curve to which Green's Theorem applies, then

$$\oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0$$

19. Let D be a region to which Green's Theorem applies and $\mathbf{n}$ the outward unit normal vector to D. S tuppose $f(x,y)$ is a function of class C^2. Show that

$$\iint_D \nabla^2 f\,dA = \oint_{\partial D} \frac{\partial f}{\partial n}\,ds$$

where $\nabla^2 f$ denotes the Laplacian of f and $\partial f/\partial n$ denotes $\nabla f \cdot \mathbf{n}$.

4.4 Surface Integrals

Parametrization:

Standard Normal Vector:

Surface Area:

Surface Integral

Orientability:

4.4 | Exercises

1. Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrized surface given by $X(s,t) = (s^2 - t^2, s+t, s^2+3t)$. Determine a normal vector to this surface at the point $X(2,-1) = (3,1,1)$. Find the equation of the tangent plane to the surface at this point. Give an equation for the surface of the form $z = f(x,y)$.

2. Evaluate $\displaystyle\iint_S z^3\, dS$, where S is the sphere parametrized by $X : [0,2\pi] \times [0,\pi] \to \mathbb{R}^3$ where $X(s,t) = (a\cos s \sin t, a\sin s \sin t, a\cos t)$.

3. Let S be the closed cylinder of radius 3 with axis along the z–axis, the top face at $z = 15$ and bottom at $z = 0$. Find $\displaystyle\iint_S z\, dS$.

4. Find $\displaystyle\iint_S (4-z)\, dS$, where S is the surface given by $X(x,y) = (x, y, 4-x^2-y^2)$.

5. Find $\displaystyle\iint_X \mathbf{F} \cdot dS$, where $\mathbf{F} = \langle x, y, z-2y \rangle$ and $X(s,t) = (s\cos t, s\sin t, t)$, $0 \le s \le 1$ and $0 \le t \le 2\pi$.

6. Evaluate $\displaystyle\iint_S (x^3\,\mathbf{i} + y^3\,\mathbf{j}) \cdot dS$, where S is the closed cylinder bound by $x^2+y^2 = 4$, $z = 0$, and $z = 5$.

7. Find $\displaystyle\iint_X (x^2+y^2+z^2)\, dS$, where $X(s,t) = (s, s+t, t)$, $0 \le s \le 1$, $0 \le t \le 2$.

8. Find $\displaystyle\iint_S x^2\, dS$, wehre S is the surface of the cube $[-2,2] \times [-2,2] \times [-2,2]$.

9. Find $\displaystyle\iint_S y^2\, dS$. [Hint: $\displaystyle\iint_S (x^2+y^2+z^2)\, dS$ and use symmetry.]

10. Find $\displaystyle\iint_S (z-x^2-y^2)\, dS$, where S is the surface of the cylinder bound by $x^2+y^2 = 4$, $z = -2$, and $z = 2$.

4.5 Divergence Theorem & Stokes' Theorem

Divergence Theorem:

Stokes' Theorem:

4.5 | Exercises

Divergence/Gauss' Theorem

1. Verify the Divergence Theorem for R the portion of the paraboloid $z = 9 - x^2 - y^2$ above the xy–plane and $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

2. Verify the Divergence Theorem for R the unit cube and $\mathbf{F} = (y - x)\,\mathbf{i} + (y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$.

3. Verify the Divergence Theorem for R the standard unit cube and $\mathbf{F} = y^2\,\mathbf{i} + (2xy + z^2)\,\mathbf{j} + 2yz\,\mathbf{k}$.

4. Let S be the solid cylinder of radius a and height b centered along the z-axis with bottom at $z = 0$. Let $\mathbf{F} = \langle x, y, z\rangle$. Verify Gauss' Theorem for S and $\mathbf{F}$.

5. Find the flux of $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ across the sphere given by $x^2 + y^2 + z^2 = a^2$.

6. Let S be the sphere $(x-2)^2 + (y+5)^2 + (z-1)^2 = 4$ along with its interior and $\mathbf{F} = 5x\,\mathbf{i} - 3\,\mathbf{j} - \mathbf{k}$. Calculate $\displaystyle\iint_{\partial S} \mathbf{F}\cdot d\mathbf{S}$.

7. Find the flux of $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ across the sphere given by $x^2 + y^2 + z^2 = a^2$.

8. Let $\mathbf{F} = e^y \cos z\,\mathbf{i} + \sqrt{x^3+1}\sin z\,\mathbf{j} + (x^2 + y^2 + 3)\,\mathbf{k}$ and S be $z = (1 - x^2 - y^2)e^{1-x^2-3y^3}$ for $z \geq 0$, oriented outwards. Find $\displaystyle\oiint_{\partial D} \mathbf{F}\cdot dS$.

9. Let S be the region formed by $z = x^2 + y^2$ and $z = 2$. Let $\mathbf{F} = y^{2/3}\,\mathbf{i} + \sin^3 x\,\mathbf{j} + z^2\,\mathbf{k}$. Find $\displaystyle\iint_{\partial S} \mathbf{F}\cdot d\mathbf{S}$.

10. Let $S_1 = \langle(x, y, z)\colon z = 1 - x^2 - y^2, z \geq 0\}$, $S_2 = \{(x, y, z)\colon z = 0, x^2 + y^2 \leq 1\}$, and define S to be the surface created by putting S_1 and S_2 together, appropriately outwardly oriented. Define $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$. Find $\displaystyle\iint_{\partial D} \mathbf{F}\cdot d\mathbf{S}$, $\displaystyle\iint_{S_1} \mathbf{F}\cdot d\mathbf{S}$, and $\displaystyle\iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$. Show without direct calculation of the surface integrals that $\displaystyle\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$.

11. Let S be the boundary of the cube defined by $-2 \leq x \leq 2$, $-1 \leq y \leq 1$, and $-1 \leq z \leq 5$ and $\mathbf{F} = x^3y^3\,\mathbf{i} + 4yz\,\mathbf{j} - 3x^2y^3z\,\mathbf{k}$. Calculate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$.

12. Let R be the region formed by $x^2 + y^2 + z^2 \leq 1$. Find $\displaystyle\iiint_R z^2\,dV$.

13. Find

$$\iint_S \mathbf{F}\cdot dS$$

where S is the box with vertices $(\pm1, \pm2, \pm3)$ with outward normal and $\mathbf{F} = x^2y^3\,\mathbf{i} + y^2z^3\,\mathbf{j} + z^2x^3\,\mathbf{k}$.

14. Let S be the surface given by $z^2 = x^2 + y^2$ and $0 \leq z \leq 1$. Define $\mathbf{F} = \langle x, 2y, 3z \rangle$. Calculate $\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S}$.

15. Consider a fluid having density $\rho(\mathbf{r})$ and velocity $\mathbf{v}(\mathbf{r})$. Let V be a volume with no fluid sources or sinks bounded by a closed surface S. The mass flux is given by $\int_S \rho\, \mathbf{v} \cdot d\mathbf{S}$ so that

$$\int_S \rho \mathbf{v} \cdot d\mathbf{S} := \int_S \mathbf{J} \cdot d\mathbf{S}$$

where $\mathbf{J} = \rho\, \mathbf{v}$ is the mass current. Argue why

$$\int_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial M}{\partial t}$$

Find any integral representation for the mass M in V. Use this and the Divergence Theorem to derive the Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Interpret the equation. How general is the result?

16. Let $\mathbf{F} = F_1(x, y, z)\, \mathbf{i} + F_2(x, y, z)\, \mathbf{j} + F_3(x, y, z)\, \mathbf{k}$. Show that

$$\text{div } \mathbf{F}(P) = \lim_{V \to 0} \frac{1}{V} \oiint_S \mathbf{F} \cdot dS$$

where S is a piecewise smooth, orientable, closed surface S enclosing a region D of volume V oriented outwardly and the limit is taken to shrink D down to the point P.

Stokes' Theorem

17. Verify Stokes' Theorem for $R = \{(x, y, z) \colon z = \sqrt{1 - x^2 - y^2}, z \geq 0\}$ and $\mathbf{F} = \langle x, y, z \rangle$.

18. Verify Stokes' Theorem for $R = \{(x, y, z) \colon x = 0, -1 \leq y, z \leq 1\}$ and $\mathbf{F} = (2xz + 3y^2)\, \mathbf{j} + 4yz^2\, \mathbf{k}$.

19. Verify Stokes' Theorem for $\mathbf{F} = x^2\, \mathbf{i} + 2x\, \mathbf{j} + z^2\, \mathbf{k}$ and S the surface given by $\{(x, y, z) \colon 4x^2 + y^2 \leq 4, z = 0\}$.

20. Let C be the boundary of $2x + y + 2z = 2$ in the first octant, oriented counterclockwise viewed from above. Let $\mathbf{F} = \langle x + y^2, y + z^2, z + x^2 \rangle$. Find $\int_C \mathbf{F} \cdot ds$.

21. Let $S = \{(x, y, z) \colon z \leq 9 - x^2 - y^2, z \geq 5\}$ with normal vector pointing outwards. Let $\mathbf{F} = yz\, \mathbf{i} + x^2 z\, \mathbf{j} + xy\, \mathbf{k}$. Find

$$\iint_S \nabla \times \mathbf{F} \cdot dS$$

22. Verify Stokes' Theorem for R the portion of the paraboloid above the xy–plane and $\mathbf{F} = (2z - y)\,\mathbf{i} + (x + z)\,\mathbf{j} + (3x - 2y)\,\mathbf{k}$.

23. Find

$$\iint_S \nabla \times \mathbf{F} \cdot dS$$

where S is the surface $S = \{(x, y, z)\colon 1 \le z \le 5 – x^2 – y^2\}$ with outward normal and $\mathbf{F} = \langle z^2, –3xy, x^3y^3\rangle$.

24. Find

$$\oint_{\partial S} \mathbf{F} \cdot ds$$

where $\mathbf{F} = (x + 2y + 3z, x^2 + 2y^2 + 3z^2, x + y + z)$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

25. Evaluate

$$\oint_{\partial S} \mathbf{F} \cdot ds$$

where ∂S is the path $C_1 : \mathbf{x} = (t, 0, 0)$, where $0 \le t \le 2$, followed by $C_2 : \mathbf{x}(t) = 2\cos t(1, 0, 0) + 2\sin t \frac{1}{\sqrt{2}}(0, 1, 1) = (2\cos t, \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}})$ for $0 \le t \le 2\pi$, and finally $C_3 : \mathbf{x}(t) = (0, 2 - t, 2 - t)$, where $0 \le t \le 2$ and $\mathbf{F} = (z - y)\,\mathbf{i} - (x + z)\,\mathbf{j} - (x + y)\,\mathbf{k}$.

26. Find $\iint_S \nabla \times \mathbf{F} \cdot dS$, where $S = S_1 \cup S_2$, where $S_1 = \{(x, y, z)\colon x^2 + y^2 = 9, 0 \le z \le 8\}$ and $S_2 = \{(x, y, z)\colon x^2 + y^2 + (z - 8)^2 = 9, z \ge 8\}$ and $\mathbf{F} = (x^3 + xz + yz^2)\,\mathbf{i} + (xyz^3 + y^7)\,\mathbf{j} + x^2z^5\,\mathbf{k}$.

27. Calculate

$$\oint_S \nabla \times \mathbf{F} \cdot n\, dS$$

where $\mathbf{F} = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$ and S is the triangle with vertives $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$.

28. Verify that Stokes' Theorem implies Green's Theorem.

29. Find the word done by the vector field $\mathbf{F} = \langle x + z^2, y + x^2, z + y^2\rangle$ on a particle moving around the edge of the sphere of radius 2 centered at the origin lying in the first octant, oriented outwards.

30. Evaluate

$$\oint_{\partial S} \mathbf{F} \cdot ds$$

where $\mathbf{F} = \left\langle e^{-x^2} + \sin\ln(x^2 + 1) - y + z, \sin y^2 - \sqrt{1 + y^4} + 2x + z, x - y - e^z + \tan\sqrt[3]{x}\right\rangle$ and ∂S is the intersection of $x^2 + y^2 = 16$ and $z = 2x + 4y$, oriented counterclockwise viewed from above.

31. Show that $\mathbf{x}(t) = (\cos t, \sin t, \sin 2t)$ lines on the surface $z = 2xy$ and evaluate

$$\oint_S (y^3 + \cos x)\, dx + (\sin y + z^2)\, dy + x\, dz$$

where C is closed curve parametrized and oriented by the path $\mathbf{x}(t)$.

32. Calculate $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot dS$, where $\mathbf{F} = (e^{y+z} - 2y)\,\mathbf{i} + (xe^{y+z} + y)\,\mathbf{j} + e^{x+y}\,\mathbf{k}$ and S is the surface $z = e^{-(x^2+y^2)}$ and $z \geq 1/e$. [Hint: Stokes' Theorem works for *any* surface with appropriate boundary.]