Calculus III Problem Book

Caleb G. McWhorter

SU

2017

Contents

Preface

One can only learn Mathematics by doing Mathematics. It is then be necessary to solve problems—lots of them! There are 799 problems in this text. The new student to Multivariable Calculus should try to solve as many as possible. However, solving problems is not enough. Trying every problem 'type' in Calculus could be a lifetime journey. Treat the problems as small lights in the dark, illuminating the paths connecting different concepts. Whenever possible, the student should have in mind the connection between the calculus being performed and the underlying geometry. The problems throughout this text—even the subject itself—cannot be separated from underlying geometrical concepts. There is space before each problem section for brief topic notes for reference.

As for texts, the author strongly suggests *Vector Calculus* by Colley or *Calculus* by Larson and Edwards. These were a common reference when considering what problem types to integrate into the text. The problems themselves were written and compiled by the author from lecture notes of previous iterations of the course. Accordingly, these notes could have been taken or supplemented by sources the author has since forgotten. If the author has seemingly missed a reference or has committed any other error, please email him at cgmcwhor@syr.edu so that he may rectify his error!

Chapter 1

Spatial Geometry & Vectors

1.1 Basic n–Euclidean Geometry

Euclidean n–space: Define $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \colon a_i \in \mathbb{R}\}$. The case of $n = 1$ is the familiar real numbers and we drop the parenthesis. We plot these on a number line. The case of $n = 2$ we know as just the set of 'ordinary' points. We gave the familiar graphical representation of points in the coordinate plane. The case of $n = 3$ is the one we will be most interested in – three-space. Note that even if a point is in three–space, we can get a point in two–space via $(\cdot, \cdot, 0)$ and the like.

Coordinate Axes: The axes are normally labeled x, y, z (or with hats). These are also sometimes given the names i, j, k. There are eight octants. We can label the axes in any way but we want them to obey the RHR. It would be good to always draw our axes in this way for now. We can plot points and project onto any of the axes. Give the rectangular projections for a point.

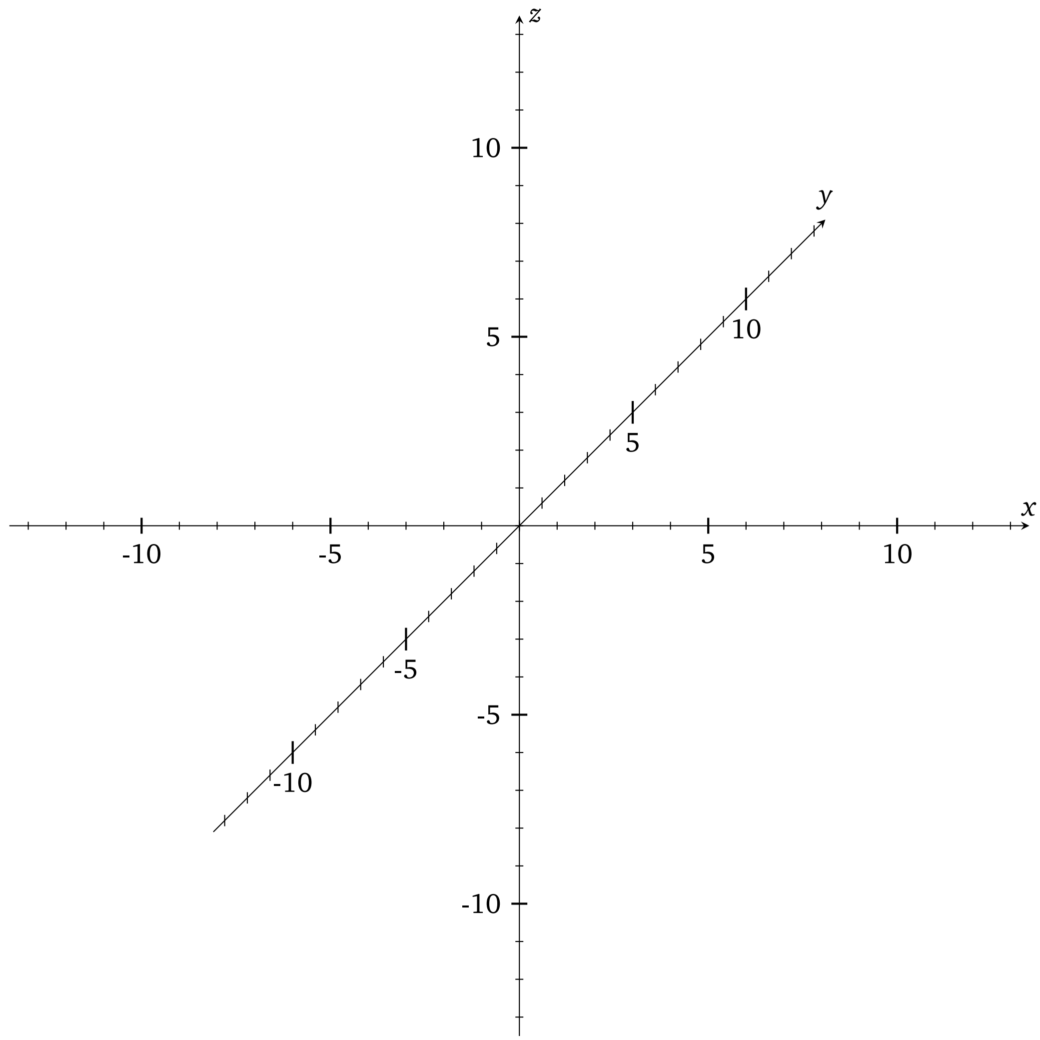
Distance: The distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

We define the distance from a point to a set of points to be the short line segment connecting the chosen point to a point in a set – if it exists. In most cases, this is a perpendicular line segment.

1.1 | Exercises

1. One the set of axes below, sketch the points $(1,2,3)$, $(6,-7,2)$, and $(-3,0,8)$. Choose a point and carefully draw dotted lines connecting the chosen point perpendicularly to the coordinate planes.



2. Draw an appropriate set of coordinate axes – labeled – and plot the points $(3,6,-2)$, $(-5,-5,5)$, and $(0,0,-7)$. Choose one of the first two points and carefully draw dotted lines connecting the chosen point perpendicularly to the coordinate planes.

3. Find the distance between the points $(2,-1,-3)$ and $(4,3,-1)$. Which point is closer to the xy–plane? Which point is closer to the yz–plane?

Ans: $2\sqrt{6}$. $(4,3,-1)$ is closer to the xy–plane and $(2,-1,-3)$ is closer to the yz–plane.

4. Find the distance between the points $(4,5,-2)$ and $(3,1,-1)$. Which points is closer to the xz–plane? Which points is closer to the yz–plane?

Ans: $3\sqrt{2}$. $(3,1,-1)$ is closer to both the xz–plane and yz–plane.

5. Consider a triangle formed by the points $A(1,0,-1)$, $B(1,-2,-1)$ and $C(1,-2,-3)$. Sketch this triangle in 3–space. Determine if the triangle is an isosceles triangle. Determine if the triangle is a right triangle. Determine if the triangle is an equilateral triangle.

Ans: $AB = 2$, $BC = 2$, $AC = 2\sqrt{2}$. So the triangle is isosceles. $\sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ so the triangle is right. The triangle is clearly not equilateral.

6. Consider a triangle formed by the points $M(-1,2,-1)$, $N(-1,2,-3)$, and $P(-1,6,-2)$. Sketch this triangle in 3–space. Determine the the triangle is an isosceles triangle. Determine if the triangle is a right triangle. Determine if the triangle is an equilateral triangle.

Ans: $MN = 2, NP = \sqrt{17}, MP = \sqrt{17}$. The triangle is isosceles. $\sqrt{\sqrt{17}^2+2^2} = \sqrt{17+4} = \sqrt{21} \neq \sqrt{17}$ so the triangle is not right. The triangle is not equilateral.

7. For the point $(3,5,4)$, determine the following:

(a) The distance to the xy–plane. Ans: 4

(b) The distance to the yz–plane. Ans: 3

(c) The distance to the xz–plane. Ans: 5

(d) The distance to the x–axis. Ans: $\sqrt{5^2+4^2} = \sqrt{41}$

(e) The distance to the y–axis. Ans: $\sqrt{3^2+4^2} = 5$

(f) The distance to the z–axis. Ans: $\sqrt{3^2+5^2} = \sqrt{34}$

8. For the point $(-1,4,2)$, determine the following:

(a) The distance to the xy–plane. Ans: 2

(b) The distance to the yz–plane. Ans: 1

(c) The distance to the xz–plane. Ans: 4

(d) The distance to the x–axis. Ans: $\sqrt{4^2+2^2} = 2\sqrt{5}$

(e) The distance to the y–axis. Ans: $\sqrt{1^2+2^2} = \sqrt{5}$

(f) The distance to the z–axis. Ans: $\sqrt{1^2+4^2} = \sqrt{17}$

9. Determine if the following three points lie along a straight line: $A(-5,7,-4)$, $B(1,1,5)$, and $C(-1,3,2)$.

Ans: $\vec{AB} = \langle 6,-6,9\rangle = 3\langle 2,-2,3\rangle$. $\vec{BC} = \langle -2,2,-3\rangle = -1\langle 2,-2,3\rangle$. $\vec{CA} = \langle -4,4,-6\rangle = -2\langle 2,-2,3\rangle$. So they lie along a straight line.

10. Determine if the following three points lie along a straight line: $M(3,-4,2)$, $N(0,-1,8)$, and $P(2,-3,4)$.

Ans: $\vec{MN} = \langle -3,3,6\rangle = -3\langle 1,-1,-2\rangle$. $\vec{NP} = \langle 2,-2,-4\rangle = 2\langle 1,-1,-2\rangle$. $\vec{PM} = \langle 1,-1,-2\rangle$. So the points lie along a straight line.

11. Find at least 6 points that have distance 3 from the point $(1,-2,6)$. What shape does the set of all points having distance 3 from the points $(1,-2,6)$ make? Sketch the shape, the points found, and the given point in the same plot.

Ans: $(4,-2,6)$, $(-2,-2,6)$, $(1,1,6)$, $(1,-5,6)$, $(1,-2,9)$, $(1,-2,3)$. The shape is a sphere.

12. Find at least 6 points that have distance 4 from the point $(2,0,-5)$. What shape does the set of all points having distance 4 from the points $(2,0,-5)$ make? Sketch the shape, the points found, and the given point in the same plot.

Ans: $(6,0,-5)$, $(-2,0,-5)$, $(2,4,-5)$, $(2,-4,-5)$, $(2,0,-1)$, $(2,0,-9)$. The shape is a sphere.

13. Show that the midpoint of the line segment connecting the points $P_1(x_1, y_1, z_1)$ and $P(x_2, y_2, z_2)$ is given by

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Ans: Let M denote the proposed midpoint.

$$\begin{aligned}
\vec{MP_1} &= \left\langle \frac{x_1-x_2}{2}, \frac{y_1-y_2}{2}, \frac{z_1-z_2}{2} \right\rangle \\
\vec{MP_2} &= -\left\langle \frac{x_1-x_2}{2}, \frac{y_1-y_2}{2}, \frac{z_1-z_2}{2} \right\rangle \\
|MP_1| &= \frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2} \\
|MP_2| &= \frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}
\end{aligned}$$

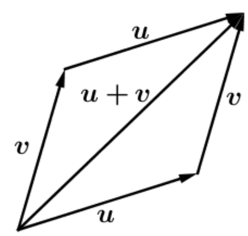
The first two calculations show that M lies along the same line. The second two calculations show that M is indeed the midpoint.

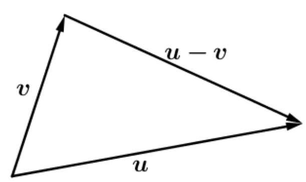
1.2 Introduction to Vectors

Vector: This term has many meanings in various contexts. A vector is a set of 'coordinates' that indicate a direction. These have a magnitude and direction. They can originate at any point. We can endow them with an inner product which actually induces a metric – hence why they have a magnitude (length). Vectors only indicate a direction and they can emanate from any point, though we often draw them from the origin. Note even if a vector is in $\mathbb{R}^3$, we get a vector in '$\mathbb{R}^2$' via $\langle \cdot, \cdot, 0\rangle$ and the like.

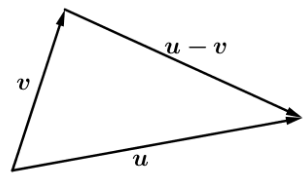
Displacement Vector: A particular vector formed by joining two points. To the vector formed by joining the point P, the initial point, to the point Q, the terminal point, is written $\vec{PQ}$. Note the arrow goes toward Q. So if $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$.

Triangle/Parallelogram Law: How to add vectors. Note that we start wherever we start. The first vector moves us the proper amount, then the second by the next proper amount. This gives the tail-to-tip method (the Triangle Law). The addition should be commutative so we get the Parallelogram Law.





Scalar Multiple: Scales the length of a vector. The sign of the scalar determines if the direction flips. This allows us to do subtraction: negative one vector then add. Or we can do it by putting the vectors tail-to-tail then connecting in the 'opposite' direction.



Parallel/Equal Vectors: Two vectors are parallel if one is a multiple of another; that is, once extended, they create parallel lines. Note they can point in opposite directions. Two vectors are equal if $\mathbf{u}-\mathbf{v}$ is the zero vector (same length and point in same direction).

Length: $\|v\| = \|\langle x, y, z\rangle\| = \sqrt{x^2 + y^2 + z^2}$. Same idea works in any dimension.

Unit Vector: A vector with length 1. Note that any nonzero vector v can be 'turned into' a vector of length 1 via $\frac{v}{\|v\|}$.

Standard Basis Vectors: $\hat{x}/\hat{i} = \langle 1, 0, 0\rangle$, $\hat{y}/\hat{j} = \langle 0, 1, 0\rangle$, $\hat{z}/\hat{k} = \langle 0, 0, 1\rangle$. Note we get all other vectors via $v = v_1\,\hat{i} + v_2\,\hat{j} + v_3\,\hat{k}$.

1.2 | Exercises

1. Show that $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$.

Ans: $\|c\mathbf{v}\| = \sqrt{\sum_{i=1}^{n}(cv_i)^2} = |c|\sqrt{\sum_{i=1}^{n} v_i^2} = |c|\,\|\mathbf{v}\|$.

2. Show that if $\mathbf{v}$ is a nonzero vector then $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

Ans: $\left\| \dfrac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \dfrac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1.$

3. Given the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ below, find

(a) $\mathbf{u}+\mathbf{v}$

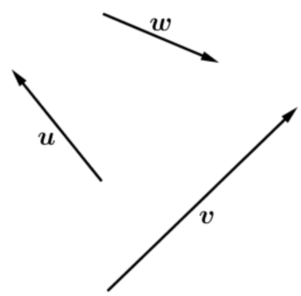
(b) $\mathbf{u}+\mathbf{w}$

(c) $\mathbf{u}-\mathbf{v}$

(d) $2\mathbf{v}$

(e) $-\frac{1}{2}\mathbf{w}$

(f) $\mathbf{u}+\mathbf{v}-\mathbf{w}$



4. Given the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ below, find

(a) $\mathbf{u}-\mathbf{v}$

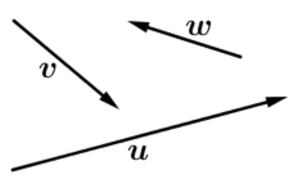
(b) $\mathbf{u}+\mathbf{v}$

(c) $\mathbf{u}-\mathbf{w}$

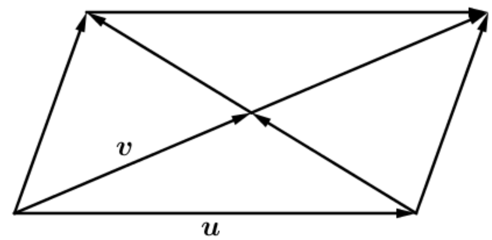
(d) $-2\mathbf{v}$

(e) $\frac{1}{2}\mathbf{w}$

(f) $\mathbf{v}+\mathbf{u}-\mathbf{w}$



5. Given the partially labeled parallelogram below, label all other vectors in the parallelogram.



6. Find and sketch the displacement vector between the two given points:

(a) $P(-1,1)$, $Q(3,-1)$ Ans: $\langle 4,-2\rangle = 2\langle 2,-1\rangle$

(b) $M(2,1)$, $N(3,5)$ Ans: $\langle 1,4\rangle$

(c) $A(4,-1)$, $B(0,4)$
Ans: $\langle -4,5\rangle$

7. Find and sketch the displacement vector between the two given points:

(a) $P(2,1,1)$, $Q(3,0,-1)$ Ans: $\langle 1,-1,-2\rangle$

(b) $M(4,2,1)$, $N(0,3,5)$ Ans: $\langle -4,1,4\rangle$

(c) $A(4,-1,1)$, $B(-1,4,-1)$
Ans: $\langle -5,5,-2\rangle$

8. Given $\mathbf{u} = \langle 1,-2\rangle$ and $\mathbf{v} = \langle 3,2\rangle$, find the following:

(a) $3\mathbf{u}$ Ans: $\langle 3,-6\rangle$

(b) $\mathbf{u}+\mathbf{v}$ Ans: $\langle 4,0\rangle$

(c) $\mathbf{u}-\mathbf{v}$ Ans: $\langle -2,-4\rangle$

(d) $2\mathbf{u}-3\mathbf{v}$ Ans: $\langle -7,-10\rangle$

(e) $||\mathbf{u}||$ Ans: $\sqrt{5}$

(f) $||\mathbf{u}+\mathbf{v}||$
Ans: 4

9. Given $\mathbf{u} = \langle 2,-1,1\rangle$ and $\mathbf{v} = \langle 3,0,1\rangle$, find the following:

(a) $-2\mathbf{u}$ Ans: $\langle -4,2,-2\rangle$

(b) $\mathbf{u}-\mathbf{v}$ Ans: $\langle -1,-1,0\rangle$

(c) $2\mathbf{u}+\mathbf{v}$ Ans: $\langle 7,-2,3\rangle$

(d) $3\mathbf{u}-\mathbf{v}$ Ans: $\langle 3,-3,2\rangle$

(e) $||\mathbf{u}||$ Ans: $\sqrt{6}$

(f) $||\mathbf{u}-\mathbf{v}||$
Ans: $\sqrt{2}$

10. Describe geometrically the collection of points $r\langle 1,3\rangle + s\langle 2,1\rangle$, where r and s are integers.
Ans: A lattice formed by integer multiples of the vectors.

11. Find a unit vector in the same direction as $3\mathbf{i}-4\mathbf{j}$.
Ans: $\frac{1}{5}\langle 3,4\rangle$

12. Find a unit vector in the same direction as $-5\mathbf{i}+12\mathbf{j}$.
Ans: $\frac{1}{13}\langle -5,12\rangle$

13. Find a unit vector in the same direction as $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$.
Ans: $\frac{1}{\sqrt{14}}\langle 2,-3,1\rangle$

14. Find a unit vector that points in the 'opposite' direction as $2\mathbf{i}-3\mathbf{j}$.
Ans: $\frac{1}{\sqrt{13}}\langle 2,-3\rangle$

15. Find a unit vector in the 'opposite direction' as $-2\mathbf{i}+5\mathbf{k}$.
Ans: $\frac{1}{\sqrt{29}}\langle -2,0,5\rangle$

16. Find the angle between the given vector and the x–axis and the y–axis: $2\mathbf{i}-2\sqrt{3}\mathbf{j}$.
Ans: x–axis: $\arccos(1/2)=\frac{\pi}{3}=60°$. y–axis: $\arccos(-\sqrt{3}/2)=\frac{5\pi}{6}=150°$

17. Find the angle between the given vector and the x–axis and the y–axis: $\dfrac{9\mathbf{i}+9\mathbf{j}}{\sqrt{2}}$.
Ans: x–axis: $\arccos(1/\sqrt{2})=\frac{\pi}{4}=45°$. y–axis: $\arccos(1/\sqrt{2})=\frac{\pi}{4}=45°$

18. Find the angle between the given vector and the x–axis and the y–axis: $2\mathbf{i}+5\mathbf{j}$.
Ans: x–axis: $\arccos(2/\sqrt{29})=68.1986°$. y–axis: $\arccos(5/\sqrt{29})=21.8014°$

19. If a vector in the plane has length 3 and makes angle $\frac{\pi}{3}$ with the positive x–axis, find the vector.
Ans: $\langle 3/2,3\sqrt{3}/2\rangle = 3\langle 1/2,\sqrt{3}/2\rangle$

20. If a vector in the plane has length 5 and makes angle $\frac{\pi}{3}$ with the negative y–axis, find the vector.
Ans: $\langle 5\sqrt{3}/2,-5/2\rangle = 5\langle \sqrt{3}/2,-1/2\rangle$

21. A rocket launches from a launch pad traveling 6000 mph east, 10000 mph north, and 4000 mph vertically. In 30 minutes, how high will the rocket be off the ground? How far East will it be? How far from the launch pad will it be? How far would you have to drive from the launch pad to look up and see the rocket straight above you?
Ans: 2000 mi. 3000 mi. 12,328.828 mi. 11,661.90378 mi (ignoring Earth curvature)

22. Complete the following parts:

(a) Write the chemical equation $CO+H_2O=H_2+CO_2$ as an equation in ordered triples. Represent it as a vector in space.

(b) Write the chemical equation $pC_3H_4O_3+qO_2=rCO_2+sH_2O$ as an equation in ordered triples with unknown coefficients.

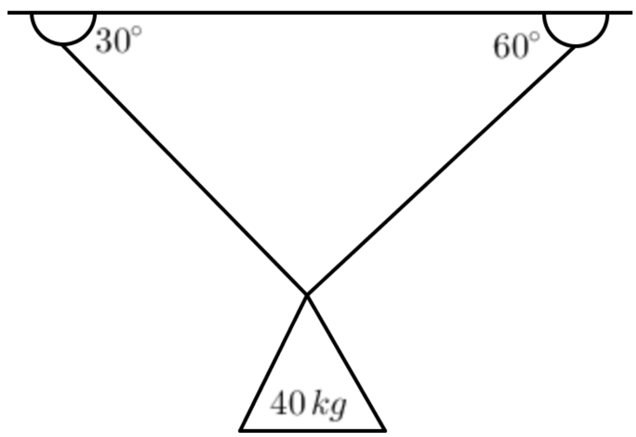
(c) Find the smallest possible integer solution for p, q, r, and s.

(d) Demonstrate the solution by plotting it in space. What are the other possible solutions? How do they relate geometrically to the vector solution you found?

23. If **u** and **v** are vectors, describe the set of points inside the parallelogram spanned by **u** and **v**. What if the vectors originate at the point (p_1, p_2, p_3)?

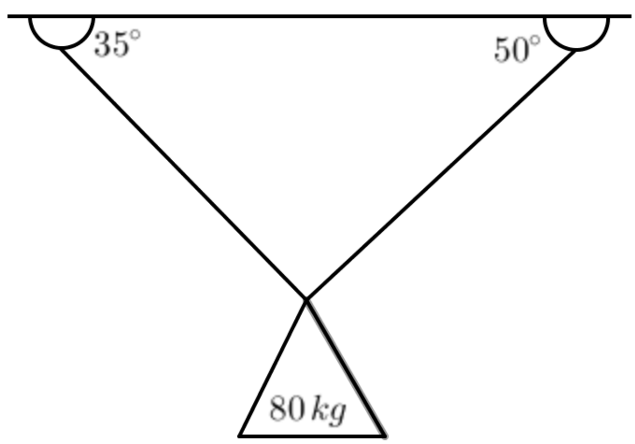
Ans: The points inside the parallelogram are of the form $s\mathbf{u} + t\mathbf{v}$, where $0 \leq s, t \leq 1$. If they originate at the point, the points are of the form $\langle p_1, p_2, p_3 \rangle + s\mathbf{u} + t\mathbf{v}$.

24. Find the tension in each wire in the diagram below.



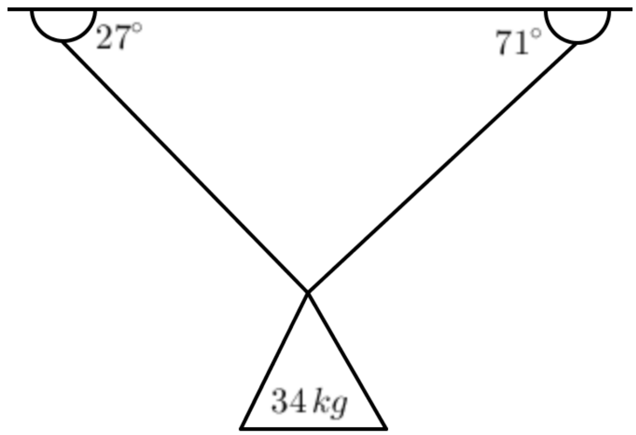
Ans: Left: 364. Right: 630.466

25. Find the tension in each wire in the diagram below.



Ans: Left: 1808.02. Right: 2304.09

26. Find the tension in each wire in the diagram below.

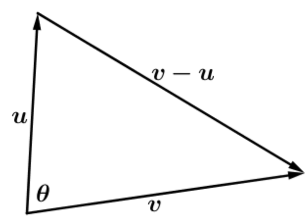


Ans: Left: 145.008. Right: 396.853

1.3 Dot Product

Dot Product: We have many operations for vectors thus far but we have no way to multiply vectors. The dot product serves as a sort of multiplication for vectors. The dot product of two vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$ is given by $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$. Note that the dot product of two vectors gives a scalar. Give the properties of the dot product. Note that $\mathbf{u} \cdot \mathbf{u} = \|u\|^2$.

Angle between Vectors: The dot product allows us to define the angle between two vectors. Observe the following diagram: Recognize the sizes as the length of the vectors: Law of Cosines gives $|\mathbf{v} - \mathbf{u}|^2 =$



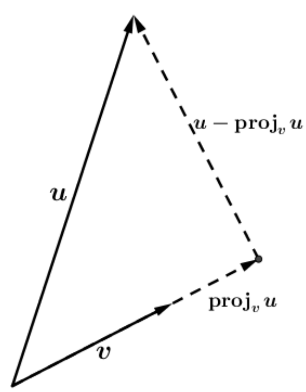
$|\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos\theta$. Now expand the left side as a dot product and we obtain $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$. Then we can define the angle between vectors as $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$. Solving for θ is a simple matter of inverse functions. This works in any dimension.

Test for Orthogonality: Two vectors are orthogonal if and only if they are perpendicular. The dot product gives a way of testing for this. So $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Projection: The projection of $\mathbf{u}$ onto $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\, \mathbf{u}$, is the vector from the tail of $\mathbf{v}$ to the intersection of the perpendicular line from the tip of $\mathbf{u}$ with the line formed by extending $\mathbf{v}$. That is, the 'shadow' cast on $\mathbf{v}$ by $\mathbf{u}$. You can project even if the resulting projection is longer than $\mathbf{v}$ or if $\mathbf{u}$ and $\mathbf{v}$ do not point in the same direction. Using simple right triangle trig and multiplication by 1, we obtain

$$|\text{proj}_{\mathbf{v}}\, \mathbf{u}| = \frac{|\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|}$$

Note the top is the absolute value (resulting from the fact that the angle could be between $\pi/2$ and π) while the bottom is length. The book calls this length $\text{comp}_{\mathbf{v}}\, \mathbf{u}$ – which we shall not use. Note that the vector you are projecting onto appears most often. If we want the projection *vector*, note we want it to point in the direction of $\mathbf{v}$. But we don't want to change the length so we multiply by a unit vector: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$. Then we have $\text{proj}_{\mathbf{v}}\, \mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$. Note that projections allow us to form and fill in *right* triangles using any two nonparallel vectors



1.3 | Exercises

1. Determine which of the following are meaningful expressions:

(a) $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$ Ans: Not meaningful

(b) $(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})$ Ans: Meaningful

(c) $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}$ Ans: Not meaningful

(d) $(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$ Ans: Meaningful

(e) $|\mathbf{v}|\,(\mathbf{u}\cdot\mathbf{w})$ Ans: Meaningful

(f) $|\mathbf{u}|\cdot(\mathbf{v}\cdot\mathbf{w})$ Ans: Not meaningful

2. Given $\mathbf{u}=\langle 2,-1,3\rangle$ and $\mathbf{v}=\langle 1,0,-2\rangle$, find

(a) $\mathbf{u}\cdot\mathbf{u}$ Ans: 14

(b) $|\mathbf{u}|$ Ans: $\sqrt{14}$

(c) $\mathbf{u}\cdot\mathbf{v}$ Ans: -4

(d) $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ Ans: $\frac{1}{5}\langle -4,0,8\rangle$

3. Given $\mathbf{u}=\langle 3,-5,1\rangle$ and $\mathbf{v}=\langle 2,-2,1\rangle$, find

(a) $\mathbf{v}\cdot\mathbf{v}$ Ans: 9

(b) $|\mathbf{u}|$ Ans: $\sqrt{35}$

(c) $\mathrm{proj}_{\mathbf{u}}\,\mathbf{v}$ Ans: $\langle 51/35,-17/7,17/35\rangle$

(d) $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ Ans: $\frac{17}{9}\langle 2,-2,1\rangle$

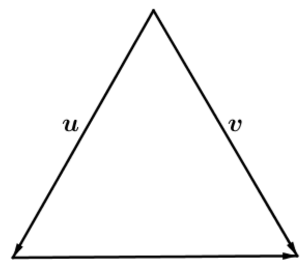
4. If $\mathbf{u}=\langle 1,2,3\rangle$ and $\mathbf{v}=\langle 3,4,5\rangle$, what is the angle between $\mathbf{u},\mathbf{v}$? Sketch these vectors and the angle between them.
Ans: $\arccos(13/(5\sqrt{7}))=10.6707°$

5. If $\mathbf{u}=\langle -1,2,0\rangle$ and $\mathbf{v}=\langle 2,0,-3\rangle$, what is the angle between $\mathbf{u},\mathbf{v}$? Sketch these vectors and the angle between them.
Ans: $\arccos(-2/\sqrt{65})=104.363°$

6. If $|\mathbf{u}|=2$, $|\mathbf{v}|=3$, and the angle between them is $\pi/6$, what is $\mathbf{u}\cdot\mathbf{v}$?
Ans: $3\sqrt{3}$

7. If $|\mathbf{u}|=4$, $|\mathbf{v}|=\sqrt{2}$, and the angle between them is $4\pi/3$, what is $\mathbf{u}\cdot\mathbf{v}$?
Ans: $-2\sqrt{2}$

8. Given the equilateral triangle below (each side is length 3), place an appropriate vector to label the other side and find $\mathbf{u}\cdot\mathbf{v}$.
Ans: 9/2



9. Recall given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can form a right triangle using the projection $\text{proj}_{\mathbf{v}}\,\mathbf{u}$. Show that $\mathbf{u}-\text{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{v}$.

Ans: $(\mathbf{u}-\text{proj}_{\mathbf{v}}\,\mathbf{u})\cdot\mathbf{v}=(\mathbf{u}-\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v})\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{|v|^2}\mathbf{v}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}=0$

10. If $\mathbf{u}=\langle 3,0,4\rangle$, find a vector $\mathbf{v}$ such that $|\text{proj}_{\mathbf{u}}\,\mathbf{v}|=\frac{1}{5}$.

Ans: If $\mathbf{v}=\langle a,b,c\rangle$, then $3a+4c=1$ so $\mathbf{v}=\langle 1,0,-1\rangle$ works.

11. A truck drags a wood pallet across the ground. The rope attaching the pallet to the truck makes an angle of $\pi/6$ with the ground and the tension in the rope is 1000 N. How much work does the truck do pulling the pallet 3 km?

Ans: $1,500,000\sqrt{3}$ J

12. A person pulls a sled along the ground. The tension in the rope is 10 N and the rope makes an angle of $\pi/4$ with the ground. What is the work done pulling the sled 20 m?

Ans: $100\sqrt{2}$

13. Find the acute angles between the curves $y=x^2-3x-1$ and $y=4x-11$. [The angle is defined to be the angle between their tangents at the point.]

Ans: Intersection: $(2,-3)$. Angle: $\arccos(5/\sqrt{34})=30.9638°$. Intersection: $(5,9)$. Angle: $\arccos(29/(5\sqrt{34}))=5.90614°$

14. Find the acute angles between the curves $y=x^2-8x+21$ and $y=5$ at their points of intersection. [The angle is defined to be the angle between their tangents at the point.]

Ans: $\arccos(0)=0°$

15. Find the acute angles between the curves $y=x^3+3$ and $y=x^2+4x-1$t their points of intersection. [The angle is defined to be the angle between their tangents at the point.]

Ans: Intersection: $(-2,-5)$. Angle: $\arccos(1/\sqrt{145})=85.2364°$. Intersection: $(1,4)$. Angle: $\arccos(19/\sqrt{370})=8.97263°$. Intersection: $(2,11)$. Angle: $\arccos(97/(5\sqrt{377}))=2.36137°$.

16. Find the acute angle between the curves $y=\sin\theta$ and $y=\cos\theta$ at the smallest positive θ value of intersection. [The angle is defined to be the angle between their tangents at the point.]

Ans: Intersection: $(\pi/4,1/\sqrt{2})$. Angle: $\arccos(1/3)=70.5288°$

17. Find the angle between the diagonal and an adjacent edge in a cube. Ans: One edge is $\langle 1,0,0\rangle$ and the diagonal is $\langle 1,1,1\rangle$. The angle between them is $\arccos(1/\sqrt{3})=54.7356°$.

18. Find the work done by a force given by $\mathbf{F} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ moving an object at the point $(1,0,-2)$ in a straight line to the point $(5,6,7)$?

Ans: 29 J

19. Find the work done by a force given by $\mathbf{F} = 3\mathbf{i} - 5\mathbf{k}$ moving an object at the point $(2,2,0)$ in a straight line to the point $(2,-3,4)$?

Ans: -20 J

20. Show that if $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal, then $|\mathbf{u}| = |\mathbf{v}|$.

Ans: $0 = (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$ so that $|\mathbf{v}|^2 = |\mathbf{v}|^2$ so that $|\mathbf{u}| = |\mathbf{v}|$.

21. Is it possible for $\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \operatorname{proj}_{\mathbf{a}} \mathbf{b}$? If so, under what conditions is it true?

Ans: If the projection is zero, then $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\operatorname{proj}_{\mathbf{b}\,\mathbf{a}} = \mathbf{0}$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. But then $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. Now if neither are zero, since they are equal as vectors, they must be in the same direction. Then $\mathbf{a} = k\mathbf{b}$. But then

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \operatorname{proj}_{k\mathbf{b}} \mathbf{b} = \frac{k\mathbf{b} \cdot \mathbf{b}}{k\mathbf{b} \cdot k\mathbf{b}} = \mathbf{b}$$
$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \operatorname{proj}_{\mathbf{b}} k\mathbf{b} = \frac{\mathbf{b} \cdot k\mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} = k\mathbf{b}$$

But then $k = 1$ so that $\mathbf{a} = \mathbf{b}$.

22. Suppose an object starts at point P and is pushed to point Q with constant force $\mathbf{F}$. If θ is the angle between the displacement vector, $\mathbf{d}$ and the force vector, show that the work is $\mathbf{F} \cdot \mathbf{d}$.

Ans: Use $\mathbf{F} \cdot \mathbf{d} = |\mathbf{F}|\,|\mathbf{d}| \cos\theta$ and simple geometry.

1.4 Cross Product

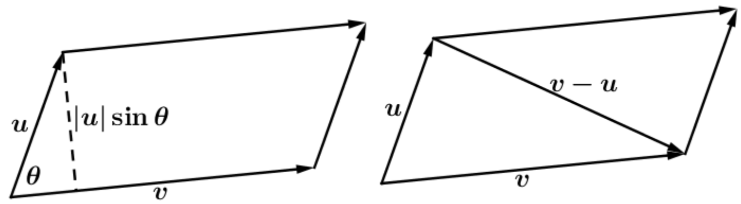
Cross Product: This is another form of vector multiplication. The cross product of the vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \times \mathbf{v}$. The result is a *vector* – unlike the dot product. This vector is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. So whereas before we tested orthogonality with the dot product, we can 'create orthogonality' with the cross product. Give the properties of cross products. You will also want to give the circle diagram with i, j, k and/or x, y, z. Show how we denote into the page and out of the page.

Determinants: Give the formula for the 2×2 determinant and show how to calculate higher ones via cofactor expansion. Show also the 'diagonal trick' with 3×3matrices. Give an example of how the cross product is calculated with determinants. Note best to choose row/column with the most zeros.

Cross Product Formula: $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,\sin\theta$

Test for 'Parallelity': Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. Note the bold zero because the result is a vector not a scalar.

Parallelogram Area: The quantity $|\mathbf{u} \times \mathbf{v}|$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. Note by filling the triangle with the difference of the vectors, we can take this area to find the area of triangles as well.



Volume of Parallelepiped: The volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. If the volume is 0, then these vectors must be coplanar.

Torque: A force $\mathbf{F}$ is acting on a body a vector 'distance' $\mathbf{r}$ from the center of rotation, the torque is $\tau = \mathbf{r} \times \mathbf{F}$.

1.4 | Exercises

1. Calculate the determinant

$$\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix}$$

Ans: 28

2. Calculate the determinant

$$\begin{vmatrix} -4 & 9 \\ 6 & 1 \end{vmatrix}$$

Ans: -58

3. Calculate the determinant

$$\begin{vmatrix} 3 & 6 & -1 \\ 4 & 0 & 3 \\ 5 & 5 & 1 \end{vmatrix}$$

Ans: 1

4. Calculate the determinant

$$\begin{vmatrix} -1 & -1 & -1 \\ 2 & 2 & -5 \\ 4 & 6 & 4 \end{vmatrix}$$

Ans: -14

5. Calculate the determinant

$$\begin{vmatrix} 1 & -3 & 4 & 1 & 0 \\ 0 & 4 & -2 & -2 & 6 \\ 7 & 1 & 3 & 1 & 1 \\ -2 & 0 & 4 & 5 & 4 \\ 3 & 4 & 0 & -1 & -4 \end{vmatrix}$$

Ans: 4188

6. Given $\mathbf{u} = \langle 1, 3, 0 \rangle$ and $\mathbf{v} = \langle -2, 5, 0 \rangle$, find $\mathbf{u} \times \mathbf{v}$.
Ans: $\langle 0, 0, 11 \rangle$

7. Given $\mathbf{u} = -5\mathbf{i} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{j} + 4\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.
Ans: $\langle -12, 20, -20 \rangle$

8. Given $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 3, 4, 5 \rangle$, find $\mathbf{u} \times \mathbf{v}$.
Ans: $\langle -2, 4, -2 \rangle$

9. Given $\mathbf{u} = \mathbf{k} - 6\,\mathbf{i}$ and $\mathbf{v} = \mathbf{k} - 2\mathbf{i} - 2\mathbf{j}$, find $\mathbf{u} \times \mathbf{v}$.
Ans: $\langle 2, 4, 12 \rangle$

10. Without using the determinant, calculate $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$ and $\mathbf{k} \times (\mathbf{j} \times \mathbf{j})$.

Ans: Both are $\mathbf{0}$

11. Without using the determinant, calculate $(\mathbf{j}-\mathbf{k}) \times (\mathbf{k}-\mathbf{i})$.
Ans: $\mathbf{i}+\mathbf{j}+\mathbf{k}$

12. If $\mathbf{u} \times \mathbf{v} = 3\mathbf{i}-7\mathbf{j}-2\mathbf{k}$, find $(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v})$.
Ans: $(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v}) = \mathbf{u} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} - \mathbf{u} \times \mathbf{v} - \mathbf{v} \times \mathbf{v} = -2(\mathbf{u} \times \mathbf{v}) = \langle -6, 14, 4 \rangle$.

13. Find two unit vectors perpendicular to both $\langle 1, 0, -2 \rangle$ and $\langle 3, 3, 1 \rangle$.
Ans: $\frac{\pm 1}{\sqrt{94}} \langle 6, -7, 3 \rangle$

14. Find two unit vectors perpendicular to both $\langle 5, 1, 2 \rangle$ and $\langle -2, -2, 6 \rangle$.
Ans: $\frac{\pm 1}{2\sqrt{330}} \langle 10, -34, -8 \rangle$

15. Find the area of the parallelogram spanned by the vectors $\langle 2, 3 \rangle$ and $\langle -3, 5 \rangle$.
Ans: 19

16. Find the area of the parallelogram spanned by the vectors $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $5\mathbf{k}-3\mathbf{i}$.
Ans: $\sqrt{155}$

17. Calculate the area of the parallelogram having vertices $(1, 1)$, $(3, 2)$, $(1, 3)$, and $(-1, 2)$.
Ans: 4

18. Calculate the area of the parallelogram having vertices $(1, 2, 3)$, $(4, -2, 1)$, $(-3, 1, 0)$, and $(0, -3, -2)$.
Ans: $5\sqrt{30}$

19. Find the volume of the parallelepiped determined by $\mathbf{u} = 2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{v} = 2\mathbf{j}-3\mathbf{k}$, and $\mathbf{w} = \mathbf{i}+\mathbf{k}$.
Ans: 7

20. Find the volume of the parallelepiped having vertices $(3, 0, -1)$, $(4, 2, -1)$, $(-1, 1, 0)$, $(3, 1, 5)$, $(0, 3, 0)$, $(4, 3, 5)$, $(-1, 2, 6)$, and $(0, 4, 6)$.
Ans: 53

21. Determine if the vectors $\mathbf{i}+5\mathbf{j}-2\mathbf{k}$, $3\mathbf{i}-\mathbf{j}$, and $5\mathbf{i}+9\mathbf{j}-4\mathbf{k}$ are coplanar.
Ans: The vectors are coplanar.

22. Find the area of the triangle having vertices $(0, 1, 2)$, $(3, 4, 5)$, and $(-1, -1, 0)$.
Ans: $3/\sqrt{2}$

23. Find the area of the triangle having vertices $(-1, -1, 0)$, $(2, 3, 4)$, and $(5, 6, 1)$.
Ans: $3\sqrt{57/2}$

24. Use the cross product of $\langle \cos\theta, \sin\theta \rangle$ and $\langle \cos\phi, \sin\phi \rangle$ to show that

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

Ans: Treating the vectors as in $\mathbb{R}^3$ and computing the cross product yields $\sin\theta\cos\phi - \cos\theta\sin\phi$. But we also have the formula $|\mathbf{u}|\,|\mathbf{v}|\sin\omega$, where ω is the angle between them. Both are unit vectors and the angle between them is $\theta - \phi$.

25. If $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = 0$, what is the geometric relationship between $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Ans: The parallelepiped formed by the vectors has no volume. Hence, the vectors must be coplanar.

26. Show that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

Ans: This is mere computation. However, $\mathbf{u}\times\mathbf{v}$ is a vector perpendicular to $\mathbf{v}$ so the dot product must be 0.

27. Show that

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Ans: This is just brute computation.

28. Show that a triangle with vertices $P(x_1, y_1)$, $Q(x_2, y_2)$, and $R(x_3, y_3)$ is given by absolute value of

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Ans: This is just brute computation.

29. Show that

$$\begin{vmatrix} x^2 & y^2 & z^2 \\ 2x & 2y & 2z \\ 2 & 2 & 2 \end{vmatrix} \neq 0$$

This is a simple example of a Wronskian, which can determine if a collection of functions is linearly independent or not.

Ans: This is just brute computation.

30. If the vertices of a parallelogram are (listed in order) $(1,0,2)$, $(1,4,3)$, $(2,1,4)$, and $(2,-3,3)$, find the area of the parallelogram. Find the area of the parallelogram projected to the xy–plane, to the xz–plane?

Ans: One could use the formula that the area of a triangle formed by $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}\sqrt{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}$. We form the vectors using the first and second point and first and third point. These are $\langle 0,4,1\rangle$ and $\langle 1,-3,1\rangle$. Then the area is the magnitude of the cross product, which is $\langle 7,1,-4\rangle$, which is $\sqrt{66}$. The projection to the xy–plane is $(1,0,0)$, $(1,4,0)$, $(2,1,0)$, and $(2,-3,0)$, with vectors $\langle 0,4,0\rangle$ and $\langle 1,-3,0\rangle$, with cross product $\langle 0,0,-4\rangle$, which has magnitude 4. The projection to the xz–plane has points $(1,0,2)$, $(1,0,3)$, $(2,0,4)$, and $(2,0,3)$, with vectors $\langle 0,0,1\rangle$, $\langle 1,0,1\rangle$, with cross product $\langle 0,1,0\rangle$, with norm 1.

31. Show that

$$(\mathbf{u}-\mathbf{v})\times(\mathbf{u}+\mathbf{v}) = 2(\mathbf{u}\times\mathbf{v})$$

Ans: $(\mathbf{u}-\mathbf{v})\times(\mathbf{u}+\mathbf{v}) = \mathbf{u}\times\mathbf{u}-\mathbf{v}\times\mathbf{v}+\mathbf{u}\times\mathbf{v}-\mathbf{v}\times\mathbf{u} = 2(\mathbf{u}\times\mathbf{v})$

32. Show that

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$$

Ans: This is just '*brutual*' computation.

33. Prove the *Jacobi Identity*:

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}+(\mathbf{v}\times\mathbf{w})\times\mathbf{u}+(\mathbf{w}\times\mathbf{u})\times\mathbf{v} = \mathbf{0}$$

Ans:

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}+(\mathbf{v}\times\mathbf{w})\times\mathbf{u}+(\mathbf{w}\times\mathbf{u})\times\mathbf{v} = ((\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u})+((\mathbf{v}\cdot\mathbf{u})\mathbf{w}-(\mathbf{w}\cdot\mathbf{u})\mathbf{v})+((\mathbf{w}\cdot\mathbf{v})\mathbf{u}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}) = \mathbf{0}$$

34. Show that

$$|\mathbf{u}\times\mathbf{v}|^2 = |\mathbf{u}|^2\,|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2$$

Ans: This is just direct computation.

35. Show that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

Ans: We know $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b}))$. We also know $\mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b})) = -\mathbf{c}\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{d})$. And then

$$-\mathbf{c}\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{d}) = -\mathbf{c}\cdot((\mathbf{a}\cdot\mathbf{d})\mathbf{b}-(\mathbf{b}\cdot\mathbf{d})\mathbf{a}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

36. Find a formula for the surface area of the tetrahedron formed by **a**, **b**, and **c**.
Ans: $\frac{1}{2}(||\mathbf{a}\times\mathbf{b}||+||\mathbf{b}\times\mathbf{c}||+||\mathbf{a}\times\mathbf{c}||+||(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})||)$

37. Let $\mathbf{r}_1,\ldots,\mathbf{r}_n$ be the vectors connecting the origin to masses $m_1,\ldots,m_n$. The center of mass of the collection of these masses is

$$\mathbf{c} = \frac{\sum_{i=1}^n m_i\mathbf{r}_i}{M},$$

where $M = \sum_{i=1}^n m_i$. Show that for any vector $\mathbf{r}$,

$$\sum_{i=1}^n m_i||\mathbf{r}-\mathbf{r}_i||^2 = \sum_{i=1}^n m_i||\mathbf{r}_i-\mathbf{c}||^2+M||\mathbf{r}-\mathbf{c}||^2$$

1.5 Functions in n–Space

Function: Ordinary one-input, one-output definition. Function of n–variables 'looks like' $f(x_1, x_2, \ldots, x_n)$.

Domain: Set of possible inputs to a function.

Partial Function: A function holding some variables constant. This could not be done for one–variable functions.

Parametric Equations: A function of one or more variables determine by a system of independent variables, called parameters. Note the following special ones:

Line Segment: $\mathbf{p}_1 t + \mathbf{p}_0(1-t); 0 \leq t \leq 1$

Circle: $\langle r\cos\theta, rr\sin\theta\rangle; 0 \leq \theta \leq 2\pi$. Note that the other way parametrizes a circle too but in the clockwise direction.

Ellipse: $\langle a\cos\theta, b\sin\theta\rangle; 0 \leq \theta \leq 2\pi$. Note that the other way parametrizes a circle too but in the clockwise direction.

Helix: $x = a\cos t$, $y = a\sin t$, and $z = bt$. Note this goes around the z–axis.

'Traditional Curves': We can parametrize any function in the plane that we are used to seeing, i.e. $y = x^2$, take $x = t$ and $y = t^2$.

Note parametric equations give many 'spiral' diagrams (spirographs) what you may have created as a kid. Eventually, we will parametrize more than just curves but surfaces as well!

1.5 | Exercises

Functions on n–Space

1. Given the function $f(x,y) = 2x - y$, find $f(3,4)$.
Ans: 2

2. Given the function $f(x,y) = \dfrac{x+y}{xy}$, find $f(-1,3)$.
Ans: $-2/3$

3. Given the function $f(x,y,z) = 2x + 3y - z$, find $f(5,-1,4)$.
Ans: 3

4. Given the function $f(x,y,z) = \dfrac{2x-y+z}{x^2+y^2+z^2}$, find $f(1,1,1)$.
Ans: $2/3$

5. Find the domain of the function $f(x,y) = x + y$. Plot this region.
Ans: The domain is the whole real plane, $\mathbb{R}^2$.

6. Find the domain of the function $f(x,y) = \dfrac{x+y}{x-y}$. Plot this region.
Ans: The whole real plane except the line $y = x$. So the regions above/below this line.

7. Find the domain of the function $f(x,y) = \dfrac{2}{xy}$. Plot this region.
Ans: The whole real plane except $x = 0$ or $y = 0$, i.e. the real plane removing the x–axis and y–axis.

8. Find the domain of the function $f(x,y) = \sqrt{2x - 3y}$. Plot this region.
Ans: The region $2x > 3y$ (equivalently, $2/3x > y$), i.e. the region below the line $y = 2/3x$.

9. Find the domain of the function $f(x,y) = \ln(x^2 - y)$. Plot this region.
Ans: The region $x^2 > y$, i.e. the region above the curve $y = x^2$.

10. Simplify the function $f(x,y) = \dfrac{x^2-y^2}{x-y}$.
Ans: $f(x,y) = x + y$

11. Simplify the function $f(x,y) = \dfrac{x^3+y^3}{x+y}$.
Ans: $f(x,y) = x^2 - xy + y^2$

12. Simplify the function $f(x,y) = \dfrac{x^3y+2y}{y^2+xy}$.
Ans: $f(x,y) = \dfrac{x^3+2}{y+x}$

13. Simplify the function $f(x,y) = \ln(x^2y^2) - \ln(xy)$.

Ans: $f(x, y) = \ln(xy)$

Parametric Equations

14. Give a parametric function for the line segment connecting the points $(1,-3)$ and $(5,2)$. Find $x(t)$ and $y(t)$. Sketch this function.

Ans: $(4t+1, 5t-3)$; $0 \le t \le 1$

15. Give a parametric function for the line segment connecting the points $(5,5)$ and $(-1,0)$. Find $x(t)$ and $y(t)$. Sketch this function.

Ans: $(5-6t, 5-5t)$; $0 \le t \le 1$

16. Give a parametric function for the line segment connecting the points $(-1,0,4)$ and $(4,4,-3)$. Find $x(t)$, $y(t)$, and $z(t)$. Sketch this function.

Ans: $(5t-1, 4t, 4-7t)$; $0 \le t \le 1$

17. Give a parametric function for the line segment connecting the points $(0,1,-2)$ and $(5,-1,3)$. Find $x(t)$, $y(t)$, and $z(t)$. Sketch this function.

Ans: $(5t, 1-2t, 5t-2)$; $0 \le t \le 1$

18. Give a parametric function for a circle of radius 3 centered at the origin. Find $x(t)$ and $y(t)$. Sketch this function.

Ans: $(3\cos t, 3\sin t)$, $0 \le t \le 1$ or $(3\cos 2\pi t, 3\sin 2\pi t)$; $0 \le t \le 1$

19. Give a parametric function for a circle of radius 4 centered at the point $(-1,3)$. Find $x(t)$ and $y(t)$. Sketch this function.

Ans: $(4\cos t - 1, 4\sin t + 3)$; $0 \le t \le 2\pi$ or $(4\cos 2\pi t - 1, 4\sin 2\pi t + 3)$; $0 \le t \le 1$

20. Give a parametric function for a circle of radius 1 centered at the point $(4,4)$. Find $x(t)$ and $y(t)$. Sketch this function.

Ans: $(\cos t + 4, \sin t + 4)$; $0 \le t \le 2\pi$ or $(\cos 2\pi t + 4, \sin 2\pi t + 4)$; $0 \le t \le 1$

21. Give a parametric function for an ellipse centered at the origin with semimajor axis 3 and semiminor axis 2. Find $x(t)$ and $y(t)$. Sketch this function.

Ans: There are many choices. One is $(3\cos t, 2\sin t)$; $0 \le t \le 2\pi$

22. Give a parametric function for an ellipse centered at the point $(-4,3)$ with semimajor axis 5 and semiminor axis 1. Find $x(t)$ and $y(t)$. Sketch this function.

Ans: There are many choices. ONe is $(\cos t - 4, 5\sin t + 3)$; $0 \le t \le 2\pi$

23. Give a parametric function for an ellipse centered at the point $(2,-1)$ with semimajor axis 6 and semiminor axis 4. Find $x(t)$ and $y(t)$. Sketch this function.

Ans: There are many choices. One is $(6\cos t + 2, 4\sin t - 1)$; $0 \le t \le 2\pi$

Vector Valued Functions

24. If $\mathbf{x}(t) = \langle t^2+1, 1-t, 4\rangle$, find $\mathbf{x}(0)$ and $\mathbf{x}(1)$.
Ans: $\langle 1,1,4\rangle$ and $\langle 2,0,4\rangle$

25. If $\mathbf{x}(t) = \cos \pi t\,\mathbf{i} + (t^3 - t + 1)\,\mathbf{j} + \ln t\,\mathbf{k}$, find $\mathbf{x}(1)$.
Ans: $\langle -1,1,0\rangle$

26. If $\mathbf{x}(t) = \langle \arctan 2t, t\cos t, \sqrt{t}\rangle$, find $\mathbf{x}'(t)$.
Ans: $\langle \frac{2}{1+4t^2}, \cos t - t\sin t, \frac{1}{2\sqrt{t}}\rangle$

27. If $\mathbf{x}(t) = (1-t^4)\,\mathbf{i} + te^{2t}\,\mathbf{j} + \frac{1}{\sqrt{t^3}}\,\mathbf{k}$, find $\mathbf{x}(t)$.
Ans: $\langle -4t^3, e^{2t} + 2te^{2t}, -\frac{1}{2t^{5/2}}\rangle$

28. If $\mathbf{a}(t) = \langle 6t, 0\rangle$, $\mathbf{v}(0) = \langle 0,-1\rangle$, and $\mathbf{x}(0) = \langle 4,1\rangle$, find $\mathbf{x}(t)$.
Ans: $\langle t^3+4, 1-t\rangle$

29. If $\mathbf{a}(t) = \langle -\pi^2 \sin \pi t, 6t, \frac{-1}{t^2}\rangle$, $\mathbf{v}(1) = \langle -\pi.2, 1\rangle$, and $\mathbf{x}(1) = \langle 0,1,0\rangle$, find $\mathbf{x}(t)$.
Ans: $\langle \sin \pi t, t^3 - t + 1, \ln t\rangle$

30. If $\mathbf{a}(t) = 2\mathbf{i} + (6t-4)\mathbf{j}$, $\mathbf{v}(0) = \mathbf{0}$, and $\mathbf{x}(0) = \mathbf{i} + 3\mathbf{k}$, find $\mathbf{x}(t)$.
Ans: $\langle t^2+1, t^3-2t^2, 3\rangle$

31. If $\mathbf{a}(t) = \langle -\sin t, 0, -\cos t\rangle$, $\mathbf{v}(0) = \langle 1,1,0\rangle$, and $\mathbf{x}(0) = \langle 0,0,1\rangle$, find $\mathbf{x}(t)$.
Ans: $\langle \sin t, t, \cos t\rangle$

32. If $\mathbf{a}(t) = 4e^{2t-2}\,\mathbf{i} - \frac{1}{t^2}\,\mathbf{j} - 24t\,\mathbf{k}$, $\mathbf{v}(1) = \mathbf{i} + \mathbf{j} - 13\,\mathbf{k}$, and $\mathbf{x}(1) = -4\,\mathbf{k}$, find $\mathbf{x}(t)$.
Ans: $\langle e^{2t-t} - t, \ln t, 1 - t - 4t^3\rangle$

1.6 Lines, Planes, & Surfaces

Line: A line in n–space is much like those in $\mathbb{R}^2$. A line only requires 2 points or a point and a slope. A line has a constant slope – constant change in each variable. However, unlike ordinary lines, there are many ways of representing lines in n–space.

Vector Form: $l(t) = \mathbf{b} + t\,\mathbf{m}$

Parametric Form: [Provided none of the m_i's are zero.]

$$\begin{cases} x = m_1 t + b_1 \\ y = m_2 t + b_2 \\ z = m_3 t + b_3 \end{cases}$$

Symmetric Form: [Provided none of the m_i are zero.]

$$\frac{x - b_1}{m_1} = \frac{y - b_2}{m_2} = \frac{z - b_3}{m_3}$$

It is also not a trivial matter to decide when two lines are even the same. As usual, two lines are parallel if they have parallel slope vectors (they need not be equal – merely parallel). However, we have a third situation, two lines can be non–parallel but not intersect. In this case, we say that the lines are skew.

Plane: A plane is a surface which when cut in the x, y, or z direction yields a line. Hence, there must be a vector which is perpendicular to it at all times. [Take a point and a vector which will determine the plane. Then the plane is generated by all vectors which when connected to the point form a vector perpendicular to the given vector.] Notice then all you need to determine a plane is a point and a normal vector – keep this in mind.

Vector Form: $\mathbf{n} \cdot \vec{P_0P} = 0$

'Traditional' Form: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. This can be rewritten as $Ax + By + Cz + D$. Do you see now how the partial functions give lines?

Parametric Form: The parametric form for a plane containing a point (c_1, c_2, c_3) with vector $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ and two nonparallel vectors $\mathbf{u}$, $\mathbf{v}$ is given by

$$x(s,t) = s\mathbf{u} + t\mathbf{v} + \mathbf{c}$$

Sphere: A sphere is formed by a center (x_0, y_0, z_0) with a set of points with fixed distance r from the point:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Cylinder: Formed by a circle extended into $\mathbb{R}^3$. But one need not restrict to 'full' cylinders. Any parabola extended into 3–space will also form an open cylinder.

Level Curve/Surface: A partial function for a surface. Formed by holding one variable fixed.

1.6 | Exercises

Spheres

1. Find the equation of a sphere with radius 4 and center $(1,0,-5)$.
Ans: $(x-1)^2+y^2+(z+5)^2=16$.

2. Find the equation of a sphere with radius 3/2 and center $(3,3,-2)$.
Ans: $(x-3)^2+(y-3)^2+(z+2)^2=9/4$.

3. Find the center and radius of the sphere $x^2+y^2+z^2-2x+6y=-6$.
Ans: Radius 2 and center $(1,-3,0)$.

4. Find the center and radius of the sphere $x^2+y^2+z^2+10-4y+2z=-21$.
Ans: Radius 3 and center $(-5,2,-1)$.

5. Find the center and radius of the sphere $x^2+y^2+z^2-x+3y-2z=-5/4$.
Ans: Radius 3/2 and center $(1/2,-3/2,1)$.

Cylinders

6. Plot the cylinder $(x-3)^2+y^2=4$.
Ans: Center $(3,0,0)$, Radius 2, about the z–axis.

7. Plot the cylinder $(z+5)^2+(y-1)^2=9$.
Ans: Center $(0,-5,1)$, Radius 3, about the x–axis.

8. Plot the cylinder $z=x^2$.
Ans: Open cylinder along the z–axis.

9. Plot the cylinder $x=(z-3)^2$.
Ans: Open cylinder along the plane $z=3$.

10. Plot the cylinder $x=(y+3)^2+2$.
Ans: Open cylinder along the plane $y=-3$.

Lines

11. Find the equation of the line through the points $(-1,0,3)$ and $(5,2,2)$.
Ans: $(6t-1,2t,3-t)$

12. Find the equation of the line through the points $(0,2,-7)$ and $(5,-6,0)$.
Ans: $(5t,2-8t,7t-7)$

13. Find the equation of the line through the points $(\pi, 6, e^2)$ and $(1, \sqrt{2}, -\frac{1}{3})$.
Ans: $((1-\pi)t+\pi, (\sqrt{2}-5)t+5, (-1/3-e^2)t+e^2)$

14. Find the equation of the line through the point $(1,2,3)$ parallel to the vector $\mathbf{i}-2\mathbf{j}+\mathbf{k}$.
Ans: $l(t)=(1,-2,1)t+(1,2,3)=(t+1,2-2t,t+3)$

15. Find the equation of the line through the point $(1,-1,0)$ and parallel to the vector $\langle -2,5,3\rangle$.
Ans: $l(t)=(-2,5,3)t+(1,-1,0)=(1-2t,5t-1,3t)$

16. Find the equation of the line through the point $(3,-1,2)$ and perpendicular to the plane $2x-3y+5z=6$.
Ans: $l(t)=(2t+3,-1-3t,5t+2)$

17. Find the equation of the line through the point $(5,0,5)$ and perpendicular to the plane $x-z=7$.
Ans: $l(t)=(t+5,0,5-t)$

18. Find the equation of the line through $(1,1,1)$ and perpendicular to the vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}-\mathbf{j}-\mathbf{k}$.
Ans: $l(t)=(1-t,t+1,1-2t)$

19. Find the equation of the line through $(-7,1,1)$ and perpendicular to the vectors $2\mathbf{x}+3\mathbf{y}$ and $5\mathbf{z}-\mathbf{y}$.
Ans: $l(t)=(15t-7,t-10,t-2)$

20. Determine if the lines l_1: $x=4t+1$, $y=4-t$, $z=3t+4$ and l_2: $x=-2-8t$, $y=2t-8$, $z=-6t-8$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.
Ans: The lines are parallel.

21. Determine if the lines l_1: $x=t-2$, $y=4-t$, $z=2t+1$ and l_2: $x=2t-4$, $y=8-2t$, $z=2t+2$ are the same skew, parallel, or intersecting. If the lines intersect, find the point of intersection.
Ans: The lines are not parallel and do not intersect.

22. Determine if the lines $l_1(t)=(7,1,20)+t(2,1,5)$ and $l_2(s)=(4,-17,-13)+s(-1,5,6)$ are the same skew, parallel, or intersecting. If the lines intersect, find the point of intersection.
Ans: The lines intersect at $(1,-2,5)$.

23. Determine if the lines $l_1 : \dfrac{x}{4}=\dfrac{y-9}{-3}=\dfrac{z+1}{4}$ and $l_2 : \dfrac{x+12}{8}=\dfrac{y-18}{-6}=\dfrac{z+13}{8}$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.
Ans: Same line. t for first line, s for second. $t=2s-3$.

24. Determine if the lines $l_1(t)=(2t+1,3t-4,5-t)$ and $l_2(s)=(2s-1,3s+1,5-s)$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.
Ans: Parallel

25. Determine if the lines l_1: $x=t$, $y=1-t$, $z=1-t$ and l_2: $x=6t$, $y=-t-1$, $z=3t+1$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

Ans: Skew

26. Determine if the lines $l_1(t) = (1-t, 5t-3, t)$ and $l_2(s) = (9-2s, 3-10s, 2s)$ are the same, skew, parallel, or intersecting. If the lines intersect, find the point of intersection.

Ans: Parallel

27. Determine if the lines $l_1(t)$: $x = 2t+5$, $y = 2t+1$, $z = 3-t$ and $l_2(s)$: $x = 11s-1$, $y = 4-s$, $z = 12s$ are perpendicular.

Ans: The lines are not perpendicular.

28. Determine if the lines $l_1(t) = (-1, 3, 6) + t(-2, 0, 1)$ and $l_2(t) = (11, -1, 15) + t(-3, 2, -6)$ are perpendicular.

Ans: The lines are not perpendicular.

29. Determine if the lines $l_1(t) = (t+6, t+7, t+9)$ and $l_2(t) = (13-4t, 3t-7, t+1)$ are perpendicular.

Ans: The lines are not perpendicular.

30. Determine if the lines $l_1(t)$: $x = 3t+1$, $y = 2t+5$, $z = t-1$ and l_2: $x = 7t$, $y = 7-14t$, $z = 6t+5$ are perpendicular.

Ans: The lines are not perpendicular.

Planes

31. Find a normal vector to the plane $2x - 3y + z = 6$ and three points on the plane.

Ans: $\langle 2, -3, 1\rangle$, $(3, 0, 0)$, $(0, -2, 0)$, $(0, 0, 6)$, $(2, -1, 1)$

32. Find a normal vector to the plane $x + y - 4z = 3$ and three points on the plane.

Ans: $\langle 1, 1, -4\rangle$, $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, -3/4)$

33. Find a normal vector to the plane $2x - 3z = 5$ and three points on the plane.

Ans: $\langle 2, 0, -3\rangle$, $(5/2, 0, 0)$, $(0, 0, -5/3)$, $(-1/2, 0, -2)$, $(5/2, \pi, 0)$

34. Find a normal vector to the plane $3y + 2z - x = 7$ and three points on the plane.

Ans: $\langle -1, 3, 2\rangle$, $(-7, 0, 0)$, $(0, 7/3, 0)$, $(0, 0, 7/2)$

35. Find a normal vector to the plane $2z - 5y = -1$ and three points on the plane.

Ans: $\langle 0, -5, 2\rangle$, $(0, 1/5, 0)$, $(0, 0, -1/2)$, $(0, 1, 2)$, $(0, -1, -3)$,$(\pi, 1/5, 0)$

36. Find the equation of the plane containing the point $(1, 2, 3)$ with normal vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

Ans: $2(x-1) - (y-2) + 3(z-3) = 0$ or $2x - y + 3z = 9$

37. Find the equation of the plane containing the point $(0, -3, 5)$ with normal vector $\langle -2, 1, -3\rangle$.

Ans: $-2(x-0) + (y+3) - 3(z-5) = 0$ or $-2x + y - 3z = -18$

38. Find the equation of the plane containing the point $(7, -2, 4)$ with normal vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Ans: $(x-7)+(y+2)-(z-4)=0$ or $x+y-z=1$

39. Find the equation of the plane containing the point $(1/2, 3, 0)$ with normal vector $\langle 2, -3, 5\rangle$.
Ans: $2(x-1/2)-3(y-3)+5z=0$ or $2x-3y-5z=-8$

40. Find the equation of the plane containing the point $(0, 5, 0)$ with normal vector $-\frac{1}{5}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{1}{4}\mathbf{k}$.
Ans: $-4x+10(y-5)+5z=0$ or $-4x+10y+5z=50$

41. Find the equation of the plane containing the point $(2, 2, -1)$ with normal vector $\langle 30, 60, -60\rangle$.
Ans: $(x-2)+2(y-2)-2(z+1)=0$ or $x+2y-2z=8$

42. Find the equation of the plane containing the points $(1, 1, 1)$, $(0, 1, 3)$, and $(-1, 2, -1)$.
Ans: $n=\langle 2, 6, 1\rangle$, $2x+6(y-1)+z=3$ or $2x+6y+z=9$

43. Find the equation of the plane containing the points $(2, 1, 3)$, $(2, 2, -5)$, and $(0, 0, 1)$.
Ans: $n=\langle 5, -8, -1\rangle$, $10(x-2)-16(y-2)-2(z+5)=0$ or $10x-16y-2z=-2$

44. Find the equation of the plane containing the points $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$.
Ans: $n=\langle 1, 1, 1\rangle$, $(x-1)+y+(z-1)=0$ or $x+y+z=2$

45. Find the equation of the plane containing the points $(0, 0, 2)$, $(5, 2, -4)$, and $(-1, -2, 4)$.
Ans: $n=\langle -2, 7, 2\rangle$, $-8(x-5)+28(y-2)+8(z+4)=0$ or $-8x+28y+8z=-16$

46. Find the equation of the plane with x–intercept a, y–intercept b, and z–intercept c.
Ans: $n=\langle bc, ac, ab\rangle$, $bcx+ac(y-b)+abz=0$ or $bcx+acy+abz=abc$

47. Find the equation of the plane containing the line $x=3t$, $y=2-2t$, $z=t+1$ and parallel to the plane $x+2y+z=2$.
Ans: $x+2y+z=5$

48. Find the equation of the plane containing the line $l(t)=(4, 8, 20)t+(1, 1, 1)$ and parallel to the plane $5y-2z=10$.
Ans: $5y-2z=3$

49. Find the equation of the plane containing the line $\dfrac{x}{-12}=\dfrac{y-1}{3}=\dfrac{z+1}{-10}$ and parallel to the plane $-2x+2y+3z=7$.
Ans: $-2x+2y+3z=-1$

50. Find the equation of the plane containing the line $x=6-28t$, $y=4t+4$, $z=5-24t$ and parallel to the plane $-x+5y+2z=16$.
Ans: $-x+5y+2z=24$

51. Find the equation of the plane containing the line $l(t)=(3-t, 2t-1, 3t+2)$ and parallel to the plane $x-y+z=-1$.

Ans: $x - y + z = 6$.

52. Find the equation of the plane perpendicular to the planes $x - 2y + 3z = 13$ and $6 + 4x - 2z = 0$ and containing the point $(1, 0, 1)$.

Ans: $4x + 14y + 8z = 12$

53. Find the equation of the plane perpendicular to the planes $2y - z = 9$ and $x + 2y = 14$ and containing the point $(7, 3, 9)$.

Ans: $-2x + y + 2z = 7$

54. Find the equation of the plane perpendicular to the planes $2x - 3y + z = 25$ and $x + y - 4z = 3$ and containing the point $(0, 6, 0)$.

Ans: $11x + 9y + 5z = 54$

55. Find the equation of the plane perpendicular to the planes $-3x + 5y + 4z = 2$ and $2x + 2y + 2z = 14$ and containing the point $(1, 2, 3)$.

Ans: $2x + 14y - 16z + 18 = 0$

56. Find the equation of the plane perpendicular to the planes $2x + y - z = 7$ and $-x + 3y + 2z = 17$ and containing the point $(1, 0, -1)$.

Ans: $5x - 3y + 7z + 2 = 0$

57. Find the equation for the line of intersection of the planes $x + 2y - z = 7$ and $x + y - z = 0$.

Ans: Using $z = 0$ to find a point for the line, $l(t) = (-1, 0, -1)t + (-7, 7, 0)$

58. Find the equation for the line of intersection of the planes $x + 2y + 2z = 16$ and $3x + z = 6$.

Ans: Using $z = 0$ to find a point for the line, $l(t) = (2, 5, -6)t + (2, 7, 0)$

59. Find the equation for the line of intersection of the planes $x + y - 3z = 1$ and $5x - y + 2z + 1 = 0$.

Ans: Using $z = 0$ to find a point for the line, $l(t) = (1, 17, -6)t + (0, 1, 0)$

60. Find the equation for the line of intersection of the planes $3x + 3y + 2z = 0$ and $2x + y + 2z = 12$.

Ans: Using $z = 0$ to find a point for the line, $l(t) = (4, -2, -3)t + (12, -12, 0)$

61. Find the equation for the line of intersection of the planes $x + 4y - z = 4$ and $3x + 2y - z = 2$.

Ans: Using $z = 0$ to find a point for the line, $l(t) = (1, 1, 5)t + (0, 1, 0)$

62. Determine if the planes $x + y + z = 5$ and $2y + 2y = 10 - 2x$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: The planes are the same.

63. Determine if the planes $x + y - 2z + 1 = 0$ and $2x + z = 2y + 8$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: The planes are skew. The angle is $\arccos(-\sqrt{2/3}/3) = 105.793°$

64. Determine if the planes $2x+y+3z+4=0$ and $4x+2y+6z=10$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: The planes are parallel.

65. Determine if the planes $x-2y+z+4=0$ and $3z-3x=0$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: The planesl are perpendicular. The angle is $90°$.

66. Determine if the planes $2x+y-z=13$ and $3x+y+4z=0$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: The planes are skew. The angle is $\arccos(\sqrt{3/13}/2)=76.1021°$

67. Determine if the planes $2x-4y+2z+8=0$ and $-5x+10y-5z=20$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: They are the same.

68. Determine if the planes $2x+y+3z=9$ and $-14x-8y+12z+2=0$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: They are perpendicular. The angle is $90°$.

69. Determine if the planes $4x+4y+z=6$ and $5x+3y+z=13$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: The lines are skew. The angle is $\arccos\sqrt{33/35}=13.8302°$

70. Determine if the planes $3x-y+2z=2$ and $3y-9x=6z-15$ are the same, parallel, perpendicular, or neither. If the planes are distinct and nonparallel, find the angle between them.

Ans: The planes are parallel.

71. Determine if the line $l(t)=(2,-1,2)t+(1,1,1)$ and the plane $2x-y+2z=9$ are perpendicular, parallel, or neither.

Ans: They are perpendicular.

72. Determine if the line $l(t)=(t+2,5-3t,5t)$ and the plane $3x+y+6z+7=0$ are perpendicular, parallel, or neither.

Ans: They are neither.

73. Determine if the line $x=3t+4$, $y=7-t$, $z=t$ and the plane $x+5y+2z=10$ are perpendicular, parallel, or neither.

Ans: They are parallel.

74. Find the equation of the line perpendicular to the plane $5x-7y+4z=12$ containing the point $(-1,4,8)$.

Ans: $l(t)=(5,-7,4)t+(-1,4,8)$

75. Find the equation of the plane perpendicular to the line $x=4-t$, $y=2t+1$, $z=5t+6$ and containing the line $l(t) = (3-8t, 11t+2, 4-6t)$.

Ans: $-1(x-3)+2(y-2)+5(z-4)$ or $-x+2y+5z = 21$

(Quadratic) Surfaces

76. Use appropriate level curves to sketch the surface given by the equation $x+y+z = 7$.

Ans: A plane.

77. Use appropriate level curves to sketch the surface given by the equation $(x-2)^2+y^2+(z+1)^2 = 9$ and describe the surface.

Ans: Sphere. Level curves in any direction are circles. This is a sphere with center $(2,0,-1)$ and radius 3.

78. Use appropriate level curves to sketch the surface given by the equation $4\frac{x}{2}+(y-1)^2+z^2 = 4$ and describe the surface.

Ans: Ellipsoid. Level curves in any direction are ellipses. This is an ellipsoid centered at $(0,1,0)$, 'radius' 1 in x, 'radius' 2 in y, and 'radius' 2 in z.

79. Use appropriate level curves to sketch the surface given by the equation $z = 4x^2+9y^2$ and describe the surface.

Ans: Elliptic paraboloid. Open: z–axis. Vertex: $(0,0,0)$.

80. Use appropriate level curves to sketch the surface given by the equation $y^2 = x$ and describe the surface.

Ans: Parabolic Cylinder. Base: $x = 0$. Open: x–axis. Sym: $y = 0$.

81. Use appropriate level curves to sketch the surface given by the equation $z^2 - x^2 - y^2 = 1$ and describe the surface.

Ans: Hyperboloid of Two Sheets. Bottom: $(0,0,\pm 1)$. Sym: $z = 0$.

82. Use appropriate level curves to sketch the surface given by the equation $z^2 = 4x^2+9y^2$ and describe the surface.

Ans: Elliptic Cone. Open: z–axis. Vertex: $(0,0,0)$.

83. Use appropriate level curves to sketch the surface given by the equation $z = 4y^2-x^2$ and describe the surface.

Ans: Hyperbolic Paraboloid. Saddle: $(0,0,0)$. Sit: x–axis.

84. Use appropriate level curves to sketch the surface given by the equation $x^2+y^2-z^2 = 1$ and describe the surface.

Ans: Hyperboloid of One Sheet. Center: $(0,0,0)$. Sym: z–axis.

85. Use appropriate level curves to sketch the surface given by the equation $2x+x^2+36y^2+4z^2 = 144y-141$ and describe the surface.

Ans: Ellipsoid. Level curves in any direction are ellipses. Equation of surface $\left(\frac{x+1}{2}\right)^2+(3(y-2))^2+z^2 = 1$ – an ellipsoid with center $(-1,2,0)$ and 'radius' 2 in the x direction, 'radius' $1/3$ in the y direction, and 'radius' 1 in the z direction.

86. Use appropriate level curves to sketch the surface given by the equation $-y = (x-1)^2 + z^2$ and describe the surface.

Ans: Elliptic Paraboloid, open downwards. Open: y–axis. Vertex: $(1,0,0)$.

87. Use appropriate level curves to sketch the surface given by the equation $2x - 3y - z = 6$.

Ans: A plane.

88. Use appropriate level curves to sketch the surface given by the equation $y = z^2 - 4z + 7$ and describe the surface.

Ans: Parabolic Cylinder. $y = (z-3)^2 + 3$. Base: $y = 3$. Open: y–axis. Sym: $z = 2$.

89. Use appropriate level curves to sketch the surface given by the equation $\dfrac{x^2}{4} + \dfrac{z^2}{9} - 1 = y^2$ and describe the surface.

Ans: Hyperboloid of One Sheet. Center: $(0,0,0)$. Sym: y–axis.

90. Use appropriate level curves to sketch the surface given by the equation $x^2 + (y+5)^2 + (z+1)^2 = 4$ and describe the surface.

Ans: Sphere. Level curves in any direction are circles. This is a sphere with center $(0,-5,-1)$ and radius 2.

91. Use appropriate level curves to sketch the surface given by the equation $x^2 = z^2 + y^2 - 2y + 1$ and describe the surface.

Ans: Elliptic Cone. Open: $y = 1, z = 0$. Vertex: $(0,1,0)$.

92. Use appropriate level curves to sketch the surface given by the equation $x + 3 = z^2 - 9y^2$ and describe the surface.

Ans: Hyperbolic Paraboloid. Saddle: $(-3,0,0)$. Sit: y–axis.

93. Use appropriate level curves to sketch the surface given by the equation $y^2 - 4x^2 - 4(z+1)^2 = 4$ and describe the surface.

Ans: Hyperboloid of Two Sheets. $\dfrac{y^2}{4} - x^2 - (z+1)^2 = 1$. Bottom: $(0,\pm 2,-1)$. Sym: $y = 0$.

94. Use appropriate level curves to sketch the surface given by the equation $y - 2z = 4$.

Ans: A plane.

95. Use appropriate level curves to sketch the surface given by the equation $x^2 - y^2 - \dfrac{z^2}{9} = 1$ and describe the surface.

Ans: Hyperboloid of Two Sheets. Bottom: $(\pm 1,0,0)$. Sym: $x = 0$.

96. Use appropriate level curves to sketch the surface given by the equation $x^2 + 36z^2 + 9y^2 = 288z - 576$ and describe the surface.

Ans: Ellipsoid. Level curves in any direction are ellipses. Equation of surface $\left(\frac{x-3}{3}\right)^2 + y^2 + (2(z-4))^2 = 1$

– ellipsoid with center $(3, 0, 4)$ and 'radius' 3 in the x direction, 'radius' 1 in the y direction, and 'radius' $1/2$ in the z direction.

97. Use appropriate level curves to sketch the surface given by the equation $x + 1 = y^2 + (z - 1)^2$ and describe the surface.

Ans: Elliptic Paraboloid. Open: x–axis. Vertex: $(-1, 0, 1)$.

98. Use appropriate level curves to sketch the surface given by $x^2 + y^2 + z^2 + 2 = 2(x + y + z)$ and describe the surface.

Ans: Sphere. Level curves in any direction are circles. This is a sphere with radius 1 and center $(1, 1, -1)$.

99. Use appropriate level curves to sketch the surface given by the equation $(y+1)^2 + (z-2)^2 - \dfrac{x^2}{2} = 1$ and describe the surface.

Ans: Hyperboloid of One Sheet. Center: $(0, -1, 2)$. Sym: x–axis.

100. Use appropriate level curves to sketch the surface given by the equation $y^2 - x^2 = z^2 - 6z - 2y + 8$ and describe the surface.

Ans: Elliptic Cone. $(y + 1)^2 = x^2 + (z - 3)^2$. Open: $x = 0, z = 3$. Vertex: $(0, -1, 3)$.

101. Use appropriate level curves to sketch the surface given by the equation $y - 2 = x^2 - z^2 + 2x + 3$ and describe the surface.

Ans: Hyperbolic Paraboloid. $y - 2 = (x + 1)^2 - z^2$. Saddle: $(-1, 2, 0)$. Sit: z–axis.

102. Use appropriate level curves to sketch the surface given by the equation $x = 4 - z^2$ and describe the surface.

Ans: Parabolic Cylinder. Base: $x = 4$. Open: $-x$–axis. Sym: $z = 0$.

103. Use appropriate level curves to sketch the surface given by the equation $9x^2 + 9y^2 + z^2 = 54x + 36y - 108$ and describe the surface.

Ans: Ellipsoid. Level curves in any direction are ellipses. Equation of the surface $(x-3)^2 + (y-2)^2 + \left(\frac{z}{3}\right)^2$ – ellipsoid center $(3, 2, 0)$ and 'radius' 1 in the x direction, 'radius' 1 in y direction, and 'radius' 3 in the z direction.

104. Use appropriate level curves to sketch the surface given by the equation $x^2 - y^2 - z^2 = 4x - 4$ and describe the surface.

Ans: Elliptic Cone. $(x - 2)^2 = \dfrac{y^2}{4} + z^2$. Open: $y = 0 = z$. Vertex: $(2, 0, 1)$.

105. Use appropriate level curves to sketch the surface given by the equation $x + 4y = 3y - z + 1$.

Ans: A plane.

106. Use appropriate level curves to sketch the surface given by the equation $z = x^2 + 3$ and describe the surface.

Ans: Parabolic Cylinder. Base: $z = 3$. Open: z–axis. Sym: $y = 0$.

107. Use appropriate level curves to sketch the surface given by $4x^2 + 4y^2 + 4z^2 + 16z = 8z - 19$ and describe the surface.

Ans: Sphere. Level curves in any direction are circles. This is a sphere with radius $(-2, 0, 1)$ and radius $1/2$.

108. Use appropriate level curves to sketch the surface given by the equation $y^2 - 2 = x^2 + z^2 + 2(x + y + z)$ and describe the surface.

Ans: Hyperboloid of One Sheet. $(x+1)^2 + (z+1)^2 - (z-1)^2 = 1$. Center: $(-1, 1, -1)$. Sym: $y = 1$.

109. Use appropriate level curves to sketch the surface given by the equation $(z+2)^2 - y^2 - (x-3)^2 = 9$ and describe the surface.

Ans: Hyperboloid of Two Sheets. Bottom: $(3, 0, \pm 3)$. Sym: $z = 0$.

110. Use appropriate level curves to sketch the surface given by the equation $x = \dfrac{y^2}{4} + z^2$.

Ans: Elliptic Paraboloid. Open: x–axis. Vertex: $(0, 0, 0)$.

111. Use appropriate level curves to sketch the surface given by the equation $x = z^2 - \dfrac{y^2}{9}$ and describe the surface.

Ans: Hyperbolic Paraboloid. Saddle: $(0, 0, 0)$. Sit: y–axis.

112. Use appropriate level curves to sketch the surface given by $x^2 + y^2 + {}^2 + 8x + 16 = 16$ and describe the surface.

Ans: Sphere. Level curves in any direction are circles. This is a sphere with center $(-4, 0, 0)$ and radius 4.

113. Use appropriate level curves to sketch the surface given by the equation $z^2 - x^2 - y^2 = 2y + 2z - 5x + 25$ and describe the surface.

Ans: Elliptic Cone. $(z-1)^2 = (y+1)^2 + (x-5)^2$. Open: $y = -1, x = 5$. Vertex: $(5, -1, 1)$.

114. Use appropriate level curves to sketch the surface given by the equation $x + 2z = 4$.

Ans: A plane.

115. Use appropriate level curves to sketch the surface given by the equation $x^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z+3}{2}\right)^2 = 1$ and describe the surface.

Ans: Ellipsoid. Level curves in any direction are ellipses. Equation of the surface $x^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z+3}{2}\right)^2 = 1$ – ellipsoid with center $(0, 0, -3)$ and 'radius' 1 in the x direction, 'radius' 3 in the y direction, and 'radius' 2 in the z direction.

116. Use appropriate level curves to sketch the surface given by the equation $z^2 + (y-3)^2 = x^2 + 9$ and describe the surface.

Ans: Hyperboloid of One Sheet. Center: $(0, 3, 0)$. Sym: x–axis.

117. Use appropriate level curves to sketch the surface given by the equation $\dfrac{y^2}{4} - 3x^2 - z^2 = 1$ and describe the surface.

Ans: Hyperboloid of Two Sheets. Bottom: $(0, \pm 2, 0)$. Sym: $y = 0$.

118. Use appropriate level curves to sketch the surface given by the equation $y = 1 - 6z - z^2$ and describe the surface.

Ans: Parabolic Cylinder. $y = 10 - (z+3)^2$. Base: $y = 10$. Open: $-y$–axis. Sym: $z = -3$.

119. Use appropriate level curves to sketch the surface given by the equation $z + 3 = (x-1)^2 - (y+2)^2$ and describe the surface.

Ans: Hyperbolic Paraboloid. Saddle: $(1, -2, -3)$. Sit: y–axis.

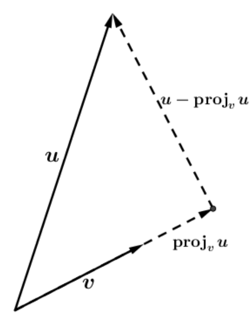
120. Use appropriate level curves to sketch the surface given by the equation $z + 1 = (y-3)^2 + (x+1)^2$ and describe the surface.

Ans: Elliptic Paraboloid. Open: z–axis. Vertex: $(-1, 3, -1)$.

1.7 Computing Distances

Distance Point–Point: We have already seen and done this before: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Distance Point–Line: Find the distance from $P(2,1,3)$ and $l(t) = (2,3-2) + t(-1,1,-2)$. There are two methods:



Method 1. We have $\mathbf{v} = (-1,1,-2)$. We need $\mathbf{u}$ so take the displacement vector to any point on the line (there is an easy one): $\mathbf{u} = (2,1,3) - (2,3,-2) = (0,-2,5)$. We have

$$\mathrm{proj}_{\mathbf{u}}\, \mathbf{v} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = (2,-2,4)$$

Then the distance is $d = |\mathbf{u} - \mathrm{proj}_{\mathbf{u}}\, \mathbf{v}| = |(-2,0,1)| = \sqrt{5}$.

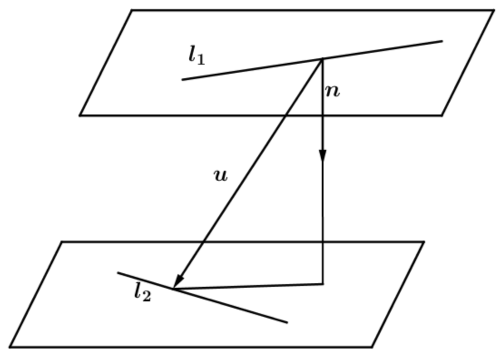
Method 2. Using right triangle trig, we have $\sin\theta = \dfrac{d}{|\mathbf{u}|}$. But $d = |\mathbf{u}|\sin\theta = \dfrac{|\mathbf{v}|}{|\mathbf{v}|}|\mathbf{u}|\sin\theta = \dfrac{|\mathbf{v}\times\mathbf{u}|}{|\mathbf{v}|}$. Now

$$\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ 0 & -2 & 5 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$

so that we must have $d = \dfrac{|\mathbf{i}+5\mathbf{j}+2\mathbf{k}|}{|-\mathbf{i}+\mathbf{j}-2\mathbf{k}|} = \dfrac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$.

Distance Point–Plane: You have a normal vector for a plane. Find a point on the plane and form the displacement vector from the given point to the found point. Then simply take the projection to find the distance.

Distance Line–Line: Find the distance from $l_1(t) = (0, 5, -1) + t(2, 1, 3)$ and $l_2(t) = (-1, 2, 0) + t(1, -1, 0)$. We have two points, form their displacement vector: $\mathbf{u} = (-1, 2, 0) - (0, 5, -1) = (-1, -3, 1)$.



The lines must lie in parallel planes. We need to find the common normal:

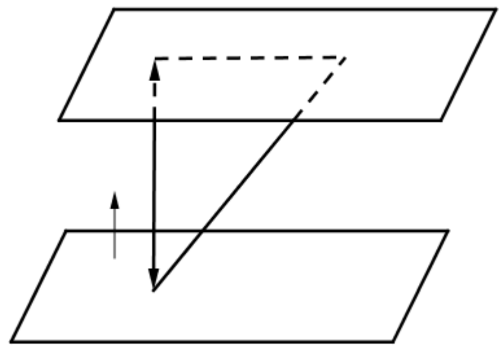
$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$$

Then we have

$$\mathrm{proj}_{\mathbf{u}}\, \mathbf{n} = \left(\frac{\mathbf{n} \cdot \mathbf{u}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} = \frac{-15}{27}\,(3, 3, -3) = -\frac{5}{3}\,(1, 1, -1)$$

so that $d = |\mathrm{proj}_{\mathbf{n}}\, \mathbf{u}| = \frac{5}{3}\sqrt{3}$.

Distance Plane–Plane: Find the distance between $2x-2y+z=5$ and $2x-2y+z=20$. (Why are these parallel?) We immediately have $\mathbf{n}=(2,-2,1)$. Find any point on the each plane (there are easy ones):



$(0,0,5)$ and $(0,0,20)$ will do. Form their displacement vector: $\mathbf{u}=(0,0,20)-(0,0,5)=(0,0,15)$. Then

$$\operatorname{proj}_{\mathbf{n}}\mathbf{u}=\left(\frac{\mathbf{n}\cdot\mathbf{u}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}=-\frac{15}{9}(2,-2,1)=-\frac{5}{3}(2,-2,1)$$

so that $d=|\operatorname{proj}_{\mathbf{n}}\mathbf{u}|=5$.

Distance Surface–Surface: We will need max/min and Lagrange Multipliers to do this. This will come later.

1.7 | Exercises

Distance between Points

1. Find the distance between the points $(-1,3,7)$ and $(9,-2,3)$.
Ans: $\sqrt{141}$

2. Find the distance between the points $(0,4,-10)$ and $(3,1,3)$.
Ans: $\sqrt{187}$

3. Find the distance between the points $(12,1,5)$ and $(7,4,6)$.
Ans: $\sqrt{35}$

Distance between Point & Line

4. Find the distance from the point $(-2,4,-2)$ to the line $l(t) = (-1,2,0) + t(4,5,3)$.
Ans: 3

5. Find the distance from the point $(4,9,4)$ to the line $l : x = 2-5t,\ y = t+5, z = 6t+3$.
Ans: $\sqrt{21}$

6. Find the distance from the point $(-1,2,-2)$ to the line $l : \dfrac{x+2}{5} = \dfrac{y+2}{3} = \dfrac{z-2}{4}$.
Ans: 9

7. Find the distance from the point $(2,9,4)$ to the line $l : x = 2-5t,\ y = t+5,\ z = 3-4t$.
Ans: $\sqrt{17}$

8. Find the distance from the point $(2,7,-3)$ to the line $l(t) = (t+4, 1-3t-6t)$.
Ans: 15

9. Find the distance from the point $(-4,6,-1)$ to the line $l : \dfrac{x}{4} = \dfrac{y-4}{5} = \dfrac{z}{5}$.
Ans: $\sqrt{21}$

10. Find the distance from the point $(-3,-1,5)$ to the line $l(t) = (-2,1,7) + t(4,4,-6)$.
Ans: 3

11. Find the distance from the point $(1,9,4)$ to the line $l : y = 4, x-2 = 3-z$.
Ans: $3\sqrt{3}$

12. Find the distance from the point $(5,7,3)$ to the line $l(t) = (6t, 4-t, -4t-3)$.
Ans: 23

Distance between Points & Plane

13. Find the distance from the point $(1,1,-3)$ to the plane $x+2y+2z+4=0$.
Ans: 1

14. Find the distance from the point $(2,2,0)$ to the plane $2x+5y+4z=1$.
Ans: $\sqrt{5}$

15. Find the distance from the point $(-4,0,2)$ to the plane $2x-4y+4z=3$.
Ans: $\frac{1}{2}$

16. Find the distance from the point $(1,3,1)$ to the plane $2y-x+z=0$.
Ans: $\sqrt{6}$

17. Find the distance from the point $(0,0,7)$ to the plane $3x-4z=1$.
Ans: $\frac{27}{5}$

18. Find the distance from the point $(3,0,3)$ to the plane $4x-2y+5z=3$.
Ans: $2\sqrt{5}$

19. Find the distance from the point $(5,5,3)$ to the plane $2x+y-2z+5=0$.
Ans: $\frac{4}{3}$

20. Find the distance from the point $(1,1,-1)$ to the plane $x-2y+z+4=0$.
Ans: $\sqrt{6}$

21. Find the distance from the point $(-2,1,2)$ to the plane $4x+3z+2=0$.
Ans: $\frac{4}{5}$

Distance between Skew Lines

22. Find the distance between the lines $l_1(t)=(3,3,1)+t(0,-1,5)$ and $l_2(t)=(-2,1,6)+t(2,2,1)$.
Ans: 3

23. Find the distance between the lines $l_1\colon\ x+5=\frac{y}{2}=\frac{z+4}{-2}$ and $l_2\colon\ z=5, \frac{x+2}{4}=\frac{y-6}{3}$.
Ans: $3\sqrt{5}$

24. Find the distance between the lines $l_1\colon\ x=-2t,\ y=-4t-1,\ z=3$ and $x=4t-2,\ y=1-3t,\ z=1$.
Ans: 2

25. Find the distance between the lines $l_1(t)=(5t+5,1,-5t-2)$ and $l_2(t)=(2t+6,4-2t,-3)$.
Ans: $\sqrt{3}$

26. Find the distance between the lines $l_1\colon\ \dfrac{x-2}{4}=\dfrac{y-1}{2}=\dfrac{z+2}{5}$ and $l_2\colon\ y=1, \dfrac{x+3}{-4}=\dfrac{z-2}{-4}$.
Ans: 6

27. Find the distance between the lines $l_1\colon\ x=2t,\ y=-t,\ z=t+5$ and $l_2\colon\ x=7-t,\ y=3t+9,$ $z=4-3t$.
Ans: $4\sqrt{2}$

28. Find the distance between the lines $l_1(t)=(1,-3t-1,4)$ and $l_2(t)=(-4,3-t,3-t)$.
Ans: 5

29. Find the distance between the lines $l_1\colon\ x=4t-2,\ y=t+3,\ z=5t-1$ and $l_2\colon\ x=8-3t,\ y=4t,$ $z=t-3$.
Ans: $3\sqrt{3}$

30. Find the distance between the lines $l_1(t)=(1,1,2)+t(4,0,-4)$ and $l_2(t)=(3,0,4)+t(1,0,2)$.
Ans: 1

Distance between Parallel Planes

31. Find the distance between the planes $-7x-6y+6z=4$ and $7x+6y=6z+7$.
Ans: 1

32. Find the distance between the planes $9x+4y+z=12$ and $9x+4y+z=-2$.
Ans: $\sqrt{2}$

33. Find the distance between the planes $8x+4y+z=-11$ and $8x+4y+z=7$.
Ans: 2

34. Find the distance between the planes $11x+5y-z=9$ and $11x+5y-z=-12$.
Ans: $\sqrt{3}$

35. Find the distance between the planes $2x+y+2z+25=0$ and $2x+y+2z+16=0$.
Ans: 3

36. Find the distance between the planes $x+y+z+11=0$ and $x+y+z+23=0$.
Ans: $4\sqrt{3}$

37. Find the distance between the planes $-6x+2y-3z=12$ and $-6x+2y-3z=5$.
Ans: 1

38. Find the distance between the planes $5x+4y+2z+10=0$ and $5x+4y+2z+25=0$.
Ans: $\sqrt{5}$

39. Find the distance between the planes $x+2y+2z+13=0$ and $x+2y+2z+25=0$.

Ans: 4

Distance Formulas

40. Show that the distance from a point p and a line l with direction vector $\mathbf{u}$ containing a point q is given by

$$d = \frac{\|\vec{pq} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

41. Show that the distance from a point (x_0, y_0, z_0) to a plane $Ax + By + Cz + D = 0$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

42. If $l_1(t) = \mathbf{b}_1 + t\mathbf{a}$ and $l_2(t) = \mathbf{b}_2 + t\mathbf{a}$ are parallel lines in $\mathbb{R}^3$, show the distance between them is given by

$$d = \frac{\|\mathbf{a} \times (\mathbf{b}_2 - \mathbf{b}_1)\|}{\|\mathbf{a}\|}$$

43. If $l_1(t) = \mathbf{b}_1 + t\mathbf{a}_1$ and $l_2(t) = \mathbf{b}_2 + t\mathbf{a}_1$ are skew lines in $\mathbb{R}^3$, show that the distance between them is given by

$$d = \frac{|(\mathbf{a}_1 \times \mathbf{a}_2) \cdot (\mathbf{b}_2 - \mathbf{b}_1)|}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}$$

44. Show that the distance between parallel planes with normal vector $\mathbf{n}$ is given by

$$d = \frac{|\mathbf{n} \cdot (\mathbf{x}_2 - \mathbf{x}_1)|}{\|\mathbf{n}\|}$$

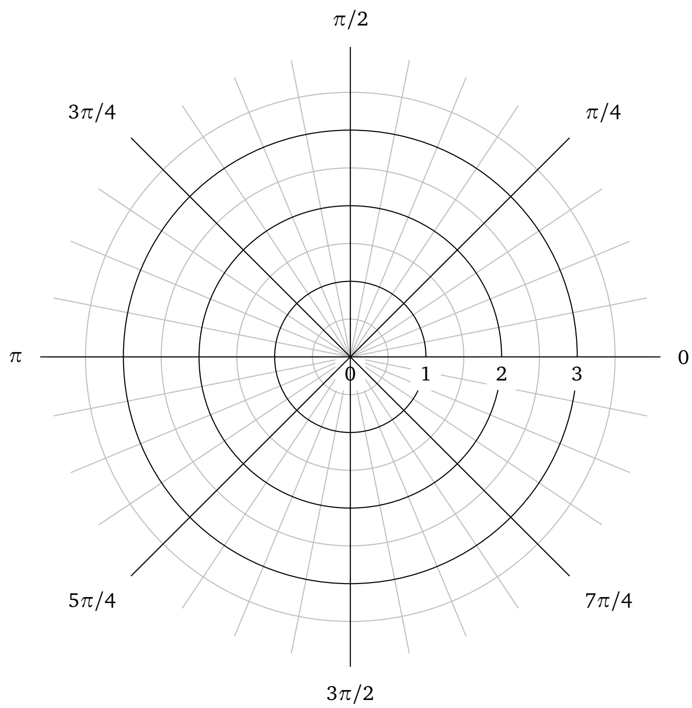
where $\mathbf{x}_i$ is the position vector on the ith plane.

45. Show that the distance between parallel planes $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$ is

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

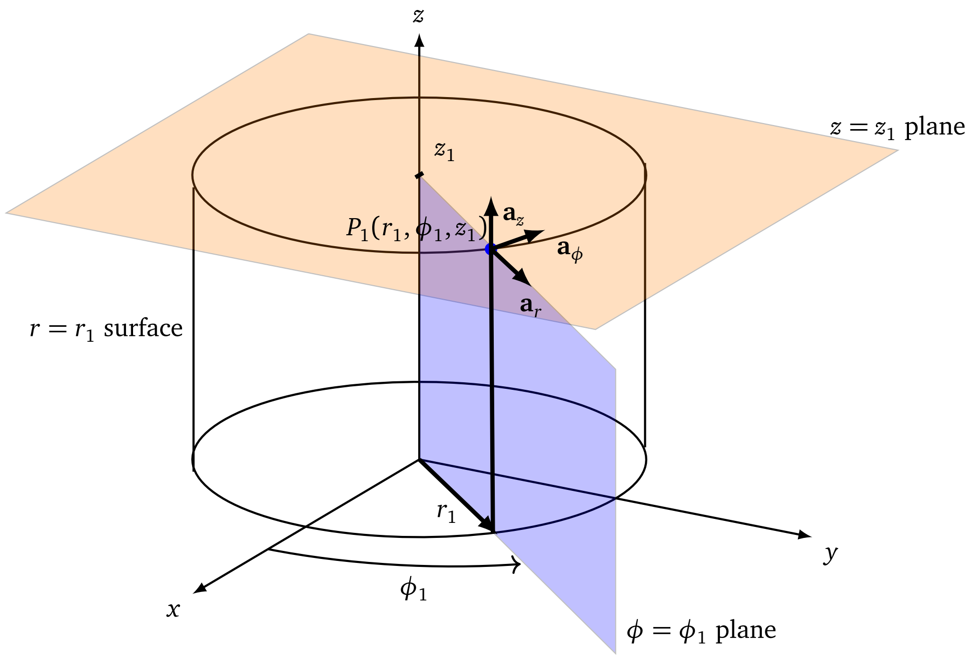
1.8 Cylindrical & Spherical Coordinates

Polar Coordinates: Gives coordinates in terms of an angle and a distance from the origin. We have special plots for these:



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$



Cylindrical Coordinates: Simply 'three–dimensional polar coordinates.' Find a coordinate by an angle, radius, and a height.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \\ z = z \end{cases}$$

Observe even by the formulas, there is little difference between the two.

Spherical Coordinates: Find a point by a angle horizontally, an angle vertically, and a distance.

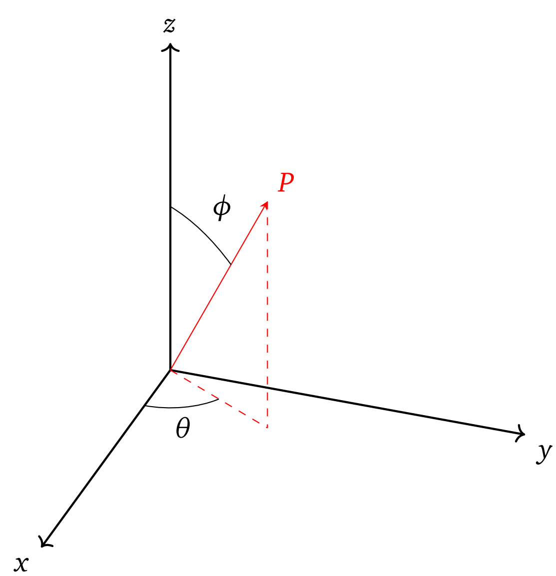
$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\phi = \dfrac{\sqrt{x^2+y^2}}{z} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

$$\begin{cases} r = \rho \sin\phi \\ \theta = \theta \\ z = \rho \cos\phi \end{cases}$$

$$\begin{cases} \rho^2 = x^2 + y^2 \\ \tan\phi = \dfrac{r}{z} \\ \theta = \theta \end{cases}$$

Note the difference between the angle ϕ in Mathematics and in the 'hard' Sciences.



1.8 | Exercises

1. Convert the following Cartesian coordinates to polar coordinates. Plot these points in both Cartesian and polar coordinates.

(a) $(-\sqrt{2}, -\sqrt{2})$ Ans: $(2, 5\pi/4)$

(b) $(-2, 2\sqrt{3})$ Ans: $(4, 2\pi/3)$

(c) $(3, 3\sqrt{3})$ Ans: $(6, \pi/3)$

2. Convert the following polar coordinates to Cartesian coordinates. Plot these points in both Cartesian and polar coordinates.

(a) $(4\sqrt{2}, 7\pi/4)$ Ans: $(4, -4)$

(b) $(4, 4\pi/3)$ Ans: $(-2, -2\sqrt{3})$

(c) $(2\sqrt{2}, 3\pi/4)$ Ans: $(-2, 2)$

3. Convert the following Cartesian coordinates to cylindrical coordinates. Plot these points in both Cartesian and cylindrical coordinates.

(a) $(-1, \sqrt{3}, 2)$ Ans: $(2, 2\pi/3, 2)$

(b) $(\sqrt{3}, -3, 1)$ Ans: $(2\sqrt{3}, 5\pi/3, 1)$

(c) $(-5, -5, -4)$ Ans: $(5\sqrt{2}, 5\pi/4, -4)$

4. Convert the following cylindrical coordinates to Cartesian coordinates. Plot these points in both cylindrical coordinates and Cartesian coordinates.

(a) $(4, 5\pi/6, -2)$ Ans: $(-2\sqrt{3}, 2, -2)$

(b) $(10, -\pi/4, -6)$ Ans: $(5\sqrt{2}, -5\sqrt{2}, -6)$

(c) $(1, 4\pi/3, 3)$ Ans: $(-1/2, -\sqrt{3}/2, 3)$

5. Convert the following Cartesian coordinates to spherical coordinates. Plot these points in both spherical and Cartesian space.

(a) $(1, -1, \sqrt{6})$ Ans: $(2\sqrt{2}, 7\pi/4, \pi/6)$

(b) $(0, \sqrt{3}, 1)$ Ans: $(2, \pi/2, \pi/3)$

(c) $(\sqrt{2}, -\sqrt{2}, 2)$ Ans: $(2\sqrt{2}, 7\pi/4, \pi/4)$

6. Convert the following spherical coordinates to Cartesian coordinates. Plot these points both in spherical and Cartesian space.

(a) $(4, \pi/3, \pi/6)$ Ans: $(1, \sqrt{3}, 2\sqrt{3})$

(b) $(16, 4\pi/3, 3\pi/4)$ Ans: $(-4\sqrt{2}, -4\sqrt{6}, -8\sqrt{2})$

(c) $(20, -\pi/3, \pi/4)$ Ans: $(5\sqrt{2}, -5\sqrt{6}, 10\sqrt{2})$

7. Convert the following spherical coordinates to cylindrical coordinates. Plot these points both in spherical and cylindrical coordinates.

(a) $(2, \pi/3, \pi/6)$ Ans: $(1, \pi/3, \sqrt{3})$

(b) $(1, 4\pi/3, \pi/2)$ Ans: $(1, 7\pi/6, 0)$

(c) $(4, \pi/2, 5\pi/6)$ Ans: $(2, \pi/2, -2\sqrt{3})$

8. Convert the following cylindrical coordinates to spherical coordinates. Plot these points in both cylindrical coordinates and spherical coordinates.

(a) $(4, \pi/2, 0)$ Ans: $(4, \pi/2, \pi/2)$

(b) $(5\sqrt{3}, 5\pi/6, 5)$ Ans: $(10, 5\pi/6, \pi/3)$

(c) $(1, -\pi/4, -\sqrt{3})$ Ans: $(2, 7\pi/4, 5\pi/6)$

9. Describe the curve given by $r = 3$ in polar coordinates. Sketch this.
Ans: Circle of radius 3 centered at the origin.

10. Describe the curve given by $\theta = 3\pi/4$ in polar coordinates. Sketch this.
Ans: Ray from the origin at 45° 'above' negative x–axis.

11. Describe the curve given by $|\,\theta\,| = \pi/4$ in polar coordinates. Sketch this.
Ans: Ray from the origin at 45° above and below 'positive' x–axis.

12. Describe the surface given by $r = 2$ in cylindrical coordinates. Sketch this.
Ans: Cylinder of radius 2 centered at the origin.

13. Describe the surface given by $\theta = \pi/3$ in cylindrical coordinates. Sketch this.
Ans: Plane 'emanating' from the z–axis at angle $\theta = \pi/3$.

14. Describe the surface given by $\rho = 5$ in spherical coordinates. Sketch this.
Ans: Sphere of radius 5 centered at the origin.

15. Describe the surface given by $\phi = \pi/4$ in spherical coordinates. Sketch this.
Ans: Cone making an angle of 45° with the positive z–axis.

16. Describe the surface given by $\theta = \pi$ in spherical coordinates. Sketch this.
Ans: Plane 'emanating' from the z–axis at angle π, i.e. along the 'negative' x–axis.

17. Describe the curve given by $r^2 = 6r\cos\theta$ in polar coordinates. Sketch this. What if this were in cylindrical coordinates?

Ans: Substitute and complete square. Circle of radius 3 centered at $(3, 0)$.

18. Describe the surface given by $z = 2r$ in cylindrical coordinates. Sketch this.

Ans: Substitute in. A cone about the 'positive' z–axis making an angle of $26.5651°$ with the positive z–axis.

19. Describe the surface given by $\rho\cos\phi = 2\rho\sin\phi$ in spherical coordinates. Sketch this.

Ans: Divide and obtain $\tan\phi = 1/2$. So $\phi = \arctan(1/2)$, or a cone making an angle of $26.5651°$ with the 'positive' z–axis (see the previous problem).

20. Describe the surface given by $\rho = 2a\cos\phi$ in spherical coordinates. Sketch this.

Ans: Substitute in and complete square. Sphere with radius a and center $(0, 0, a)$.

Chapter 2

Partial Derivatives & their Applications

2.1 Limits

Limit: We say that $f(x)$ has limit L at a if for $\epsilon > 0$, there is $\delta > 0$ such that for all $|x-a| < \delta$, we have $|f(x)-L| < \epsilon$. Overall, limits in multivariable Calculus are more complicated because there are more directions to 'worry' about. Moreover, it can become increasingly difficult to picture what is going on.

Take for example the following $\lim_{(x,y,z)\to(1,-1,2)}(3x-5y+2z) = 12$. Given $\epsilon > 0$, we want δ so that if $0 < \|(x,y,z)-(1,-1,2)\| < \delta$, we have $|3x-5y+2z-12| < \epsilon$. Now $\|(x,y,z)-(1,-1,2)\| = \sqrt{(x-1)^2+(y+1)^2+(z-2)^2}$ so we merely need find the radius of some sphere. We have

$$\begin{aligned}\sqrt{(x-1)^2} &= |x-1|\\ \sqrt{(y+1)^2} &= |y+1|\\ \sqrt{(z-2)^2} &= |z-2|\end{aligned}$$

Now if $\sqrt{(x-1)^2+(y+1)^2+(z-2)^2} < \delta$, so too are each of the above. But...

$$\begin{aligned}|3x-5y+2z-12| &= |(3(x-1)-5(y+1)+2(z-2)|\\ &\leq 3|x-1|+5|y-1|+2|z-2|\\ &< 3\delta+5\delta+2\delta = 10\delta < \epsilon\end{aligned}$$

so that if we choose $\delta = \epsilon/10$, then $\epsilon = 10\delta$ so that we have the necessary inequality – seen above.

Squeeze Theorem: If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ and $f(x) \leq g(x) \leq h(x)$, then $\lim_{x\to a} g(x) = L$.

'Scholastic Approach': If you can plug it in, then you're done because we hand you continuous functions. Otherwise, try the limit from a few different directions, i.e. using a few different curves (horizontal line, vertical line, slanted line, and possible a polynomial path). Note, the curves should end at the point you want to end up at! If the limit is different at any stage, then you pontificate the limit may exist and will be that number. Now you need to prove it is. Try factoring or some 'algebraic trick' first. If this does not work, it 'must' be Squeeze Theorem by academic prestidigitation.

2.1 | Exercises

For Exercises 1–18, find the limit or explain why the limit does not exist.

1. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} x^2+4xy-yz^4+y^2-3z^2+4$

Ans: The limit is 0.

2. $\displaystyle\lim_{(x,y,z)\to(1,0,1)} x^2+4xy-z^2+3\sin(xyz)+5$

Ans: The limit is 5.

3. $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{e^{x-z}}{x+y+1}$

Ans: The limit is 1/3.

4. $\displaystyle\lim_{(x,y,z)\to(3,2,6)} \ln\left(2x-3y+e^{z-x}\right)$

Ans: The limits is 3.

5. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{2x+2y}{x+y}$

Ans: The limit is 2.

6. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{x^2+2xy+y^2}{x+y}$

Ans: The limit is 0.

7. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{y^2+2x^2}$

Ans: Observe that $y^2 \le y^2+2x^2$ so that $\frac{y^2}{y^2+2x^2} \le 1$. But then we have $0 \le \left|\frac{y^2\sin^2 x}{y^2+2x^2}\right| \le |\sin^2 x| \to 0$ so that the limit is 0 by Squeeze Theorem.

8. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Ans: Along $x=0$, the limit is 0. Along $y=0$, the limit is 1. Therefore, the limit does not exist.

9. $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x-1}{(x-1)^2+y^2}$

Ans: Along $x=1$, the limit is 0. Along $y=0$, the limit does not exist. Therefore, the limit does not exist.

10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Ans: Along both $x=0$ and $y=0$, the limit is 0. However, along $x=y$, the limit is 1/2. Therefore, the limit does not exist.

11. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x^3-y^3}{x-y}$

Ans: As $x^3 - y^2 = (x-y)(x^2+xy+y^2)$, the limit is 0.

12. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{x^2-y^2}{x+y}$

Ans: As $x^2-y^2=(x-y)(x+y)$, the limit is 0.

13. $\displaystyle\lim_{(x,y)\to(2,0)} \frac{x^2-y^2-4x+4}{x^2+y^2-4x+4}$

Ans: Along $x=2$, the limit is -1. Along the path $y=0$, the limit is 1. Therefore, the limit does not exist.

14. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4}$

Ans: Along $y=0$, the limit is 0. Along the line $x=y$, the limit is $1/2$. Therefore, the limit does not exist.

15. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y}{\sqrt{x}-\sqrt{y}}$

Ans: As $x-y=(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})$, the limit is 0.

16. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Ans: As $x-y=(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})$, the limit is 0.

17. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{3x^2+4y^2+5z^2}{x^2+y^2+z^2}$

Ans: Taking $x\to0$, $y\to0$, $z\to0$, we get 5 while taking $z\to0$, $y\to0$, $x\to0$, we obtain 3. Therefore, the limit does not exist.

18. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy-xz-yz}{x^2+y^2+z^2}$

Ans: Along the line $l(t)=(t,t,t)$, the limit is $1/3$ while along the path $(0,0,t)\to(0,0,0)$ the limit is 1. Therefore, the limit does not exist.

19. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$ exists along every straight line through the origin but that the limit does not exist. [Hint: For the second part, you may want to try a polynomial path.]

Ans: Any straight line path through the origin is of the form $y=mx$ (except $x=0$ where the limit is clearly 0). Then we have $\frac{x^4y^4}{(x^2+y^4)^3}=\frac{m^4x^8}{(x^2+m^4x^4)^3}=\frac{m^4x^8}{x^6(1+m^4x^2)}=\frac{m^4x^2}{(1+m^4x^2)^3}\to0$. However, the limit does not exist as along the path $x=y^2$, the limit is $1/8$. Therefore, the limit does not exist.

20. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to0}\frac{\sin x}{x}$

(b) $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{\sin(x+y)}{x+y}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$

Ans:

(a) By l'Hôpital?s rule or a geometric argument, we know $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$.

(b) Let $h = x + y$. Then $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{\sin(x+y)}{x+y} = \lim_{h\to 0} \frac{\sin h}{h} = 1$.

(c) Again, let $h = xy$. Then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = \lim_{h\to 0} \frac{\sin h}{h} = 1$.

Evaluate the following limits. [Note: It in some cases it may be easier to change coordinates]

21. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$

Ans: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = \lim_{r\to 0} \frac{r^3 \sin\theta\cos\theta}{r^2} = \lim_{r\to 0} r\, \frac{\sin 2\theta}{2} = 0$

22. $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

Ans: $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0} r^2 \ln r^2 = \lim_{r\to 0} \frac{2\ln r}{1/r^2} \overset{\text{L.H.}}{=} \lim_{r\to 0} \frac{2/r}{-2/r^3} = \lim_{r\to 0} -r^2 = 0$

23. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$

Ans: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2 + r^2\sin\theta\cos\theta}{r^2} = \lim_{r\to 0}\left(1 + \frac{\sin 2\theta}{2}\right)$. But then the limit depends on the angle and thus does not exist.

24. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Ans: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$. Thus, the limit depends on θ and hence does not exist.

25. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}$

Ans: $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \lim_{\rho\to 0} \frac{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}{\rho} = \lim_{\rho\to 0} \rho\sin^2\phi = 0.$

26. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$

Ans: $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\rho\sin\phi\cos\theta\cdot\rho\sin\phi\sin\theta\cdot\rho\cos\phi}{\rho^2} = \lim_{\rho\to 0} \rho\sin^2\phi\cos\phi\cos\theta\sin\theta =$ 0.

27. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xz}{x^2+y^2+z^2}$

Ans: $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xz}{x^2+y^2+z^2} = \lim_{\rho^2\sin\phi\cos\phi\cos\theta} \rho^2 = \lim_{\rho\to 0} \sin\phi\cos\phi\cos\theta = \cos\theta\frac{\sin 2\phi}{2}$. Therefore, the limit depends on the angles and hence does not exist.

28. Use the definition of the limit to show that $\lim_{(x,y,z)\to(2,-1,3)}(2x+5y-z)=-4$.

Ans: A simple proof.

29. Use the definition of the limit to show that $\lim_{(x,y,z)\to(1,1,2)}(x-y+3z)=6$.

Ans: A simple proof.

30. Use the definition of the limit to show that $\lim_{(x,y,z)\to(0,-1,3)}(7x-4y-z)=1$.

Ans: A simple proof.

31. Use the definition of the limit to show that $\lim_{(x,y,z)\to(3,-2,5)}(x+y+z)=0$.

Ans: A simple proof.

32. Show that $\lim_{(x,y,)\to(0,0}\dfrac{x^3+y^3}{x^2+y^2}=0$ by completing the following steps:

(a) Show $|x|\le\|(x,y)-(0,0)\|$ and $|y|\le\|(x,y)-(0,0)\|$.

(b) Show that $|x^3+y^3|\le 2(x^2+y^2)^{1/3}$. [Hint: The Triangle Inequality.]

(c) Show that $\left|\dfrac{x^3+y^3}{x^2+y^2}\right|<2\delta$ if $\|(x,y)-(0,0)\|<\delta$.

(d) Use the proceeding parts to show that the limit is 0.

Ans:

(a) We have $\|(x,y)-(0,0)\|=\|(x,y)\|=\sqrt{x^2+y^2}\ge\sqrt{x^2}=|x|$. The second part follows mutatis mutandis.

(b) Using the Triangle Inequality and the previous part, we have $|x^3+y^3|\le|x^3|+|y^3|=|x|^3+|y|^3\le 2(\sqrt{x^2+y^2})^3=2(x^2+y^2)^{3/2}$.

(c) If $0<\|(x,y)\|<\delta$, then by the previous part, we have

$$\left|\frac{x^3+y^3}{x^2+y^2}\right|\le\left|\frac{2(x^2+y^2)^{3/2}}{x^2+y^2}\right|=2\sqrt{x^2+y^2}=2\|(x,y)\|<2\delta$$

(d) Choose $\delta=\epsilon/2$ then $|f(x,y)|<\epsilon$ and then the limit is as stated.

2.2 Partial Derivatives

Ordinary Derivative: We want a good notion of differentiation. Recall the ordinary derivative:

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

We want the local rate of change of a function about some point. However, the situation is more complicated because we have many more directions we can approach from leaving us open to many more possible rates of change. This makes it harder for the limit to exist. Before defining a notion of derivative, we can look at the simpler partial derivative.

Partial Derivative: Literally, the derivative of a partial function. For example, taking $f(x, y)$ and creating a partial function by holding y fixed, we obtain

$$\frac{\partial f}{\partial x}(x, y) := f_x(x, y) := \lim_{h\to 0} \frac{f(x+h, y)-f(x)}{h}$$

We can do the same thing for y. Note this is also denoted $D_1 f$ – but much less often. These partial derivatives give us the slopes in the x and y directions. Image cutting the surface $z = f(x, y)$ by planes in the x and y direction. Then f_x, f_y give us the slopes of the curves in these partial directions, respectively. One might wonder if we cut in other directions but this will need to the more general notion of directional derivatives later.

Of course, then there is nothing stopping us from doing this process again and again to form higher order partial derivatives. Note that $f_{yx} := \dfrac{\partial^2 f}{\partial x \partial y} := \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y} f$. Be careful of the order. Of course, this immediately begs the question, is it the case that $f_{xy} = f_{yx}$? The general answer is no, but there is a special case.

Clairaut's Theorem: If f is defined on a disk containing (a, b) and f_{xy}, f_{yx} are continuous, then $f_{xy} = fyx$. That is, if there is some 'room' about the point which you are looking at and the partials are continuous, then one needn't be concerned about the order.

Higher Derivatives: The general notion of multivariable derivatives is more complicated than the simpler notion of partial derivatives. First, consider a function $f(x_1, x_2, \ldots, x_n)$. Define a matrix of partial derivatives

$$Df(x_1, x_2, \ldots, x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Chain Rule: Recall the chain rule in one variable: $(g \circ f)'(x) = g'(f(x)) = f'(x)$. The same idea works in higher dimension. You can also think of this as changes at lower levels inducing change at higher levels and also change at higher levels resulting from changes at lower levels. So if $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $\mathbf{x}(t) \in X$, then $\dfrac{df}{dt}(t_0) = \dfrac{\partial f}{\partial x}(\mathbf{x}_0)\dfrac{dx}{dt}(t_0) + \dfrac{\partial f}{\partial y}(\mathbf{x}_0)\dfrac{dy}{dt}(t_0)$. Overall, think of $z = f(x,y)$, where x, y depend on t. Then if we change t, we most likely change z. So we have

$$`\Delta z' = \underbrace{\Delta z}_{\text{From } x} + \underbrace{\Delta z}_{\text{From } y} = z_x \cdot \Delta x + z_y \cdot \Delta y = z_x x'(t_0) + z_y y'(t_0)$$

which replacing the pieces properly gives the exact formula. This type of logic will always work – no need for 'fancy' Theorems.

2.2 | Exercises

Partial Derivatives

In Exercises 1–10, given the function f, find as many of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ as possible.

1. $f(x,y) = x^2y^3 + x - y + 1$

Ans:

(a) $\frac{\partial f}{\partial x} = 2xy^3 + 1$

(b) $\frac{\partial f}{\partial y} = 3x^2y^2 - 1$

(c) $\frac{\partial^2 f}{\partial x^2} = 2y^3$

(d) $\frac{\partial^2 f}{\partial y^2} = 6x^2y$

(e) $\frac{\partial^2 f}{\partial x \partial y} = 6xy^2$

(f) $\frac{\partial^2 f}{\partial y \partial x} = 6xy^2$

2. $f(x,y) = x\sqrt{y} - y\sqrt{x}$

Ans:

(a) $\frac{\partial f}{\partial x} = \sqrt{y} - \frac{y}{2\sqrt{x}}$

(b) $\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}} - \sqrt{x}$

(c) $\frac{\partial^2 f}{\partial x^2} = \frac{y}{4x^{3/2}}$

(d) $\frac{\partial^2 f}{\partial y^2} = -\frac{x}{4y^{3/2}}$

(e) $\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{2\sqrt{y}} - \frac{1}{2\sqrt{x}}$

(f) $\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{2\sqrt{y}} - \frac{1}{2\sqrt{x}}$

3. $f(x,y) = \frac{x}{y}$

Ans:

(a) $\frac{\partial f}{\partial x} = \frac{2x}{y^3}$

(b) $\frac{\partial f}{\partial y} = -\frac{3x^2}{y^4}$

(c) $\frac{\partial^2 f}{\partial x^2} = \frac{2}{y^3}$

(d) $\frac{\partial^2 f}{\partial y^2} = \frac{12x^2}{y^5}$

(e) $\frac{\partial^2 f}{\partial x \partial y} = -\frac{6x}{y^4}$

(f) $\frac{\partial^2 f}{\partial y \partial x} = -\frac{6x}{y^4}$

4. $f(x,y) = x \ln y$

Ans:

(a) $\frac{\partial f}{\partial x} = \ln y$

(b) $\frac{\partial f}{\partial y} = \frac{x}{y}$

(c) $\frac{\partial^2 f}{\partial x^2} = 0$

(d) $\frac{\partial^2 f}{\partial y^2} = -\frac{x}{y^2}$

(e) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{1}{y}$

(f) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{1}{y}$

5. $f(x, y) = x \ln(xy)$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \ln(xy) + 1$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{x}{y}$

(c) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{x}$

(d) $\dfrac{\partial^2 f}{\partial y^2} = -\dfrac{x}{y^2}$

(e) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{1}{y}$

(f) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{1}{y}$

6. $f(x, y) = \arctan(xy)$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{y}{1 + x^2 y^2}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{x}{1 + x^2 y^2}$

(c) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{-2xy^3}{(1 + x^2 y^2)^2}$

(d) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{-2x^3 y}{(1 + x^2 y^2)^2}$

(e) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{1 - x^2 y^2}{(1 + x^2 y^2)^2}$

(f) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{1 - x^2 y^2}{(1 + x^2 y^2)^2}$

7. $f(x, y) = y e^{xy}$

Ans:

(a) $\dfrac{\partial}{\partial x} = y^2 e^{xy}$

(b) $\dfrac{\partial}{\partial y} = e^{xy}(1 + xy)$

(c) $\dfrac{\partial^2}{\partial x^2} = y^3 e^{xy}$

(d) $\dfrac{\partial^2}{\partial y^2} = x e^{xy}(2 + xy)$

(e) $\dfrac{\partial^2}{\partial x \partial y} = y e^{xy}(2 + xy)$

(f) $\dfrac{\partial^2}{\partial y \partial x} = y e^{xy}(2 + xy)$

8. $f(x, y) = x^y$

Ans:

(a) $\dfrac{\partial f}{\partial x} = y x^{y-1}$

(b) $\dfrac{\partial f}{\partial y} = x^y \ln(x)$

(c) $\dfrac{\partial^2 f}{\partial x^2} = y x^{y-2}(1 - y)$

(d) $\dfrac{\partial^2 f}{\partial y^2} = x^y \ln(x)^2$

(e) $\dfrac{\partial^2 f}{\partial x \partial y} = x^{y-1}(1 + y \ln x)$

(f) $\dfrac{\partial^2 f}{\partial y \partial x} = x^{y-1}(1 + y \ln x)$

9. $f(x,y) = \dfrac{y}{1-xy}$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{y^2}{(1-xy)^2}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{1}{(1-xy)^2}$

(c) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{-2y^3}{(1-xy)^3}$

(d) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{-2x}{(1-xy)^3}$

(e) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{2y}{(1-xy)^3}$

(f) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{2y}{(1-xy)^3}$

10. $f(x,y) = e^{2y}\sin(\pi x)$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \pi e^{2y}\cos(\pi x)$

(b) $\dfrac{\partial f}{\partial y} = 2e^{2y}\sin(\pi x)$

(c) $\dfrac{\partial^2 f}{\partial x^2} = -\pi^2 e^{2y}\sin(\pi x)$

(d) $\dfrac{\partial^2 f}{\partial y^2} = 4e^{2y}\sin(\pi x)$

(e) $\dfrac{\partial^2 f}{\partial x \partial y} = 2\pi e^{2y}\cos(\pi x)$

(f) $\dfrac{\partial^2 f}{\partial y \partial x} = 2\pi e^{2y}\cos(\pi x)$

In Exercises 11–20, given the function f, find as many of $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial f}{\partial z}$, $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial z^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\dfrac{\partial^2 f}{\partial y \partial z}$, $\dfrac{\partial^2 f}{\partial x \partial z}$ as possible.

11. $f(x,y,z) = x^4yz^2 - x^3z + yx^2 + 4$

Ans:

(a) $\dfrac{\partial f}{\partial x} = x(-3xz + y(2+4x^2z^2))$

(b) $\dfrac{\partial f}{\partial y} = x^2 + x^4z^2$

(c) $\dfrac{\partial f}{\partial z} = x^3(2xyz - 1)$

(d) $\dfrac{\partial^2 f}{\partial x^2} = 2(y - 3xz + 6x^2yz^2)$

(e) $\dfrac{\partial^2 f}{\partial y^2} = 0$

(f) $\dfrac{\partial^2 f}{\partial z^2} = 2x^4y$

(g) $\dfrac{\partial^2 f}{\partial y \partial x} = 2(x + 2x^3z^2)$

(h) $\dfrac{\partial^2 f}{\partial z \partial y} = 2x^4z$

(i) $\dfrac{\partial^2 f}{\partial z \partial x} = x^2(8xyz - 3)$

12. $f(x,y,z) = \dfrac{x\sqrt{y}}{z}$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{\sqrt{y}}{z}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{x}{2z\sqrt{y}}$

(c) $\dfrac{\partial f}{\partial z} = -\dfrac{x\sqrt{y}}{z^2}$

(d) $\dfrac{\partial^2 f}{\partial x^2} = 0$

(e) $\dfrac{\partial^2 f}{\partial y^2} = -\dfrac{x}{4zy^{3/2}}$

(f) $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{2x\sqrt{y}}{z^3}$

(g) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{1}{2z\sqrt{y}}$ (h) $\dfrac{\partial^2 f}{\partial z \partial y} = -\dfrac{x}{2z^2\sqrt{y}}$ (i) $\dfrac{\partial^2 f}{\partial z \partial x} = -\dfrac{\sqrt{y}}{z^2}$

13. $f(x,y,z) = \dfrac{x^z}{y}$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{zx^{z-1}}{y}$ (d) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{z(z-1)x^{z-2}}{y}$ (g) $\dfrac{\partial^2 f}{\partial y \partial x} = -\dfrac{zx^{z-1}}{y^2}$

(b) $\dfrac{\partial f}{\partial y} = -\dfrac{x^z}{y^2}$ (e) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2x^z}{y^3}$ (h) $\dfrac{\partial^2 f}{\partial z \partial y} = -\dfrac{x^z \ln x}{y^2}$

(c) $\dfrac{\partial f}{\partial z} = \dfrac{x^z \ln x}{y}$ (f) $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{x^z \ln(x)^2}{y}$ (i) $\dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{x^{z-1}(1 + z\ln x)}{y}$

14. $f(x,y,z) = \dfrac{\cos x \ln y}{\sqrt{z}}$

Ans:

(a) $\dfrac{\partial f}{\partial x} = -\dfrac{\ln y \sin x}{\sqrt{z}}$ (d) $\dfrac{\partial^2 f}{\partial x^2} = -\dfrac{\cos x \ln y}{\sqrt{z}}$ (g) $\dfrac{\partial^2 f}{\partial y \partial x} = -\dfrac{\sin x}{y\sqrt{z}}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{\cos x}{y\sqrt{z}}$ (e) $\dfrac{\partial^2 f}{\partial y^2} = -\dfrac{\cos x}{y^2\sqrt{z}}$ (h) $\dfrac{\partial^2 f}{\partial z \partial y} = -\dfrac{\cos x}{2yz^{3/2}}$

(c) $\dfrac{\partial f}{\partial z} = -\dfrac{\cos x \ln y}{2z^{3/2}}$ (f) $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{3\cos x \ln y}{4z^{5/2}}$ (i) $\dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{\ln y \sin x}{2z^{3/2}}$

15. $f(x,y,z) = e^{xyz}$

Ans:

(a) $\dfrac{\partial f}{\partial x} = yze^{xyz}$ (d) $\dfrac{\partial^2 f}{\partial x^2} = y^2z^2e^{xyz}$ (g) $\dfrac{\partial^2 f}{\partial y \partial x} = z(1 + xyz)e^{xyz}$

(b) $\dfrac{\partial f}{\partial y} = xze^{xyz}$ (e) $\dfrac{\partial^2 f}{\partial y^2} = x^2z^2e^{xyz}$ (h) $\dfrac{\partial^2 f}{\partial z \partial y} = x(1 + xyz)e^{xyz}$

(c) $\dfrac{\partial f}{\partial z} = xye^{xyz}$ (f) $\dfrac{\partial^2 f}{\partial z^2} = x^2y^2e^{xyz}$ (i) $\dfrac{\partial^2 f}{\partial z \partial x} = y(1 + xyz)e^{xyz}$

16. $f(x,y,z) = z^{y/x}$

Ans:

(a) $\dfrac{\partial f}{\partial x} = -\dfrac{yze^{y/x}}{x^2}$ (d) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{yz(2x + y)e^{y/x}}{x^4}$ (g) $\dfrac{\partial^2 f}{\partial y \partial x} = -\dfrac{z(x + y)e^{y/x}}{x^3}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{ze^{y/x}}{x}$ (e) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{ze^{y/x}}{x^2}$ (h) $\dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{e^{y/x}}{x}$

(c) $\dfrac{\partial f}{\partial z} = e^{y/x}$ (f) $\dfrac{\partial^2 f}{\partial z^2} = 0$ (i) $\dfrac{\partial^2 f}{\partial z \partial x} = -\dfrac{ye^{y/x}}{x^2}$

17. $f(x,y,z) = \dfrac{x+y}{y+z}$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{1}{y+z}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{z-x}{(y+z)^2}$

(c) $\dfrac{\partial f}{\partial z} = -\dfrac{x+y}{(y+z)^2}$

(d) $\dfrac{\partial^2 f}{\partial x^2} = 0$

(e) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2(x-z)}{(y+z)^3}$

(f) $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{2(x+y)}{(y+z)^3}$

(g) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{-1}{(y+z)^2}$

(h) $\dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{2x+y-z}{(y+z)^3}$

(i) $\dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{-1}{(y+z)^2}$

18. $f(x,y,z) = \dfrac{x+e^{2y}-z}{xy}$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{z-e^{2y}}{x^2y}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{z-x+e^{2y}(2y-1)}{xy^2}$

(c) $\dfrac{\partial f}{\partial z} = \dfrac{-1}{xy}$

(d) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2(e^{2y}-z)}{x^3y}$

(e) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2(x+e^{2y}(1+2(y-1)y)-z)}{xy^3}$

(f) $\dfrac{\partial^2 f}{\partial z^2} = 0$

(g) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{e^{2y}(1-2y)-z}{x^2y^2}$

(h) $\dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{1}{xy^2}$

(i) $\dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{1}{x^2y}$

19. $f(x,y,z) = y \arctan(xyz)$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{zy^2}{1+x^2y^2z^2}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{xyz}{1+x^2y^2z^2} + \arctan(xyz)$

(c) $\dfrac{\partial f}{\partial z} = \dfrac{xy^2}{1+x^2y^2z^2}$

(d) $\dfrac{\partial^2 f}{\partial x^2} = -\dfrac{2xy^4z^3}{)1+x^2y^2z^2)^2}$

(e) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2xz}{(1+x^2y^2z^2)^2}$

(f) $\dfrac{\partial^2 f}{\partial z^2} = -\dfrac{2x^3y^4z}{(1+x^2y^2z^2)^2}$

(g) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{2yz}{(1+x^2y^2z^2)^2}$

(h) $\dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{2xy}{(1+x^2y^2z^2)^2}$

(i) $\dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{y-x^2y^4z^2}{(1+x^2y^2z^2)^2}$

20. $f(x,y,z) = \ln y^4 e^{\pi x} \sin(3z)$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \pi e^{\pi x} \ln(y^4) \sin(3z)$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{4e^{\pi x} \sin(3z)}{y}$

(c) $\dfrac{\partial f}{\partial z} = 3e^{\pi x} \ln(y^4) \cos(3z)$

(d) $\dfrac{\partial^2 f}{\partial x^2} = \pi^2 e^{\pi x} \ln(y^4) \sin(3z)$

(e) $\dfrac{\partial^2 f}{\partial y^2} = -\dfrac{4e^{\pi x} \sin(3z)}{y^2}$

(f) $\dfrac{\partial^2 f}{\partial z^2} = -9e^{\pi x} \ln(y^4) \sin(3z)$

(g) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{4\pi e^{\pi x} \sin(3z)}{y}$

(h) $\dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{12e^{\pi x} \cos(3z)}{y}$

(i) $\dfrac{\partial^2 f}{\partial z \partial x} = 3\pi e^{\pi x} \ln(y^4) \cos(3z)$

Chain Rule

21. If $T(x,y) = x^2e^y - xy^3$ and $x = \cos t$ and $y = \sin t$, find $\dfrac{dT}{dt}$.

Ans: $-2\cos t\sin te^{\sin t} + \cos^3 te^{\sin t} + \sin^4 t - 3\cos^2 t\sin^2 t$

22. If $u(x,y,z) = xe^{yz}$, where $x(t) = e^t$, $y(t) = t$, and $z(t) = \sin t$, find $\dfrac{du}{dt}$.

Ans: $e^{t(1+\sin t)}(1 + \sin t + t\cos t)$

23. If $z = f(x,y)$, where $x = g(t)$ and $y = h(t)$, find a formula for $\dfrac{dz}{dt}$ using $g(t)$ and $h(t)$.

Ans: $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

24. If $z = x^2y^3 + y\cos x$, where $x = \ln t^2$ and $y = \sin(4t)$, find $\dfrac{dz}{dt}$.

Ans: $\dfrac{4\sin^3(4t)\ln t^2 - 2\sin 4t\sin\ln t^2}{t} + 4\cos(4t)(3\sin^2(4t)(\ln t^2)^2 + \cos\ln t^2)$

25. If $z = e^{2t}\sin(3\theta)$, where $r(s,t) = st - t^2$ and $\theta(s,t) = \sqrt{s^2+t^2}$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Ans: $\dfrac{\partial z}{\partial s} = t(2e^{2(st-t^2)}\sin(3\sqrt{s^2+t^2})) + \dfrac{3se^{2(st-t^2)}\cos(3\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}$ and $\dfrac{\partial z}{\partial t} = (s-2t)(2e^{2(st-t^2)}\sin(3\sqrt{s^2+t^2})) + \dfrac{3te^{2(st-t^2)}\cos(3\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}$

26. If $z(x,y) = x^2y - y^2$ and $x(t) = t^2$ and $y(t) = 2t$, then find $\dfrac{dz}{dt}$.

Ans: $10t^4 - 8t$

27. If $z = ye^{x^2}$, where $x(u,v) = \sqrt{uv}$ and $y(u,v) = \dfrac{1}{v}$, find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Ans: $z_u = e^u$ and $z_v = 0$.

28. If a curve is given by $F(x,y) = 0$, show that the slope of the tangent line is given by

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Ans: Simple use of chain rule.

29. If a curve is given by $F(x,y) = 0$ and $g(x,y)$ is defined along the curve, find $\dfrac{dg}{dx}$.

Ans: $\dfrac{\partial g}{\partial x} - \dfrac{\partial g}{\partial y}\dfrac{F_x}{F_y}$

30. If a ship is traveling along a path given by $\mathbf{x}(t) = (t, \cos t, \sin t)$, describe the path and find the temperature of the ship, $H(x,y,z)$, as a function of time.

Ans: Helix about the x–axis. $\dfrac{dH}{dt} = H_x - \sin tH_y + \cos tH_z$

2.3 Arclength, Tangent Planes, & Differential Geometry

Arclength: Imagine a curve being traced out in time. We measure where the curve is at in small increments of time. Then we can form the line segments connecting each point to the next, find the length of each segment, and add these up to find an approximation for the length of the curve. Note that $\sum \Delta s = \sum \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}$. Using a bit of work – MVT – we obtain

$$L(\mathbf{x}(t)) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_a^b \|x'(t)\|\, dt$$

Reduce down to the 2–dimensional case via $(x(t), y(t), 0)$ and factor out the dx so that it reduces down to ordinary Calculus II formula.

Unit Tangent: The unit tangent is $\mathbf{T} = \dfrac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$. It gives the direction a curve is moving. Give a simple diagram example.

Curvature: The curvature κ in $\mathbb{R}^3$ is the angular rate of change of the direction of $\mathbf{T}$ per unit change in distance along the path. Since this is per unit time, the curvature is an intrinsic quantity.

$$\kappa(t) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\left\|\frac{d\mathbf{T}}{dt}\right\|}{\frac{ds}{dt}}$$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{x}'(t)|}$$

The curvature of the curve given by $\mathbf{x}(t)$ is

$$\kappa(t) = \frac{|\mathbf{x}'(t) \times \mathbf{x}''(t)|}{|\mathbf{x}'(t)|^3}$$

(Principal) Unit Normal Vector: The principal normal vector measures the change in the unit tangent vector.

$$\mathbf{N} = \frac{\mathbf{T}'(t)}{|\mathbf{T}')(t)|}$$

It gives the direction in which the curve is ‘turning’.

Binormal Vector: The binormal vector is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. This creates a moving coordinate system for the curve.

Tangent Plane: Simply a plane but we need to match the slope in the x, y, and z directions. Once again, we can reduce to the 2–dimensional case if need be.

$$z = z_0 + f_x(x - x_0) + f_y(y - y_0)$$

We can use this to approximate function values as in the Calculus I case (this should remind the reader of Taylor Series. Also mention the idea of Chain Rule. Do you see the idea of approximation to function (change in z) and how we have defined it?

Total Differential: Same idea as approximation with the tangent plane. The change in f due to change in x times the change in x, f due to change in y times the change in y, f due to change in z times the change in z, then add to get total change.

$$df = f_x \Delta x + f_y \Delta y + f_z \Delta z$$

2.3 | Exercises

Derivatives & Tangents

1. If $x(t) = 2t + 1$ and $y(t) = 3 - t$, what is $\dfrac{dy}{dx}$?
Ans: $-1/2$

2. If $x(t) = t^2 - t + 1$ and $y(t) = 4 - t^2$, what is $\dfrac{dy}{dx}$?
Ans: $\dfrac{2t}{1-2t}$

3. If $x(t) = \cos t$ and $y(t) = t \sin t$, what is $\dfrac{dy}{dx}$?
Ans: $-\csc t(t\cos t + \sin t)$

4. If $x(t) = \ln t$ and $y(t) = \tan t$, what is $\dfrac{dy}{dx}$?
Ans: $t \sec^2 t$

5. If $x(t) = \sqrt{t^3}$ and $y(t) = \dfrac{e^t}{t}$, what is $\dfrac{dy}{dx}$?
Ans: $\dfrac{2\sqrt{t^3}e^t(t-1)}{3t^4}$

6. Find $\dfrac{d^2y}{dx^2}$ if $x(t) = 4t^2 + 3$ and $y(t) = 1 - t$.
Ans: 0

7. Find $\dfrac{d^2y}{dx^2}$ if $x(t) = \sec t$ and $y(t) = \tan t$.
Ans: $2\sec t$

8. Find $\dfrac{d^2y}{dx^2}$ if $x(t) = \ln t$ and $y(t) = t^{3/2}$.
Ans: $\dfrac{3\sqrt{t}}{4}$

9. Find $\dfrac{d^2y}{dt^2}$ if $x(t) = e^t$ and $y(t) = t\cos^2 t$.
Ans: $e^{-t}(-2t\cos^2 t - 4\cos t \sin t + 2t\sin^2 t) = -2e^{-t}(t\cos 2t + \sin 2t)$

10. Find $\dfrac{d^2y}{dt^2}$ if $x(t) = t - \dfrac{1}{t}$ and $y(t) = \arctan t$.
Ans: $\dfrac{-2t}{(1+1/t^2)(1+t^2)^2} = \dfrac{-2t^3}{(1+t^2)^3}$

11. Find the tangent line to the curve $x(t) = t^2 + 3t + 1$, $y(t) = 4t + 7$ when $t = 2$.
Ans: $y = \dfrac{4x+17}{7}$

12. Find the tangent line to the curve $x(t) = \cos t$, $y(t) = \sin t$ when $t = \dfrac{\pi}{4}$.

Ans: $y = \sqrt{2} - x$

13. Find the tangent line to the curve $C(t) = (\ln(t+1), \arctan(t-1))$ when $t = 0$.
Ans: $y = \dfrac{4x + \pi}{8}$

14. Find the tangent line to the curve $x(t) = \dfrac{3t}{1+t^3}$, $y(t) = \dfrac{3t^2}{1+t^3}$ when $t = 1$.
Ans: $y = 3 - x$

15. Find the tangent line to the curve $x(t) = \sec t$, $y(t) = \tan t$ when $t = \dfrac{\pi}{3}$.
Ans: $y = \dfrac{2x}{\sqrt{3}}$

Arclength

16. Find the length of the curve $l(t) = (1, 2)$, where $0 \leq t \leq \sqrt{\pi}$.
Ans: 0

17. Find the length of the curve $x(t) = 4t - 1$, $y(t) = 5 - 3t$, where $1 \leq t \leq 3$.
Ans: 10

18. Find the length of the curve $x(t) = \dfrac{2t^{3/2}}{3}$, $y(t) = 1 - t$, where $0 \leq t \leq 1$.
Ans: $\dfrac{2(2\sqrt{2} - 1)}{3}$

19. Find the length of the curve $x(t) = \sin t$, $y(t) = \cos t$, where $0 \leq t \leq \pi$.
Ans: 2π

20. Find the length of the curve $x(t) = \arctan t$, $y(t) = t^2$, where $-1 \leq t \leq 3$.
Ans: 10.7583

21. Find the length of the curve $x(t) = \sqrt{2}t$, $y(t) = \dfrac{t^2}{2}$, $z(t) = \ln t$, where $1 \leq t \leq 4$.
Ans: $\dfrac{15 + 4\ln 2}{2}$

22. Find the length of the curve $\mathbf{x}(t) = (r\cos t, r\sin t, st)$, where $0 \leq t \leq 2\pi$.
Ans: $2\pi\sqrt{r^2 + s^2}$

23. Find the length of the curve $\mathbf{x}(t) = \mathbf{i} + \dfrac{t^2}{2}\mathbf{j} + t\mathbf{k}$, where $0 \leq t \leq 1$.
Ans: $\dfrac{\sqrt{2} + \ln(1 + \sqrt{2})}{2}$

24. Find the length of the straight line path connecting (x_0, x_1) and (y_0, y_1) two different ways.
Ans: One use the distance formula. The other parametrize a straight line segment connecting the points

and find its length.

25. For a curve $y = f(x)$ such that $f'(x)$ exists and is continuous for $x \in [a, b]$, show that the length of the curve between $(a, f(a))$ and $(b, f(b))$ is exactly

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Ans: The standard proof or simply use the parametrization $x = t,\ y = f(t)$.

Tangent Plane

26. Find the tangent plane for the function $f(x, y) = x^2 y + x + y + 1$ at the point $(1, -2, 0)$.
Ans: $0 - 3(x - 1) + 2(y + 2) = 7 - 3x + 2y$

27. Find the tangent plane for the function $f(x, y) = \dfrac{x}{y}$ at the point $(-2, 3, -\frac{2}{3})$.

Ans: $-\dfrac{2}{3} + \dfrac{1}{3}(x + 2) + \dfrac{2}{9}(y - 3) = \dfrac{3x + 2y - 6}{9}$

28. Find the tangent plane for the function $f(x, y) = \sin x \cos y$ at the point $(\pi/2, \pi/2, 0)$.
Ans: $0 - (y - \pi/2) = \dfrac{\pi - 2y}{2}$

29. Find the tangent plane for the function $f(x, y) = \dfrac{x + y}{x + 2}$ at the point $(1, 0, 1)$.
Ans: $\dfrac{1}{3} + \dfrac{2}{9}(x - 1) + \dfrac{1}{3}(y - 0) = \dfrac{1 + 2x + 3y}{9}$

30. Find the tangent plane for the function $f(x, y) = x \arctan(xy)$ at the point $(1, 0, 0)$.
Ans: y

31. Find the tangent plane for the function $f(x, y) = x\sqrt{y} - \dfrac{1}{\sqrt{x^3}}$ at the point $(1, 1, 0)$.

Ans: $0 + \dfrac{5}{2}(x - 1) + \dfrac{1}{2}(y - 1) = \dfrac{5x + y - 6}{2}$

32. Find the tangent plane for the function $f(x, y) = e^{x-1}\ln(xy + 2)$ at the point $(1, -1, 3)$.
Ans: $3 - (x - 1) + (y + 1) = 5 - x + y$

33. Find the tangent plane for the function $f(x, y) = y\sin(\pi xy) + \dfrac{1}{2}$ at the point $(1, 1/2, 1)$.

Ans: $1 + 0(x - 1) + (y - 1/2) = \frac{1}{2} + y$

34. Find the tangent plane for the function $f(x, y) = \sqrt{xy}$ at the point $(2, 18, 6)$.
Ans: $6 + \dfrac{3}{2}(x - 2) + \dfrac{1}{6}(y - 18) = \dfrac{9x + y}{6}$

35. Find the tangent plane for the function $f(x, y) = y2^{xy} - 2$ at the point $(0, 1, -1)$.

Ans: $-1+\ln 2(x-0)+1(y-1)=y+\ln 2x-2$

Total Differentials

36. Find the total differential for the function $f(x,y)=xy$.
Ans: $df=y\,dx+x\,dy$

37. Find the total differential for the function $f(x,y)=x^3y^2+2$.
Ans: $df=3x^3y^2\,dx+2x^3y\,dy$

38. Find the total differential for the function $f(x,y)=x+y+\dfrac{x}{y}$.
Ans: $df=\left(1+\dfrac{1}{y}\right)dx+\left(1-\dfrac{x}{y^2}\right)dy$

39. Find the total differential for the function $f(x,y)=x\arctan y$.
Ans: $df=\arctan y\,dx+\dfrac{x}{1+y^2}\,dy$

40. Find the total differential for the function $f(x,y)=ye^{xy}$.
Ans: $df=y^2e^{xy}\,dx+e^{xy}(1+xy)\,dy$

41. Find the total differential for the function $f(x,y,z)=x^2y+y^2z+4$.
Ans: $df=2xy\,dx+(x^2+2yz)\,dy+y^2\,dz$

42. Find the total differential for the function $f(x,y,z)=\dfrac{xy^2}{\sqrt{z^3}}$.
Ans: $df=\dfrac{y^2}{z^{3/2}}\,dx+\dfrac{2xy}{z^{3/2}}\,dy-\dfrac{3xy^2}{2z^{5/2}}\,dz$

43. Find the total differential for the function $f(x,y,z)=e^x\sin y+\cos z$.
Ans: $df=e^x\sin y\,dx+e^x\cos y\,dy-\sin z\,dz$

44. Find the total differential for the function $f(x,y,z)=x\arctan(yz^2)$.
Ans: $df=\arctan(yz^2)\,dx+\dfrac{xz^2}{1+y^2z^4}\,dy+\dfrac{2xyz}{1+y^2z^4}\,dz$

45. Find the total differential for the function $f(x,y,z)=\dfrac{2+\ln xz}{y}$.
Ans: $df=\dfrac{1}{xy}\,dx-\dfrac{2+\ln xz}{y^2}\,dy+\dfrac{1}{yz}\,dz$

Approximations

46. Approximate $(10.2)^3(7.9)^2$.
Ans: $10^3\cdot 8^2=64000$. $df=2240$. Approximation: 66240. Actual: 66230. Error: 10.0087

47. Approximate $\dfrac{3.2^3}{\sqrt{4.1\cdot 8.8}}$.

Ans: $\dfrac{3^3}{\sqrt{4\cdot 9}} = 9/2$. $df = 0.89375$. Approximation: 5.39375. Actual: 5.45528. Error: -0.0615253

48. Approximate $\sin(\pi - 0.2)\cos(0.1)$.

Ans: $\sin(\pi)\cos(0) = 0$. $df = 0.2$. Approximation: 0.2. Actual: 0.197677. Error: 0.00232319

49. Approximate $7.9\cos(2.2(\pi - 0.1))$.

Ans: $8\cos(2\pi) = 8$. $df = -0.1$. Approximation: 7.9. Actual: 7.25054. Error: 0.649461.

Differential Geometry

50. Compute $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ for the curve $\mathbf{x}(t) = 5\cos 3t\,\mathbf{i} + 6t\,\mathbf{j} + 5\sin 3t\,\mathbf{k}$.

Ans: $\mathbf{T} = \dfrac{1}{\sqrt{261}}\langle -15\sin 3t, 6, 15\cos 3t\rangle$, $\mathbf{N} = \langle -\cos 3t, 0, -\sin 3t\rangle$, $\mathbf{B} = \dfrac{1}{\sqrt{29}}\langle -2\sin 3t, -52\cos 3t\rangle$, $\kappa = 5/29$.

51. Compute $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ for the curve $\mathbf{x}(t) = \left(t, \dfrac{(t+1)^{3/2}}{3}, \dfrac{(1-t)^{3/2}}{3}\right)$, $-1 < t < 1$.

Ans: $\mathbf{T} = \sqrt{\dfrac{2}{3}}(1, \tfrac{1}{2}\sqrt{t+1}, \dfrac{-1}{2}\sqrt{1-t})$, $\mathbf{N} = \dfrac{1}{\sqrt{2}}(0, \sqrt{1-t}, \sqrt{t+1})$, $\mathbf{B} = \dfrac{1}{\sqrt{3}}(1, -\sqrt{t+1}, \sqrt{1-t})$, $\kappa = \dfrac{1}{3\sqrt{2(1-t^2)}}$.

52. Compute $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ for the curve $\mathbf{x}(t) = (e^{2t}\sin t, e^{2t}\cos t, 1)$.

Ans: $\mathbf{T} = \dfrac{1}{\sqrt{5}}(2\sin t + \cos t, 2\cos t - \sin t, 0)$, $\mathbf{N} = \dfrac{1}{\sqrt{5}}(2\cos t - \sin t, -2\sin t - \cos t, 0)$, $\mathbf{B} = (0, 0, -1)$, $\kappa = \dfrac{1}{e^{2t}\sqrt{5}}$.

53. Show that a circle has constant curvature.

Ans: Simple application of the formula. $\kappa = 1/R$.

54. Show that a helix has constant curvature.

Ans: Simple application of the formula. $\kappa = 1/R$.

2.4 Gradients & Directional Derivatives

Vector Field: A map which assigns to each point of the space a vector. It gives direction of a force or the like.

Del Operator: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

Gradient: We want something that computes all the change of f at once – taking $\hat{x}$, $\hat{y}$, and $\hat{z}$ into account at once. The gradient turns a scalar field into a vector field. Taking $f(x, y, z)$, we have

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

This easily generalizes to higher dimensions.

Divergence: Turns a vector field into a scalar field.

$$\nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

It gives the 'net mass flow' at a point or the 'flux density' at a point, i.e. flow in/out. Positive means more flow out while negative means more flow in than out. If $\nabla \cdot \mathbf{F} = 0$, we say that $\mathbf{F}$ is incompressible or solenoidal.

Curl: Curl measures the 'circulation' of a vector field. Imagine a twig moving in the current on a lake or in a stream. Imagine the current keeps the twig in a closed loop path. The curl does not see this. The curl measures how quickly and in what direction the twig rotates about itself as it moves in its path. A vector field is said to be irrotational if $\nabla \times \mathbf{F} = \mathbf{0}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

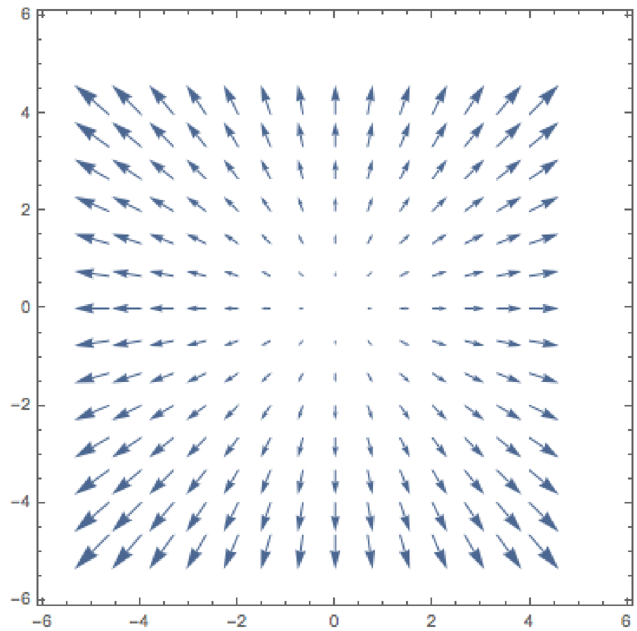
Note that $\nabla \times \nabla f = \mathbf{0}$ for all f of class C^2. That is, gradient fields are irrotational. Furthermore, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$; that is, curl's are always incompressible vector fields.

Normal Line: Perpendicular line.

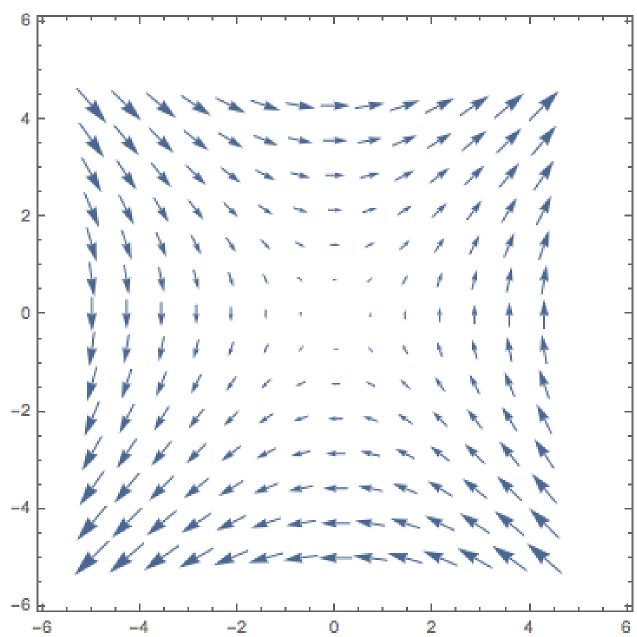
2.4 | Exercises

Vector Fields

1. Plot the vector field $\langle x, y\rangle$. Find the curl of this vector field.
Ans: 0

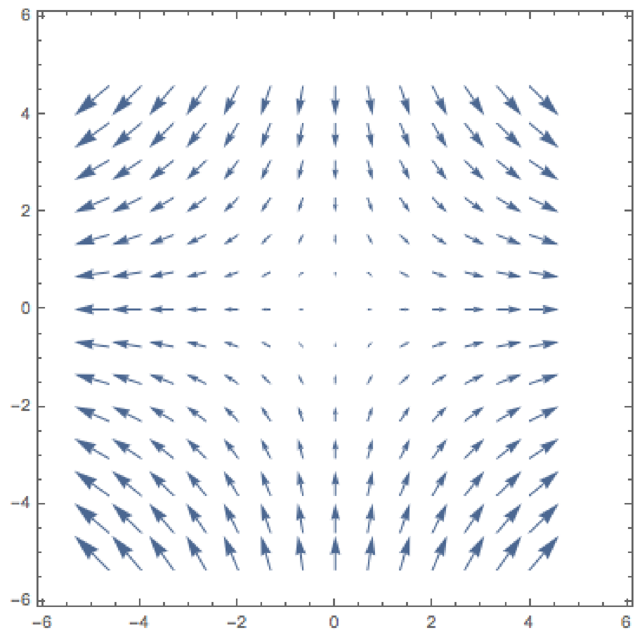


2. Plot the vector field $y\mathbf{i}+x\mathbf{j}$. Find the curl of this vector field.
Ans: 0

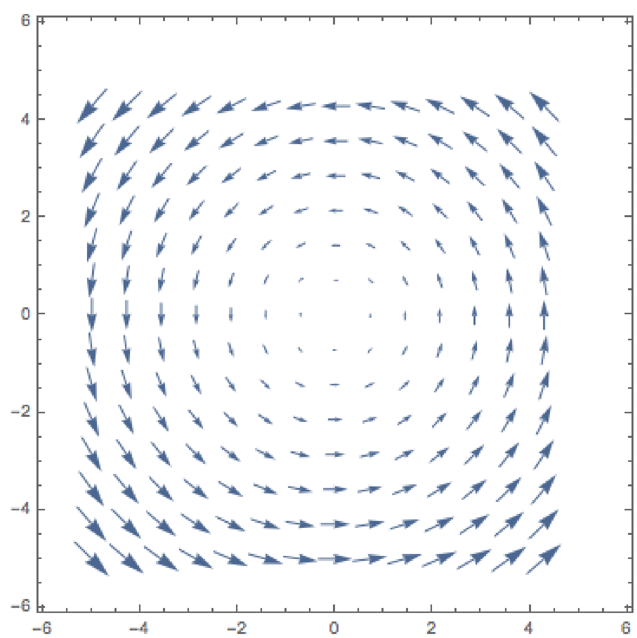


3. Plot the vector field $\langle x, -y\rangle$. Find the curl of this vector field.

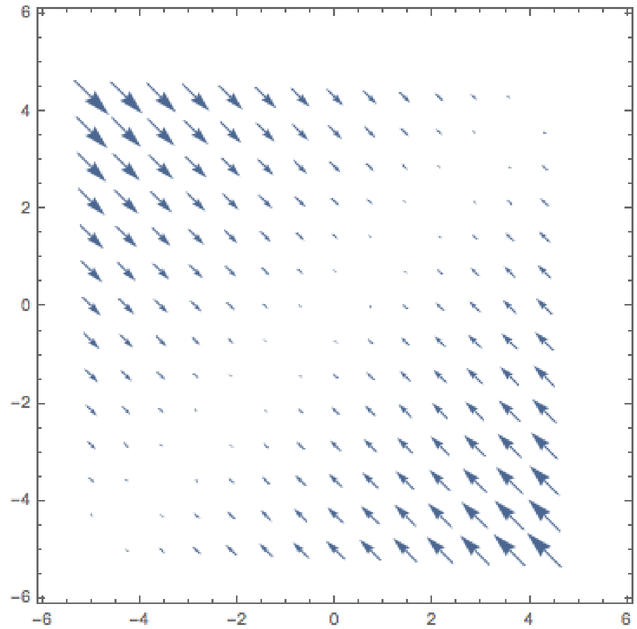
Ans: 0



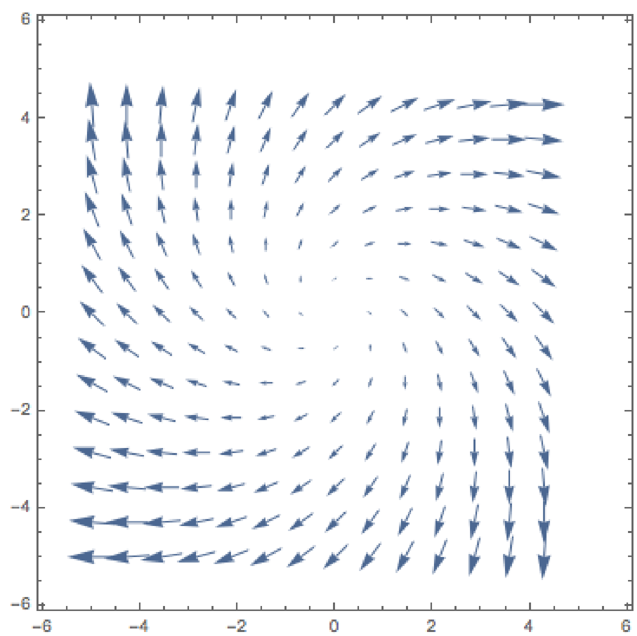
4. Plot the vector field $\langle -y, x\rangle$. Find the curl of this vector field.
Ans: 2



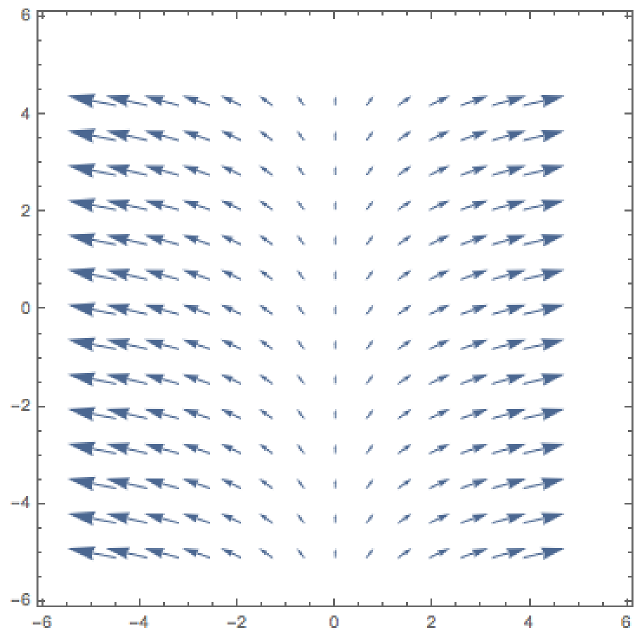
5. Plot the vector field $(y-x)\mathbf{i}+(x-y)\mathbf{j}$. Find the curl of this vector field.
Ans: 0



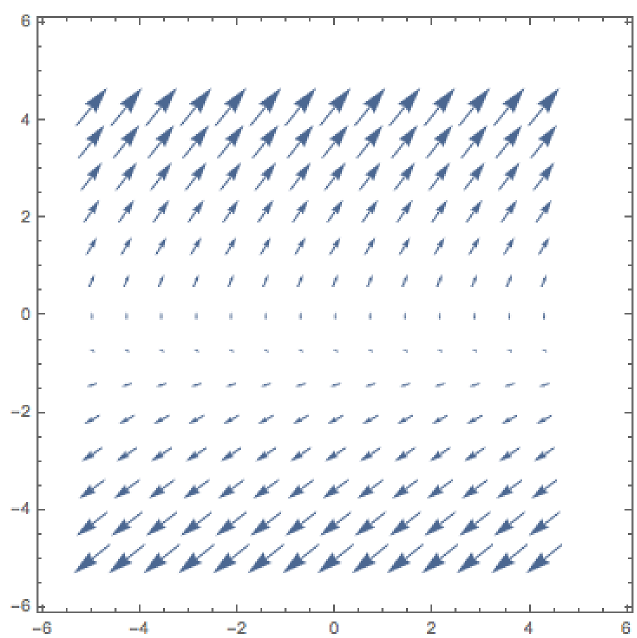
6. Plot the vector field $\langle x + y, y - x \rangle$. Find the curl of this vector field.
Ans: -2



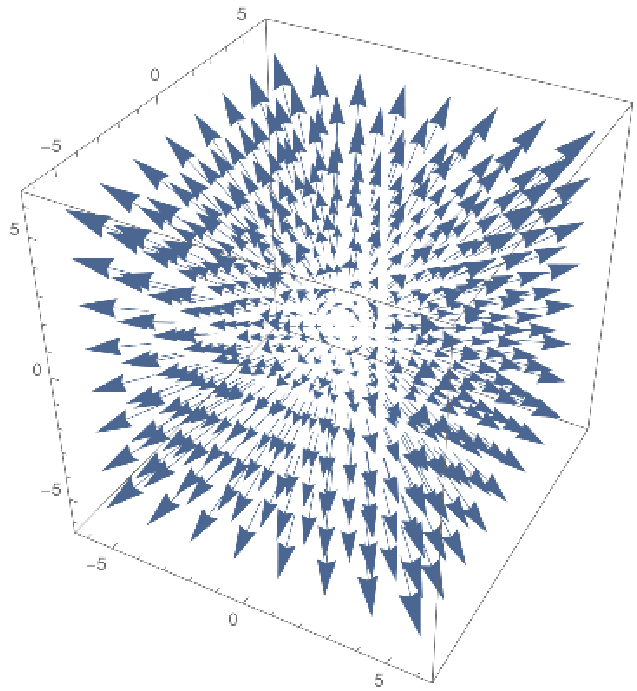
7. Plot the vector field $x\mathbf{i} + \mathbf{j}$. Find the curl of this vector field.
Ans: 0



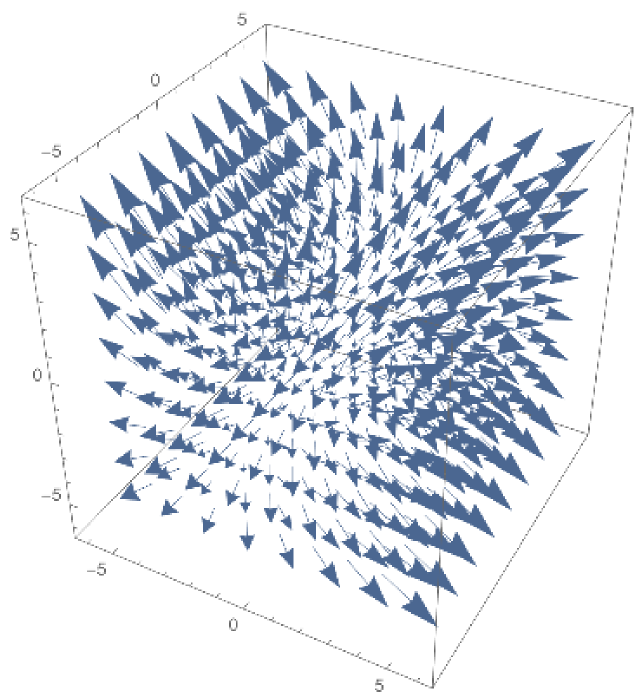
8. Plot the vector field $\langle y, y+1 \rangle$. Find the curl of this vector field.
Ans: -1



9. Plot the vector field $\langle x, y, z \rangle$. Find the curl of this vector field.
Ans: 0



10. Plot the vector field $\langle x, 1, z\rangle$. Find the curl of this vector field.
Ans: 0



Directional Derivatives

Find the directional derivative of the function f at the given point in the direction given by the angle indicated by the angle θ.

11. $f(x,y)=x^2y$, $(1,1)$, $\theta=0$
Ans: 2

12. $f(x,y)=\dfrac{x+1}{2-y}$, $(2,-2)$, $\theta=\frac{\pi}{2}$
Ans: 3/16

13. $f(x,y)=x\arctan y$, $(1,\sqrt{3})$, $\theta=\dfrac{2\pi}{3}$
Ans: $\dfrac{\sqrt{3}}{8}-\dfrac{\pi}{6}$

14. $f(x,y)=\ln(xy)$, $(1,2)$, $\theta=\dfrac{5\pi}{4}$
Ans: $\dfrac{-3}{2\sqrt{2}}$

15. $f(x,y)=xye^y$, $(-1,0)$, $\theta=\dfrac{11\pi}{6}$
Ans: 1/2

In Exercises 16–25, find the directional derivative of the given function f at the point $\mathbf{x}$ in a direction parallel to the vector $\mathbf{u}$.

16. $f(x,y)=x^2+xy+y^2$, $\mathbf{x}=(1,-1)$, $\mathbf{u}=\dfrac{2\mathbf{i}-\mathbf{j}}{\sqrt{5}}$
Ans: $\|\mathbf{u}\|=1$, $3/\sqrt{5}$

17. $f(x,y)=x\sin y$, $\mathbf{x}=(1,\frac{\pi}{2})$, $\mathbf{u}=2\mathbf{i}+3\mathbf{j}$
Ans: $\|\mathbf{u}\|=\sqrt{13}$, $2/\sqrt{13}$

18. $f(x,y)=\dfrac{xy\sin x}{y}$, $\mathbf{x}=(\pi,10)$, $\mathbf{u}=\langle -1,1\rangle$
Ans: $\|\mathbf{u}\|=\sqrt{2}$, $\pi/\sqrt{2}$

19. $f(x,y)=e^x+\dfrac{x}{y}$, $\mathbf{x}=(1,2)$, $\mathbf{u}=3\mathbf{i}-2\mathbf{j}$
Ans: $\|\mathbf{u}\|=\sqrt{13}$, $\dfrac{3e+2}{\sqrt{13}}$

20. $f(x,y)=\arctan(xy)$, $\mathbf{x}=(\frac{1}{2},2)$, $\mathbf{u}=\langle 1,0\rangle$
Ans: $\|\mathbf{u}\|=1$, -1

21. $f(x,y,z)=xe^y+x^2e^z+y^3e^z$, $\mathbf{x}=(0,1,0)$, $\mathbf{u}=\dfrac{\mathbf{k}-\mathbf{i}}{\sqrt{2}}$
Ans: $\dfrac{1-e}{\sqrt{2}}$

22. $f(x,y,z)=xyz\sin(yz)$, $\mathbf{x}=(1,\frac{\pi}{2},2)$, $\mathbf{u}=\langle -1,0,-1\rangle$.
Ans: $\|\mathbf{u}\|=\sqrt{2}$, $\dfrac{\pi^2}{2\sqrt{2}}$

23. $f(x,y,z) = xz^2 + \sqrt{x^2+y^2}$, $\mathbf{x} = (3,4,1)$, $\mathbf{u} = 3\mathbf{u} + 4\mathbf{j} - \mathbf{k}$
Ans: $\|\mathbf{u}\| = \sqrt{14}$, $-\sqrt{\dfrac{2}{175}} = \dfrac{-1}{5}\sqrt{\dfrac{2}{7}}$

24. $f(x,y,z) = e^{x^2+y^2+z^2}$, $\mathbf{x} = (1,-1,1)$, $\mathbf{u} = \langle 1,-1,1\rangle$
Ans: $\|\mathbf{u}\| = \sqrt{3}$, $2\sqrt{3}e^3$

25. $f(x,y,z) = \dfrac{\sqrt{2}\,x\,e^y}{z+1}$, $\mathbf{x} = (2,0,1)$, $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$
Ans: $\|\mathbf{u}\| = 5\sqrt{2}$, $9/10$

Tangent Planes

In Exercises 26–35, find the tangent plane to the given function at the given point.

26. $f(x,y) = x^2y + x + y - 2$, $\mathbf{x} = (2,3)$
Ans: $13(x-2) + 5(y-3) = 0$ or $13x + 5y = 41$

27. $f(x,y) = xe^y$, $\mathbf{x} = (5,0)$
Ans: $x + 5y = 5$

28. $f(x, y0 = \tan^{-1}(xy)$, $\mathbf{x} = (1,-1)$
Ans: $\dfrac{x-1}{2} + \dfrac{y+1}{2} = 0$ or $\dfrac{1}{2}(2 - x + y) = 0$ or $x - y = 2$

29. $f(x,y) = \dfrac{x^2}{y^3}$, $\mathbf{x} = (3,-1)$
Ans: $-6(x-3) - 27(y+1) = 0$ or $-3(3 + 2x + 9y) = 0$ or $2x + 9y = -3$

30. $f(x,y) = \dfrac{x\sin y}{x+1}$, $\mathbf{x} = (1, \frac{\pi}{4})$
Ans: $\dfrac{y - \frac{\pi}{4}}{\sqrt{2}} = 0$

31. $f(x,y,z) = xy + xz + yz$, $\mathbf{x} = (1,2,3)$
Ans: $5(x-1) + 4(y-2) + 3(z-3) = 0$ or $5x + 4y + 3z = 22$

32. $f(x,y,z) = \dfrac{3x+y}{x+2z}$, $\mathbf{x} = (1,-1,1)$
Ans: $\dfrac{7}{9}(x-1) + \dfrac{1}{3}(y+1) - \dfrac{4}{9}(z-1) = 0$ or $\dfrac{1}{9}(7x + 3y - 4z) = 0$ or $7x + 3y - 4z = 0$

33. $f(x,y,z) = \dfrac{xe^z}{y+2}$, $\mathbf{x} = (-3,1,0)$
Ans: $\dfrac{1}{3}(x+3) + \dfrac{1}{3}(y-1) - z = 0$ or $\dfrac{1}{3}(x + y - 3z + 2) = 0$ or $x + y - 3z = -2$

34. $f(x,y,z) = \sqrt{\dfrac{xy}{z^3}}$, $\mathbf{x} = (4,1,1)$

Ans: $\frac{1}{4}(x-4)+(y-3)-3(z-1)-1=0$ or $\frac{x}{4}+y-3z+1=0$ or $x+4y-12z=-4$

35. $f(x,y,z)=\dfrac{\ln(xz)}{y}$, $\mathbf{x}=(1,1,1)$
Ans: $x+z=2$

Normal Lines

36. Find the normal line to the curve given by $y=2x-3$ at the point $(-1,2)$.
Ans: $(-2t-1,t+2)$

37. Find the normal line to the curve given by $x^2-y^3=1$ at the point $(3,2)$.
Ans: $(6t+3,2-12t)$

38. Find the normal line to the curve given by $x^3-x^2-x+2=y^2$ at the point $(2,2)$.
Ans: $(7t+2,2-4t)$

39. Find the normal line to the curve given by $x^2-xy^2+y^3=1$ at the point $(2,-1)$.
Ans: $(3t+2,7t-1)$

40. Find the normal line to the curve given by $x^2y=y^2x+6$ at the point $(2,-1)$.
Ans: $(2-5t,8t-1)$

41. Find the normal line to the surface given by $x-3y+z=4$ at the point $(1,0,3)$.
Ans: $(t+1,-3t,t+3)$

42. Find the normal line to the surface given by $x^2+y^2+z^2=9$ at the point $(3,0,0)$.
Ans: $(6t+3,0,0)$

43. Find the normal line to the surface given by $z=x^2+y^2$ at the point $(-1,3,10)$.
Ans: $(2t-1,3-6t,t+10)$

44. Find the normal line to the surface given by $x^2+y^2=z^2+1$ at the point $(1,1,1)$.
Ans: $(2t+1,2t+1,1-2t)$

45. Find the normal line to the surface given by $z=y^2-x^2$ at the point $(4,1,-15)$.
Ans: $(8t+4,1-2t,t-15)$

Gradients

46. Find the maximum rate of change of the function $f(x,y)=e^x\cos y$ at the point $(0,\frac{\pi}{6})$ and the direction which it occurs.
Ans: $\langle\sqrt{3}/2,-1/2\rangle$, 1

47. Find the maximum rate of change of the function $f(x,y) = \sin(x+y)$ at the point $(1,-1)$ and the direction which it occurs.
Ans: $\langle 1,1\rangle$, $\sqrt{2}$

48. Find the minimum rate of change of the function $f(x,y) = \dfrac{x^2y - y^2x}{3}$ at the point $(2,-2)$ and the direction which it occurs,
Ans: $\langle -2,2\rangle$, $-4\sqrt{2}$

49. Find the maximum rate of change of the function $f(x,y) = \dfrac{\ln(xy)}{x}$ at the point $(1,1)$ and the direction which it occurs.
Ans: $\langle 1,1\rangle$, $\sqrt{2}$

50. Find the maximum rate of change of the function $f(x,y) = \dfrac{x+y}{y-x}$ at the point $(2,1)$ and the direction which it occurs.
Ans: $\langle 2,-4\rangle$ or $\langle 1,-2\rangle$, $2\sqrt{5}$

51. Find the minimum rate of change of the function $f(x,y,z) = xy\sin(xz)$ at the point $(2,1,\pi)$ and the direction which it occurs.
Ans: $\langle -2\pi,0,-4\rangle$, $-\sqrt{4\pi^2+16}$

52. Find the minimum rate of change of the function $f(x,y,z) = xz\ln(yz)$ at the point $(2,1,1)$ and the direction which it occurs.
Ans: $\langle 0,-2,-2,\rangle$ or $\langle 0,-1,-1\rangle$, $-2\sqrt{2}$

53. Find the maximum rate of change of the function $f(x,y,z) = \dfrac{x^2yz^3+1}{z+2}$ at the point $(-1,-1,-1)$ and the direction which it occurs.
Ans: $\langle -2,-1,-5\rangle$, $\sqrt{30}$

54. Find the maximum rate of change of the function $f(x,y,z) = x^2y - yz^2 + xz^3$ at the point $(-1,-1,2)$ and the direction which it occurs.
Ans: $\langle 10,-3,8\rangle$, $\sqrt{173}$

55. Find the minimum rate of change of the function $f(x,y,z) = \dfrac{xy}{\ln(xz)}$ at the point $(-e,1,-1)$ and the direction which it occurs.
Ans: $\langle 0,e,e\rangle$ or $\langle 0,1,1\rangle$, $-e\sqrt{2}$

'Applied' Gradients

56. Find the points on the surface $f(x,y) = x^3+2xy+y+5$ that are parallel to the plane $5x+3y-z=0$.
Ans: The plane is $g(x,y) = 5x+3y$. The gradient of this is $\langle 5,3\rangle$. The gradient of f is $\langle 3x^2+2y, 2x+1\rangle$. Solving gives $(1,1)$.

57. Find the points on the hyperboloid $9x^2 - 45y^2 + 5z^2 - 45$ where the tangent plane to the surface is parallel to the plane $x + 5y - 2z = 7$.

Ans: $(5/4, -5/4, -9/2)$

58. Find the points on the paraboloid $z = x^2 + y^2$ where the tangent plane to the surface is parallel to the plane $x + y + z = 1$. Find the equation of the tangent planes at these points.

Ans: $(-1/2, -1/2, 1/4)$, $x + y + z = -1/2$.

59. Find the point(s) where the tangent plane to the surface $z = x^2 - 6x + y^3$ is parallel to the plane $4x - 12y + z = 7$.

Ans: $(1, 2, -3)$, $(1, -2, 5)$

60. Find the points where the tangent plane to $x^2 + y^2 + z^2 = 9$ is parallel to the plane $2x + 2y + z = 1$.

Ans: $\pm(2, 2, 1)$.

61. Describe the set of points on $x^2 + 3y^2 + z^2 + xz = 6$ where the tangent plane is parallel to the z–axis.

Ans: Vertical tangents $(x, y, 0)$. Get $\partial f/\partial z = 0$. So $x + 2z = 0$ so $x = -2z$. This gives $y^2 + z^2 = 2$. This is a cylinder. Intersecting with plane $x = -2z$ gives an ellipse.

Other

62. Show that the sum of the x, y, and z intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is constant.

Ans: Tangent plane at (x_0, y_0, z_0) is

$$\frac{x - x_0}{2\sqrt{x_0}} + \frac{y - y_0}{2\sqrt{y}} + \frac{z - z_0}{2\sqrt{z_0}}$$

Set any two coordinates to 0 and solve to find the intercepts. For instance, z intercept $x = y = 0$ and $z = \sqrt{z_0}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = \sqrt{c}\sqrt{z_0}$. Adding the three gives $\sqrt{c}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = c$.

63. Let a, b be constants and f, g be functions. Use the definition of the gradient to show that it has the following properties:

(a) $\nabla(af + bg) = a\nabla f + b\nabla g$

(b) $\nabla(fg) = f\nabla g + g\nabla f$

(c) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

(d) $\nabla f^n = nf^{n-1}\nabla f$

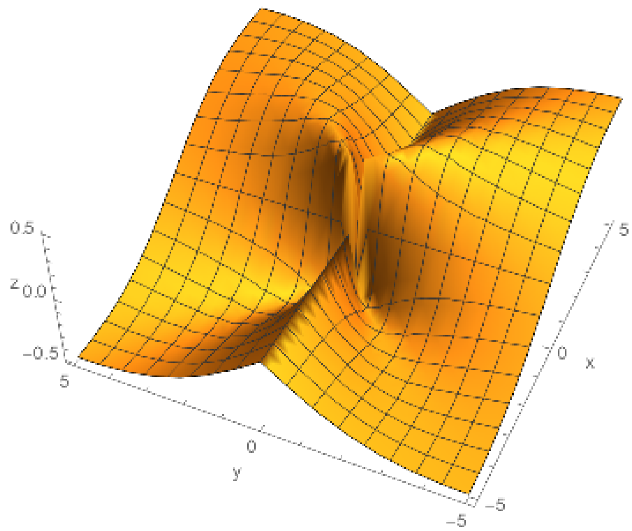
Ans: These are straightforward application of ordinary derivative rules.

64. Let $f(x, y)$ be the function given by

$$f(x, y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & \text{otherwise} \end{cases}$$

(a) Use a computer system to graph $f(x,y)$.

Ans:



(b) Use the definition of the derivative to calculate $f_x(0,0)$ and $f_y(0,0)$.

Ans:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$

(c) Use the definition of the directional derivative to determine for which unit vectors $\mathbf{u} = \langle a,b\rangle$ $D_{\mathbf{u}}f(0,0)$ exists.

Ans:

$$D_{\mathbf{u}}f(0,0) = \lim_{h\to 0} \frac{f((0,0)+h(a,b))-f(0,0)}{h} = \lim_{h\to 0} \frac{f(ha,hb)}{h} = \lim_{h\to 0} \frac{ha|hb|}{h\sqrt{h^2a^2+h^2b^2}} = \lim_{h\to 0} \frac{h|h|a|b|}{h|h|} = a|b|$$

(d) Is $f(x,y)$ differentiable at the origin? Put your result in the context of the previous parts.

Ans: One can use polar coordinates to see that this is not differentiable at the origin. So the partials exist and every directional derivative exist at $(0,0)$, but $f(x,y)$ is not differentiable there.

65. Show that the equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at the point (x_0,y_0,z_0) can be expressed as

$$x\frac{x_0}{a^2}+y\frac{y_0}{b^2}+z\frac{z_0}{c^2}=1$$

Ans: Define $f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$ so that $\nabla f(x_0,y_0,z_0)=\left\langle \frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2}\right\rangle$. Then the tangent plane is given by $0=\nabla f(x_0,y_0,z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle$. Distributing and using the equation of the surface gives the solution.

2.5 Max/Min & Lagrange Multipliers

Max/Mins: A max/min is the largest/smallest point in a sufficiently small neighborhood of the point. However, the situation in three and higher dimensions is more complicated; we can have points which are critical points but are neither max nor mins – called saddle points (why we shall see soon). Now if X is open in $\mathbb{R}^n$ and $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable, then if f has a local extremum at $\mathbf{x} \in X$, then $Df(\mathbf{x}) = \mathbf{0}$. So for 'most' functions, to find critical points, we look for derivative 0 – as usual. For $Df = 0$, we want both partials to vanish. But how can we tell what it is? We use the Hessian: a critical points $\mathbf{a}$ of a function of class C^2, the Hessian is

$$Hf(\mathbf{a}) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

each entry evaluated at $\mathbf{a}$. Let d_n be the determinant of the $n \times n$ matrix with upper left corner the entry $f_{x_1x_1}$. If $d_k > 0$ for $k = 1, 2, \ldots, n$, then f has a local min at $\mathbf{a}$. If $d_k < 0$ for k odd and $d_k > 0$ for k even, then f has a local max at $\mathbf{a}$. Otherwise, $\mathbf{a}$ is a saddle point. In the simple 2–dimensional case, this is just looking at $f_{xx}f_{yy} - f_{xy}^2$ – as in the textbook. For bounded regions, we need to look at the boundary of the region as well. But often, we will need Lagrange Multipliers for this.

Lagrange Multipliers: Often, we search for max/mins with respect to some constraint equation. Let X be open in $\mathbb{R}^n$ and $f, g : X \to \mathbb{R}$ be functions of class C^1. Let $S = \{\mathbf{x} \in X : g(\mathbf{x}) = c\}$ denote the level set of g at heigh c. Then $f\big|_S$ has an extremum at a point $\mathbf{x}_0 \in S$ such that $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$, there must be some scalar λ such that $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$.

The proof is rather technical and requires the Implicit Function Theorem. In more than one constraint, we have

$$\nabla f(\mathbf{x}_0) = \sum_{i=1}^{n} \lambda_i \nabla g_i(\mathbf{x}_0)$$

the ∇g_i being linearly independent.

Compact Regions: Extreme Value Theorem: If $X \subseteq \mathbb{R}^n$ is compact and $f : X \to \mathbb{R}$ is continuous, then f must have a global maximum and minimum on X.

1. Find the critical points.
2. Find which are in the region, classify them, and evaluate the function at these points.
3. Check the boundary.
4. Compare all the points.

2.5 | Exercises

Critical Points

1. Find and classify the critical points for the function $f(x,y)=x^2+xy+y^2$.
Ans: Min: $(0,0)$, $f(0,0)=0$

2. Find and classify the critical points for the function $f(x,y)=x^2+xy+y^2+2x-2y+5$.
Ans: Min: $(-2,2)$, $f(-2,2)=1$

3. Find and classify the critical points for the function $f(x,y)=\ln(x^2+y^2+\pi)$.
Ans: Min: $(0,0)$, $f(0,0)=\ln(\pi)$

4. Find and classify the critical points for the function $f(x,y)=x^2-xy^2+y^3$.
Ans: Saddle: $(0,0)$, $f(0,0)=0$. Saddle: $(9/2,3)$, $f(9/2,3)=27/4$

5. Find and classify the critical points for the function $f(x,y)=(x+y)(1-xy)$.
Ans: 2 complex solutions. Saddle: $(1/\sqrt{3},1/\sqrt{3})$, $f(1/\sqrt{3},1/\sqrt{3})=4/(3\sqrt{3})$. Saddle: $(-1/\sqrt{3},-1/\sqrt{3})$, $f(-1/\sqrt{3},-1/\sqrt{3})=-4/(3\sqrt{3})$

6. Find and classify the critical points for the function $f(x,y)=e^x\sin y$.
Ans: There are none.

7. Find and classify the critical points for the function $f(x,y)=\cos x\sin y$.
Ans: Saddle: $(\pi/2+n\pi,m\pi)$, $f=0$. Max/Min: $(n\pi,\pi/2+m\pi)$, $|f|=1$

8. Find and classify the critical points for the function $f(x,y)=x+y+\ln(xy)$.
Ans: Max: $(-1,-1)$, $f(-1,-1)=-2$

9. Find and classify the critical points for the function $f(x,y)=(x^2-y^2)e^{-x}$.
Ans: Saddle: $(0,0)$, $f(0,0)=0$. Max: $(2,0)$, $f(2,0)=4/e^2$

10. Find and classify the critical points for the function $f(x,y)=\dfrac{x^2y^2-8x+y}{xy}$.
Ans: Max: $(-1/2,4)$, $f(-1/2,4)=-6$

11. Find and classify the critical points of the function $f(x,y)=3x^2y+y^3-3x^2-3y^2+1$.
Ans: Max: $(0,0)$, $f(0,0)=1$. Min: $(0,2)$, $f(0,2)=-3$. Saddle: $(\pm1,1)$, $f(\pm1,1)=-1$

12. Find and classify the critical points for the function $f(x,y,z)=x^3+xy^2+x^2+y^2+3z^2$.
Ans: Min: $(0,0,0)$, $f(0,0,0)=0$. Saddle: $(-2/3,0,0)$, $f(-2/3,0,0)=4/27$

13. Find and classify the critical points for the function $f(x,y,z)=x^2+y^2+2z^2+xz$.
Ans: Min: $(0,0,0)$, $f(0,0,0)=0$

14. Find and classify the critical points for the function $f(x,y,z)=x^3+xy+yz+y^2$.

Ans: Saddle: $(0,0,0)$, $f(0,0,0)=0$

15. Find and classify the critical points for the function $f(x,y,z)=x^2+y^2-z^2$.
Ans: Saddle: $(0,0,0)$, $f(0,0,0)=0$

16. Find and classify the critical points for the function $f(x,y,z)=x^2+y^2+7z^2-xy-3yz+4$.
Ans: Min: $(0,0,0)$, $f(0,0,0)=4$

17. Find and classify the critical points for the function $f(x,y,z)=xy+xz+2yz+\frac{1}{x}$.
Ans: Saddle: $(-1,1/2,1/2)$, $f(-1,1/2,1/2)=-3/2$

18. Find and classify the critical points for the function $f(x,y,z)=e^z(z^2-y^2-2x^2)$.
Ans: Saddle: $(0,0,0)$, $f(0,0,0)=0$. Max: $(-2,0,0)$, $f(-2,0,0)=-8$

19. Show that the function $f(x,y)=x^3y^3$ has a critical point at the origin. Show that the Hessian fails to give any information about $f(x,y)$ at the point $(0,0)$. Use other methods to determine the behavior of the function $f(x,y)$ at $(0,0)$.
Ans: Saddle.

20. Show that the function $f(x,y)=x^2y^2$ has a critical point at the origin. Show that the Hessian fails to give any information about the behavior $f(x,y)$ at the point $(0,0)$. Use other methods to determine the behavior of the function $f(x,y)$ at $(0,0)$.
Ans: Min.

21. Show that the function $f(x,y)=e^{-(x^2+y^2)}$ has a critical point at the origin. Show that the Hessian fails to give any information about the function $f(x,y)$ at the point $(0,0)$. Use other methods to determine the behavior of the function $f(x,y)$ at $(0,0)$.
Ans: Max.

22. Show that the function $f(x,y,z)=x^2y^4z^3$ has a critical point at the origin. Show that the Hessian fails to give any information about the function $f(x,y)$ at the point $(0,0,0)$. Use other methods to determine the behavior of the function $f(x,y)$ at $(0,0,0)$.
Ans: Saddle.

Lagrange Multipliers

23. If x and y are such that $x+2y=4$, find the maximum and minimum values of $f(x,y)=x^2+y^2-2x-2y$.
Ans: Min: $(6/5,7/5)$, $f=-9/5$

24. Find the maximum and minimum values of $f(x,y)=xy$ along the curve $3x^2+y^2=6$.
Ans: Max: $(\pm1,\pm\sqrt{3})$, $f=\sqrt{3}$. $Min:(\pm1,\mp\sqrt{3})$, $f=-\sqrt{3}$

25. Find the critical values of $f(x,y)=x^2+y^2-2x-2y$ along a circle of radius 2 centered at the origin.

Ans: Min: $(\sqrt{2},\sqrt{2})$, $f=4-4\sqrt{2}$. Max: $(-\sqrt{2},-\sqrt{2})$, $f=4+4\sqrt{2}$

26. A electrons orbit about a nucleus is given by $x^2+y^2=1$. The energy of the particle is given by $f(x,y)=3xy+2$. Find the maximum and minimum energy states of the electron.

Ans: Max: $(1/\sqrt{2},1/\sqrt{2})$, $(-1/\sqrt{2},-1/\sqrt{2})$, $f=7/2$. Min: $(-1/\sqrt{2},1/\sqrt{2})$, $(1/\sqrt{2},-1/\sqrt{2})$, $f=1/2$

27. Find the point(s) on the curve $x^2+xy=1$ to the origin.

Ans: $\lambda=-2\pm2\sqrt{2}$. $(\pm1/\sqrt[4]{2},(-1+\sqrt{2})/\sqrt[4]{2})$

28. Find the point(s) on the curve $x^2+4xy-5x+2y^2-3y=0$ closest to the point $(2,1)$.

Ans: $(1.96881134711, 0.89528903068)$

29. Find the points on $z^2=xy+1$ closest to the origin.

Ans: $(0,0,1)$ and $(0,0,-1)$, last coordinate distance.

30. An observatory is being constructed. The base will consist of a right circular cylinder of height h with a half sphere of radius r sitting atop it. If the material for the half sphere top costs \$20/m^2, the siding costs \$8/m^2, and the bottom costs \$5/m^2, what ratio of height to diameter minimizes costs if the total volume of the structure must be 200π cubic meters.

Ans: $h/(2r)=\dfrac{58\pi/3\lambda}{2\cdot16\pi/3\lambda}=29/16$

31. Find the maximum volume of a rectangular box that is contained in the ellipsoid $x^2+9y^2+4z^2=9$, assuming the edges of the box are parallel to the coordinate axes.

Ans: (x,y,z) octant 1 corner. Volume $2x\cdot2y\cdot2z$. Point max is $(\sqrt{3},1/\sqrt{3},\sqrt{3}/2)$ and max volume $4\sqrt{3}$.

32. A rectangular box with the top of the box removed is made from 12 square ft of cardboard. What is the maximum possible the box can be constructed to have?

Ans: $(2,2,1)$, maximum volume 4.

33. Find the critical points of $f(x,y,z)=x+y+2z$ if $x^2+y^2+z^2=3$.

Ans: Max: $(1/\sqrt{2},1/\sqrt{2},\sqrt{2})$, $f=3\sqrt{2}$. Min: $(-1/\sqrt{2},-1/\sqrt{2},-\sqrt{2})$, $f=-3\sqrt{2}$

34. On the surface $x^2+2y^2+3z^2=1$, find the maximum and minimum values of $f(x,y,z)=x^2-y^2$.

Ans: Max: $(\pm1,0,0)$, $f=1$. Min: $(0,\pm1/\sqrt{2},0)$, $f=-1/2$. Saddle: $(0,0,\pm1/\sqrt{3})$, $f=0$

35. Let S be the surface created by intersecting $z^2=x^2+y^2$ and $z=x+y+2$. Find the points on S nearest and farthest from the origin.

Ans: Two constraint. Nearest: $(-2+\sqrt{2},-2+\sqrt{2},-2+2\sqrt{2})$ with distance $24-16\sqrt{2}$ and Farthest $(-2-\sqrt{2},-2-\sqrt{2},-2-2\sqrt{2})$ with distance $24+16\sqrt{2}$.

Compact Regions

36. Find the absolute maximum and minimum of the function $f(x,y)=x^2+4y^2-2x^2y+4$ over the region $D=\{(x,y):\,-1\le x\le 1,-1\le y\le 1\}$.

Ans: Min: $(0, 0, 4)$. Max: $(\pm 1, -1, 11)$

37. Find the absolute maximum and minimum of the function $f(x, y) = 2x^2 - y^2 + 6y$ over $D = \{(x, y)\colon x^2 + y^2 \leq 16\}$.

Ans: Min: $(0, -4, -40)$. Max: $(\pm\sqrt{15}, 1, 35)$

38. Find the absolute maximum and minimum of the function $f(x, y) = 5 - 3x + 4y$ on the triangular region with vertices $(0, 0)$, $(4, 0)$, and $(4, 5)$.

Ans: Min: $(4, 0, -7)$. Max: $(4, 5, 13)$

39. Find the absolute maximum and minimum of the function $f(x, y) = x^3 - xy + y^2 - x$ on $D = \{(x, y)\colon x, y \geq 0, x + y \leq 2\}$.

Ans: Min: $(2/3, 1/3, -13/27)$. Max: $(2, 0, 6)$

40. Find the absolute maximum and minimum of the function $f(x, y) = x^2 + y^2 - x$ over the square with vertices $(\pm 1, \pm 1)$.

Ans: Min: $(1/2, 0, -1/4)$. Max: $(-1, 1, 3)$

41. Find the absolute maximum and minimum of the function $f(x, y) = e^{xy}$ over $D = \{(x, y)\colon 2x^2 + y^2 \leq 1\}$.

Ans: Min: $(\pm 1/2, \mp 1/\sqrt{2}, e^{-1/(2\sqrt{2})})$. Max: $(\pm 1, \pm 1/\sqrt{2}, e^{1/(2\sqrt{2})})$

42. Find the absolute maximum and minimum of the function $f(x, y) = xy^3$ over $D = \{(x, y)\colon x, y \geq 0, x^2 + y^2 \leq 1\}$.

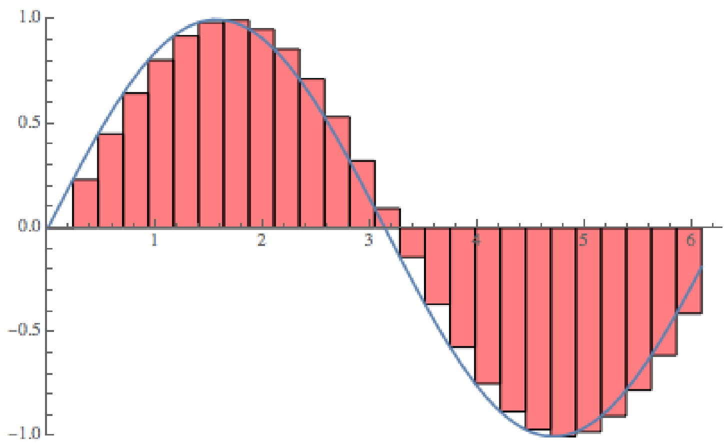
Ans: For circle piece, change to polar. $f(t) = \sin^2 t(4\cos^2 t - 1)$. $\sin^2 t = 0$ when $t = 0$ so $(1, 0)$. But $y = 0$ already gave that. $4\cos^2 t - 1 = 0$ so $t = \pi/3$ giving $(1/\sqrt{2}, 3/\sqrt{2})$. Min: All $(x, 0)$, $(0, y)$, value 0. Max: $(1/\sqrt{2}, 3/\sqrt{2}, 27/4)$.

Chapter 3

Multiple Integration

3.1 Double Integrals

Double Integral: Recall the idea of the ordinary Riemann integral was to break up an interval into pieces which would serve as widths of rectangles. Then a point is chosen from each interval to serve as the height for that rectangle. One then adds the areas. As these widths get smaller, the sum converges to a single value (for 'nice' functions) that we define as the Riemann integral. We do the same thing for the



double integral. We split a region into small 'rectangles' by splitting the x–interval and y–interval into pieces. This gives lots of regions from which we choose a point to serve as the height of the rectangle. We take the sum of these rectangular prisms. These converge (for 'nice' functions) to what we define as the double integral. Note that $\iint_R f(x,y)\,dA$ calculates the 'directed' volume 'under' $f(x,y)$ over the region R. Though the idea is the same, double integrals are more tedious. Before the interval was given to us. Here, we have to break up the region. Furthermore, sometimes we will have to switch the order of integration. In both cases, it is all about drawing a picture and choosing 'slices.' Note by notation prestidigitation $\iint_R dA =$ Area of R.

3.1 | Exercises

Rectangular Regions

1. Sketch the region of integration and evaluate the integral $\int_0^2 \int_0^3 x + y\ dx\ dy$.

Ans: 15

2. Sketch the region of integration and evaluate the integral $\int_{-1}^3 \int_2^5 x - y\ dy\ dx$.

Ans: -30

3. Sketch the region of integration and evaluate the integral $\int_0^1 \int_{-1}^1 3x^2 - 2y\ dy\ dx$.

Ans: 2

4. Sketch the region of integration and evaluate the integral $\int_1^2 \int_3^6 x^2 + y^2\ dx\ dy$.

Ans: 198

5. Sketch the region of integration and evaluate the integral $\int_{-1}^1 \int_0^1 \int_0^\pi x \sin y\ dy\ dx$.

Ans: 1

6. Sketch the region of integration and evaluate the integral $\int_{\pi/4}^{\pi/2} \int_0^\pi \cos x \sin y\ dy\ dx$.

Ans: $2 - \sqrt{2}$

7. Sketch the region of integral and evaluate the integral $\int_0^3 \int_0^1 x^2 e^y\ dy\ dx$.

Ans: $9(e-1)$

8. Sketch the region of integration and evaluate the integral $\int_{-1}^1 \int_{-\pi}^\pi y + y^2 \cos x\ dx\ dy$.

Ans: 0

9. Sketch the region of integration and evaluate the integral $\int_0^1 \int_0^1 \sqrt[3]{xy}\ dx\ dy$.

Ans: $9/16$

10. Sketch the region of integration and evaluate the integral $\int_0^1 \int_0^1 15\sqrt{x+y}\ dx\ dy$.

Ans: $8(2\sqrt{2}-1)$

11. Sketch the region of integration and evaluate the integral $\int_0^1 \int_0^1 x\, e^{xy}\ dy\ dx$.

Ans: $e - 2$

12. Sketch the region of integration and evaluate the integral $\int_1^2 \int_1^e \frac{\ln y}{xy}\ dy\ dx$.

Ans: $\ln \sqrt{2}$

13. Sketch the region of integration and evaluate the integral $\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\ dy\ dx$.

Ans: $\frac{2}{15}(2\sqrt{2}-1)$

14. Sketch the region of integration and evaluate the integral $\int_1^2 \int_1^2 \frac{x}{y} + \frac{y}{x}\, dy\, dx$.

Ans: $\ln 8$

'Irregular' Regions

15. Sketch the region of integration and evaluate the integral $\int_0^2 \int_0^{x^2} y\, dy\, dx$.

Ans: 16/5

16. Sketch the region of integration and evaluate the integral $\int_{-2}^0 \int_{x^2}^4 y\, dy\, dx$.

Ans: 64/5

17. Sketch the region of integration and evaluate the integral $\int_0^1 \int_{2x}^{8x} y + 1\, dy\, dx$.

Ans: 13

18. Sketch the region of integration and evaluate the integral $\int_{-1}^3 \int_0^x 3y\, dy\, dx$.

Ans: 14

19. Sketch the region of integration and evaluate the integral $\int_0^1 \int_x^{2x} xy\, dy\, dx$.

Ans: 3/8

20. Sketch the region of integration and evaluate the integral $\int_0^3 \int_0^{3+2x-x^2} 2x - 2y + 1\, dy\, dx$.

Ans: 9/10

21. Sketch the region of integration and evaluate the integral $\int_{-3}^3 \int_{x^2-9}^{x^2} \frac{1}{9}\, dy\, dx$.

Ans: 6

22. Sketch the region of integration and evaluate the integral $\int_{-2}^7 \int_0^{(y+2)/3} 3 - x\, dx\, dy$.

Ans: 27

23. Sketch the region of integration and evaluate the integral $\int_0^1 \int_y^{\sqrt[3]{y}} x^2 + 1\, dx\, dy$.

Ans: 1/3

24. Sketch the region of integration and evaluate the integral $\int_0^2 \int_{(4x-10)/5}^{(15-x)/5} \frac{y+5}{7}\, dy\, dx$.

Ans: 33/5

25. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2 \int_{-e^x}^{e^x} 3y^2 \, dy \, dx$.

Ans: $\dfrac{2}{3}(e^6 - 1)$

26. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_{2^x}^{3^x} \frac{2}{y} \, dy \, dx$.

Ans: $\ln(3/2)$

27. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-4}^0 \int_0^{\sqrt{4-x^2}} 2y \, dy \, dx$.

Ans: $-16/3$

28. Sketch the region of integration and evaluate the integral $\displaystyle\int_{-3}^3 \int_{-\sqrt{9-y^2}}^0 5x^3 \, dx \, dy$.

Ans: -324

29. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_0^1 xy\sqrt{x^2+y^2} \, dy \, dx$.

Ans: $\dfrac{2(2\sqrt{2}-1)}{15}$

30. Sketch the region of integration and evaluate the integral $\displaystyle\int_{\pi/4}^{5\pi/4} \int_{\cos x}^{\sin x} dy \, dx$.

Ans: $2\sqrt{2}$

31. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2 \int_{y^2}^4 dx \, dy$.

Ans: $16/3$

32. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^{\pi/2} \int_0^{2\cos\theta} r \, dr \, d\theta$.

Ans: $\pi/2$

33. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^{\pi} \int_0^{\sin x} (1+\cos x) \, dy \, dx$.

Ans: 2

34. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_0^x \sqrt{1-x^2} \, dy \, dx$.

Ans: $1/3$

35. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^2 \int_x^2 x\sqrt{1+y^3} \, dy \, dx$.

Ans: $26/9$

36. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{2+y^3}\,dy\,dx$.

Ans: $\ln 5$

37. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_{2x}^2 4e^{y^2}\,dy\,dx$.

Ans: e^4-1

38. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^1 \int_y^1 \sin x^2\,dx\,dy$.

Ans: $\sin^2(1/2)$

39. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\,dy\,dx$.

Ans: $\dfrac{\ln 33}{10}$

40. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\,dy\,dx$.

Ans: 1

41. Sketch the region of integration and evaluate the integral $\displaystyle\int_0^{\ln 2} \int_{-1}^1 \tan x\sqrt{e^y+1}\,dx\,dy$.

Ans: Separate and note $\tan x$ odd. 0

42. Change the order of integration in $\displaystyle\int_0^1 \int_1^{e^y} f(x,y)\,dx\,dy$.

Ans: $\displaystyle\int_1^e \int_{\ln x}^1 f(x,y)\,dy\,dx$

43. Evaluate $\iint_R xy\,dA$, where R is the region enclosed by $y=\dfrac{x}{2}$, $y=\sqrt{x}$, $x=2$, and $x=4$.

Ans: $\int_2^4 \int_{x/2}^{\sqrt{x}} xy\,dy\,dx = 11/6$

44. Evaluate $\iint_R (2x-y^2)\,dA$, where R is the region enclosed by $x+y=1$, $y=x+1$, and $y=$.

Ans: $\int_1^3 \int_{1-y}^{y-1} (2x-y^2)\,dx\,dy = -68/3$.

45. Evaluate $\iint_R (4-y)\,dA$, where R is the region enclosed by the circle $x^2+y^2=4$.

Ans: $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx = 16\pi$.

Areas

46. Use a double integral to show that the area between the functions $f(x), g(x)$ between $x=a$ and

$x = b$, where $f(x) \geq g(x)$ on $[a, b]$, is given by

$$\int_a^b f(x) - g(x)\, dx$$

Ans: A simple derivation.

47. Use a double integral to find the area of the rectangle with vertices $(3,3)$, $(7,3)$, $(7,-4)$, and $(3,-4)$.

Ans: $\displaystyle\int_{-4}^{3}\int_{3}^{7} dx\, dy = 28$

48. Use a double integral to find the area of the region bound by $x = 0$, $y = 0$, and $y = 9 - x^2$.

Ans: $\displaystyle\int_{0}^{3}\int_{0}^{9-x^2} dy\, dx = 18$

49. Use a double integral to find the area bound by $y = \sqrt{x}$, $x = 0$, and $y = 2$.

Ans: $\displaystyle\int_{0}^{4}\int_{\sqrt{x}}^{2} dy\, dx = 8/3$

50. Use a double integral to find the area bound by $y = x^3$ and $y = \sqrt{x}$.

Ans: $\displaystyle\int_{0}^{1}\int_{x^3}^{\sqrt{x}} dy\, dx = 5/12$

51. Use a double integral to find the area between $y = x^3$ and $y = x^2$ in Quadrant 1.

Ans: $\displaystyle\int_{0}^{1}\int_{x^3}^{x^2} dy\, dx = 8/3$

52. Use a double integral to find the area below the parabola $y = 4x - x^2$ and above both the line $y = 6 - 3x$ and the x–axis.

Ans: $\displaystyle\int_{1}^{2}\int_{6-3x}^{4x-x^2} dy\, dx + \int_{2}^{4}\int_{0}^{4x-x^2} dy\, dx = 15/2$

53. Use a double integral to find the area bound by $x = 0$, $y = 4$, $y = -4$, and $x = y^2$.

Ans: 128/3

Volumes

54. Find the volume of the solid region formed by the region above the plane $z = 4 - x - y$ and below the rectangle $\{(x, y) \colon 0 \leq x \leq 1, 0 \leq y \leq 2\}$.

Ans: $\displaystyle\int_{0}^{2}\int_{0}^{1} (4 - x - y)\, dx\, dy$

55. Find the volume of the region bound by $z = 2 - x - 2y$ and the coordinate axes.

Ans: $\displaystyle\int_{0}^{2}\int_{0}^{(2-x)/2} 2 - x - 2y\, dy\, dx = 2/3$

56. Find the volume of the region bound above by $z = xy^2$ and below by the region in the plane formed by $y = x^3$, $y = -x^2$, $x = 0$, and $x = 1$.

Ans: $\displaystyle\int_0^1\int_{-x^2}^{x^3} xy^2\,dy\,dx = 19/264$

57. Find the volume of the solid bound by $z=0$, $x=0$, $y=0$, $x=\sqrt[3]{y}$, $x=2$ but below the function $f(x,y)=e^{x^4}$.

Ans: $\displaystyle\int_0^2\int_0^{x^3} e^{x^4}\,dy\,dx = \frac{e^{16}-1}{4}$

58. Find the volume of the region bound by $x^2+y^2=1$ and $y^2+z^2=1$ in the first octant.

Ans: Note $z=0$ and $z=\sqrt{1-y^2}$ then $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\,dy = 2/3$

59. Find the volume of the sphere $x^2+y^2+z^2=a^2$.

Ans: $\displaystyle 8\int_0^a\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy = \frac{4}{3}\pi a^3$

60. Find the volume of the region below $z=4-x^2-2y^2$ and above $z=0$.

Ans: $\displaystyle\int_{-2}^2\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (4-x^2-2y^2)\,dy\,dx = 4\sqrt{2}\pi$

61. Find the volume of the region above the plane $z=1-y$ and below the paraboloid $z=1-x^2-y^2$.

Ans: $\displaystyle\int_0^1\int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1-x^2-y^2)\,dx\,dy - \int_0^1\int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1-y)\,dx\,dy = \frac{\pi}{32}$

Other Double Integral Problems

62. Evaluate the improper integral $\displaystyle\int_1^\infty\int_0^{1/x} y\,dy\,dx$.

Ans: $1/2$

63. Evaluate the improper integral $\displaystyle\int_0^3\int_0^\infty \frac{x^2}{1+y^2}\,dy\,dx$.

Ans: $9\pi/2$

64. Evaluate the improper integral $\displaystyle\int_0^\infty\int_0^\infty xye^{-(x^2+y^2)}\,dx\,dy$.

Ans: $\dfrac{1}{4}(1-e^{-9})$

65. Is the following true or false, if it is true explain why and if it is false give an example to show it.

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

Ans: False

66. Is the following true or false, if it is true explain why and if it is false give an example to show it.

$$\int_0^1 \int_0^{2x} f(x,y)\,dy\,dx = \int_0^2 \int_{y/2}^1 f(x,y)\,dx\,dy$$

Ans: True

67. Is the following true or false, if it is true explain why and if it is false give an example to show it.

$$\int_0^1 \int_0^{x} f(x,y)\,dy\,dx = \int_0^1 \int_0^y f(x,y)\,dx\,dy$$

Ans: False

3.2 Triple Integrals

Triple Integrals: Same idea, higher dimension. The hardest part is setting up the integral. Work hard to get the three–dimensional picture, eliminate one variable, and then it's just a double integral!

3.2 | Exercises

Rectangular Prism Regions

1. Evaluate the integral $\displaystyle\int_1^3 \int_{-1}^1 \int_0^1 3\,dx\,dy\,dz$.

Ans: 12

2. Evaluate the integral $\displaystyle\int_{-1}^2 \int_3^4 \int_2^5 dy\,dz\,dx$.

Ans: 9

3. Evaluate the integral $\displaystyle\iiint_R dV$, where $R = [-1,3] \times [0,1] \times [0,5]$.

Ans: 20

4. Evaluate the integral $\displaystyle\iiint_R xyz\,dV$ over the region R given by $[0,1] \times [1,2] \times [2,3]$.

Ans: $15/8$

5. Evaluate the integral $\displaystyle\int_0^1 \int_0^1 \int_0^1 (y^2 + z^2)\,dz\,dy\,dx$.

Ans: $2/3$

6. Evaluate the integral $\displaystyle\int_0^1 \int_{x^2}^x \int_0^{xy} 6\,dz\,dy\,dx$.

Ans: $1/4$

7. Evaluate $\iiint_R (x^2 e^y + xyz)\,dV$, where R is the region $[-2,3] \times [0,1] \times [0,5]$.

Ans: $\dfrac{!75}{3}(e-1) + \dfrac{125}{8}$

Irregular Regions

8. Evaluate the integral $\displaystyle\iiint_R x\,dV$, where R is the region bound by the coordinate axes and $x + y + z = 4$.

Ans: $\displaystyle\int_0^4 \int_0^{4-x} \int_0^{4-x-y} x\,dz\,dy\,dx = 32/3$

9. Evaluate the integral $\iint_R xy\ dV$, where R is the region enclosed by $z = x + y$, $z = 0$, $y = x^2$, and $x = y^2$.

Ans: $\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy\ dz\ dy\ dx = 3/28$

10. Evaluate the integral $\iiint_R 2x\ dV$, where R is the region lying under $2x + 3y + z = 6$ and in the first octant.

Ans: $\displaystyle\int_0^3 \int_0^{-2/3x+2} \int_0^{6-2x-3y} 2x\ dz\ dy\ dx = 9$

11. Evaluate $\iiint_R (1 + xy)\ dV$, where R is the bound by the coordinate planes ant $x + y + z = 1$.

Ans: $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1 + xy)\ dz\ dy\ dx = 7/40$

12. Evaluate $\iiint_R (2x - y + z)\ dV$, where R is the region bound by the cylinder $z = y^2$, $x = 0$, $x = 1$, $y = -2$, $y = 2$, and $z = 0$.

Ans: $\displaystyle\int_0^1 \int_{-2}^{2} \int_0^{y^2} (2x - y + z)\ dz\ dy\ dx = 176/15$

13. Evaluate $\iiint_R y\ dV$, where R is the region bounded by $x + y + z = 2$, $x^2 + z^2 = 1$, and $y = 0$.

Ans: $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-x-z} y\ dy\ dz\ dx = \frac{9\pi}{4}$

14. Evaluate $\iiint_R 8xyz\ dV$, where R is the region bounded by $y = x^2$, $y + z = 9$, and the xy–plane.

Ans: $\displaystyle\int_{-3}^3 \int_{x^2}^{9} \int_0^{9-y} 8xyz\ dz\ dy\ dx = 0$ integrating odd x over interval symmetric in x.

15. Evaluate $\iiint_R z\ dV$, where R is the region in the first octant bounded by $y^2 + z^2 = 9$, $y = x$, $x = 0$, and $z = 0$.

Ans: $\displaystyle\int_0^3 \int_x^3 \int_0^{\sqrt{9-y^2}} z\ dz\ dy\ dx = 81/8$

16. Evaluate $\iiint_R (1 - z^2)\ dV$, where R is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

Ans: $\displaystyle\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} (1 - z^2)\ dz\ dy\ dx = 1/10$

17. Evaluate $\iiint_R 3x\ dV$, where R is the region in the first octant bounded by $z = x^2 + y^2$, $x = 0$, $y = 0$, and $z = 4$.

Ans: $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 3x \, dz \, dy \, dx = 64/5$

18. Evaluate $\displaystyle\iiint_R (x+y) \, dV$, where R is the region bounded by $z = x^2 + y^2$, $x = 0$, $y = 0$, and $z = 4$.

Ans: $\displaystyle\int_0^3 \int_0^{3-x} \int_{-\sqrt{3-x^2/3}}^{\sqrt{3-x^2/3}} (x+y) \, dz \, dy \, dx = \frac{81\sqrt{3}\pi}{16}$

19. Evaluate $\displaystyle\iiint_R z \, dV$, where R is the region bounded by $z = 0$, $x^2 + 4y^2 = 4$, and $z = x + 2$.

Ans: $\displaystyle\int_{-2}^2 \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_0^{x+2} z \, dz \, dy \, dx = 5\pi$

Volume Integrals

20. Show that a circular cylinder with base $x^2 + y^2 = r^2$ and height h has volume $\pi r^2 h$.

Ans: $\displaystyle\int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_0^h 1 \, dz \, dy \, dx = \pi r^2 h$ or $\displaystyle\int_0^{2\pi} \int_0^r \int_0^h r \, dz \, dr \, d\theta$

21. Find the volume of a cone of height h with base radius r.
Ans: Use $r^2 z^2 = h^2 x^2 + h^2 y^2$ so when $z = h$, radius is r. Just cone upside down. Then $\frac{h}{a}\sqrt{x^2+y^2} \le z \le h$. Then $\displaystyle\int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{(h/r)\sqrt{x^2+y^2}}^h dz \, dy \, dx = 1/3\pi h r^2$ or if radius a, then $\displaystyle\int_0^{2\pi} \int_0^a \int_{hr/a} r \, dz \, dr \, d\theta$ or $\displaystyle\int_0^{\arctan(r/h)} \int_0^{2\pi} \int_0^{h\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

22. Find the volume of the tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,1,0)$, and $(0,0,3)$.

Ans: Plane $3x + 6y + 2z = 6$. $\displaystyle\int_0^2 \int_0^{1-x/2} \int_0^{3-3x/2-3y} dz \, dy \, dx = 1$

23. Find $\displaystyle\iiint_R \frac{24xy}{13} \, dV$, where R is the region where $0 \le z \le 1 + x + y$ and above the region in the plane bound by $y = \sqrt{x}$, $y = 0$, and $x = 1$.

Ans: $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} \frac{24xy}{13} \, dz \, dy \, dx = 5/7$

24. Find $\displaystyle\iiint_R f(x,y,z) \, dV$, where $f(x,y,z) = 3x - 2y$ and R is the region bounded by the coordinate planes and the plane $2x + 3y + z = 6$ in the first octant.

Ans: $\displaystyle\int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3y} 3x - 2y \, dz \, dy \, dx = 15/2$

25. Compute the integral $\displaystyle\iiint_D f(x,y,z) \, dV$, where $f(x,y,z) = 2x + z$ and D is the region bound by the surfaces $z = x^2$ and $z = 2 - x^2$ from $0 \le y \le 3$.

Ans: $\displaystyle\int_0^3 \int_{-1}^1 \int_{x^2}^{2-x^2} (2x + z) dz \, dx \, dy = 8$

26. Evaluate the integral $\iiint_R (x^2+y^2+z^2)\,dV$, where R is the region bounded by $x+y+z=1$, $x=0$, $y=0$, and $z=0$.

Ans: $\displaystyle\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x^2+y^2+z^2)\,dz\,dy\,dx = 1/20$

27. Find the volume of the region bound by the surfaces $z=x^2+y^2$, $z=0$, $x=0$, $y=0$, and $x+y=1$.

Ans: $\displaystyle\int_0^1\int_0^{1-x}\int_0^{x^2+y^2} 1\,dz\,dy\,dx = 1/6$

28. Find the volume of the region bound by the planes $z=x+y$, $z=10$, $x=0$, and $y=0$.

Ans: $\displaystyle\int_0^{10}\int_0^{10-x}\int_{x+y}^{10} 1\,dz\,dy\,dx = 500/3$

29. Find the volume of the region beneath $z^2=xy$ but above the region in the plane bound by $x=y$, $y=0$, and $x=4$.

Ans: $\displaystyle\int_0^4\int_0^x\int_0^{\sqrt{xy}} dz\,dy\,dx = 128/9$

30. Find the volume of the solid sitting in the first octant bound by the coordinate planes and $z=x^2+y^2+9$ and $y=4-x^2$.

Ans: $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^{x^2+y^2+9} dz\,dy\,dx = 2512/35$

31. Find the volume of the solid bounded by $z=4x^2+y^2$ and the cylinder $y^2+z=2$.

Ans: $\displaystyle\int_{-1}^1\int_{-\sqrt{(1-y^2)/2}}^{\sqrt{(1-y^2)/2}}\int_{4x^2+y^2}^{2-y^2} dz\,dx\,dy = \frac{\pi}{\sqrt{2}}$

3.3 Change of Variables

Change of Variables: Often times the nature of the region or of the integrand calls for a change of variables. Such transformations stretch/shrink and twist the region of integration. Note if A is a 2×2 matrix with $\det A \neq 0$ and $T = A$, then T is one-to-one and takes parallelograms to parallelograms and their vertices to vertices.Furthermore, area $D = |\det A|$ area D^*.

But generally, these transformations will not be so nice. Meaning the 'scaling' factor will depend on the location – enter the Jacobian.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

The weird partial notation is simply a historical convenience.

If D and D^* are two elementary regions in the xy–plane and uv–plane and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a coordinate transformation of class C^1 taking D^* to D injectively, use T to make substitution $x = x(u,v)$ and $y = y(u,v)$, then

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

The same idea works in higher dimensions.

'Standard Transformations':
Polar: $dx\,dy = r\,dr\,d\theta$

Cylindrical: $dx\,dy\,dz = r\,dr\,d\theta\,dz$

Spherical: $dx\,dy\,dz = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

3.3 | Exercises

General Change of Variable Integrals

1. Describe the image of $[0,1]\times[0,1]$ under the transformation $T(u,v)=(3u,-v)$.
Ans: $[0,3]\times[-1,0]$

2. What is the image of $[0,1]\times[0,1]$ under the transformation $T(u,v)=\left(\frac{u-v}{\sqrt{2}},\frac{u+v}{\sqrt{2}}\right)$.
Ans: This is a rotation of 45° counterclockwise.

3. If

$$T(u,v)=\begin{pmatrix}2 & 3\\ -1 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

and D^* is the parallelogram with vertices $(0,0)$, $(1,3)$, $(-1,2)$, and $(0,5)$. What is $D=T(D^*)$?
Ans: Non-zero determinant. So takes parallelograms to parallelograms. So parallelogram with vertices $(0,0)$, $(11,2)$, $(4,3)$, and $(15,5)$.

4. Find $\displaystyle\iint_R y^2\,dA$, where R is the region bounded by $xy=1$, $xy=2$, $xy^2=1$, and $xy^2=2$.
Ans: $u=xy$, $v=xy^2$, $1\le u\le 2$, $1\le v\le 2$. $J=1/v$. Integral 3/4

5. Evaluate $\displaystyle\iint_R\left(\frac{x-y}{x+y+2}\right)^2dx\,dy$, where R is the square with vertices $(-1,0)$, $(0,-1)$, $(1,0)$, and $(0,1)$.
Ans: $u=x+y$, $v=x-y$. $x=\dfrac{u+v}{2}$, $y=\dfrac{u-v}{2}$. $|J|=1/2$. Integral 2/9

6. Evaluate $\displaystyle\iint_R(x+y)\,dx\,dy$, where R is the region $y=x$, $y=x+1$, $xy=1$, and $xy=2$.

Ans: $u=y-x$, $v=xy$. $|J|=\dfrac{1}{x+y}$. $\displaystyle\int_1^2\int_0^1 du\,dv=1$

7. Evaluate $\displaystyle\iint_R\sqrt{\frac{x+y}{x-2y}}\,dA$, where R is the region enclosed by $y=x/2$, $y=0$, and $x+y=1$.

Ans: $x=\dfrac{2u+v}{3}$, $y=\dfrac{u-v}{3}$, $u=x+y$, $v=x-2y$. $|J|=1/3$. $\displaystyle\int_0^1\int_0^u\frac{1}{3}\frac{u^{1/2}}{v^{1/2}}\,dv\,du=1/3$.

8. Evaluate $\displaystyle\iint_R\cos(x+2y)\sin(x-y)\,dx\,dy$, where R is the region bounded by $y=0$, $y=x$, and $x+2y=8$.

Ans: $u=x+2y$, $v=x-y$, $x=\dfrac{u+2v}{3}$, $y=\dfrac{u-v}{3}$. $|J|=1/3$. $\displaystyle\int_0^8\int_0^u\frac{1}{3}\cos u\sin v\;dv\;du=\frac{\sin 8-5-\frac{1}{4}\sin 16}{3}$

9. Evaluate $\displaystyle\iint_R e^{\frac{y-x}{y+x}}\,dA$, where R is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,2)$.

Ans: $u = y - x$, $v = y + x$.$|J| = 1/2$. $\frac{1}{2}\int_0^2 \int_{-v}^{v} e^{u/v}\,du\,dv = e - \frac{1}{e}$

10. Evaluate $\iint_R (x-y)^2\,dA$, where R is the parallelogram $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$.

Ans: Sides $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 2$. So $u = x - y$, $v = x + y$. $J = 1/2$. $\int_0^2 \int_0^2 \frac{u^2}{2}\,du\,dv = 8/3$

11. Evaluate $\iint_R x^2\,dA$, where R is the ellipse $9x^2 + 4y^2 = 36$.

Ans: $u = x/2$, $v = v/3$. $J = 6$. Integral 6π

12. Evaluate $\iint_R \frac{y}{x}\,dx\,dy$, where R is the region bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$, and $y = \frac{x}{2}$.

Ans: $u = x^2 - y^2$, $v = \frac{y}{x}$. $J = \frac{1}{2(1-v^2)}$. $\int_0^{1/2}\int_1^4 \frac{v}{2(1-v^2)}\,du\,dv = -\frac{3}{4}\ln\frac{3}{4}$

13. Use a change of variables on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ to show that its volume is $\frac{4}{3}\pi$.

Ans: $x = au$, $y = bv$, $z = cw$, Jacobian $J = abc$. Integral is $\iiint_V\,dV = \iiint_{V'} |J|\,dV = |J|\,\text{vol}\,V' = \frac{4\pi}{3}abc$

14. Find $\iint_R \cos\left(\frac{x-y}{x+y}\right)\,dx\,dy$, where R is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Ans: $u = x - y$, $v = x + y$, $|J| = 1/2$. $\frac{1}{2}\int_0^1\int_{-v}^{v} \cos(u/v)\,du\,dv = \frac{\sin 1}{2}$

15. Evaluate $\iiint_R x - y\,dV$, where R is the parallelepiped with vertices $(0,0,0)$, $(2,0,0)$, $(3,1,0)$, $(1,1,0)$, $(0,1,2)$, $(2,1,2)$, $(3,2,2)$, and $(1,2,2)$.

Ans: Front $2y - z = 0$, Back $2y - z = 2$, Left $2x - 2y + z = 0$, Right $2x - 2y + z = 4$. $u = 2x - 2y + z$, $v = 2y - z$, $w = z$. $u \in [0,4]$, $v \in [0,2]$, $w \in [0,2]$. $x = \frac{u+v}{2}$, $y = \frac{v+w}{2}$, $z = w$. $J = 1/4$. Integral 2

16. Use the substitution $u = x^2 - y^2$ and $v = \frac{y}{x}$ to evaluate $\iint_R \frac{dA}{x^2}$, where R is the region under $y = \frac{1}{x}$ in Quadrant 1 and to the right of $x^2 - y^2 = 1$.

Ans: $J = \frac{1}{2(1-v^2)}$, $\int_0^a \int_1^{1/v - v} \frac{du\,dv}{2u}$, where a is intersection of $xy = 1$ and $x^2 - y^2 = 1$. Use $y = ax$, get $ax^2 = 1$ and $x^2(1 - a^2) = 1$ so $a = 1 - a^2$ and then $a = \frac{-1+\sqrt{5}}{2}$. Integral $\frac{1}{4}(1 - \sqrt{5} + 2\text{arcsinh}(2)$

17. Show that for a polar change of coordinates, $dx\,dy = dA = r\,dr\,d\theta$.

Ans: Use the Jacobian.

18. Show that for a cylindrical change of coordinates, $dx\,dy\,dx = dV = r\,dr\,d\theta\,dz$.

Ans: Use the Jacobian.

19. Show that for a spherical change of coordinates, $dx\,dy\,dz = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$.
Ans: Use the Jacobian.

Polar Integrals

20. Evaluate $\int_R dA$, where R is *any* circle of radius R.

Ans: Translate to origin – the integral is simply its area. Then change coordinates and πR^2

21. Evaluate $\iint_R x^2 + y^2\, dA$, where R is the circle of radius 2 centered at the origin.

Ans: 8π

22. Evaluate $\iint_R x\, dA$, where $R = \{(r,\theta)\colon 1 \le r \le 2, 0 \le \theta \le \frac{\pi}{4}\}$.

Ans: $\int_1^2 \int_0^{\pi/4} r\cos\theta \cdot r\, dr\, d\theta = \dfrac{7\sqrt{2}}{6}$

23. Evaluate
$int_R \sqrt{1+x^2+y^2}\, dA$, where R is the part of the interior of $x^2+y^2=4$ in Quadrant I.

Ans: $\int_0^{\pi/2} \int_0^2 r\sqrt{r^2+1}\, dr\, d\theta = \frac{\pi}{6}(5^{3/2}-1)$

24. Evaluate $\iint_R (x^2+y^2)^{3/2}\, dA$, where R is the circle of radius 3 centered at the origin.

Ans: $\int_0^{2\pi} \int_0^3 r^4\, dr\, d\theta = \dfrac{486\pi}{5}$

25. Evaluate $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} e^{x^2+y^2}\, dx\, dy$.

Ans: $\int_{-\pi/2}^{\pi/2} \int_0^a re^{r^2}\, dr\, d\theta = \dfrac{\pi(e^{a^2}-1}{2}$

26. Evaluate $\int_0^3 \int_0^x \dfrac{dy\, dx}{\sqrt{x^2+y^2}}$.

Ans: $\int_0^{\pi/4} \int_0^{3\sec\theta} dr\, d\theta = \ln(1+\sqrt{2})$

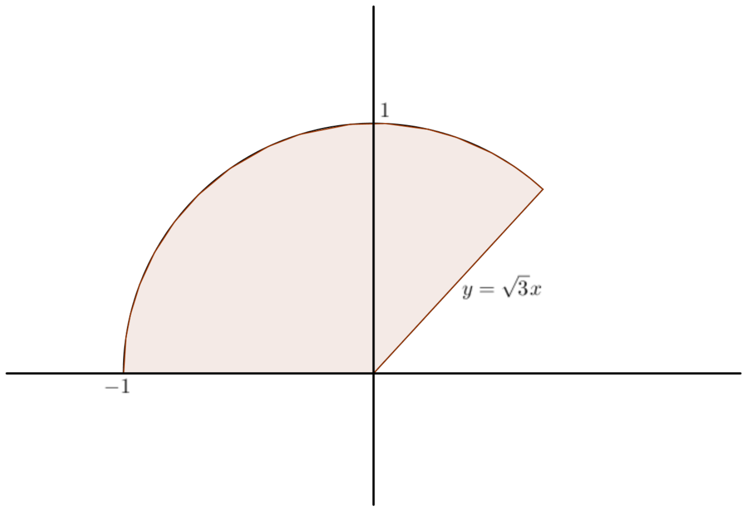
27. Evaluate $\iint_R \dfrac{dA}{\sqrt{4-x^2-y^2}}$, where R is the disk of radius 1 centered at $(0,1)$.

Ans: $x^2+(y-1)^2 = 1$. Then $r^2\cos^2\theta + (r\sin\theta - 1)^2$ if and only if $r^2 = 2r\sin\theta$ so $r = 2\sin\theta$.
$\int_0^{\pi} \int_0^{2\sin\theta} \dfrac{dr\, d\theta}{\sqrt{4-r^2}} = 2\pi - 4$

28. Evaluate $\iint_R y^2\,dA$, where R is the region between the circle of radius 1 and the square with side length 2 centered at the origin.

Ans: $\iint_{\text{Square}} y^2\,dA - \iint_{\text{disk}} y^2\,dA = \dfrac{\pi}{4}$

29. Evaluate $\iint_R \cos(x^2+y^2)\,dA$, where R is the region below.



Ans: $\displaystyle\int_{\pi/3}^{\pi}\int_0^1 r\cos r^2\,dr\,d\theta = \frac{\pi}{3}\sin 1$

30. Find $\displaystyle\iint_R \frac{x}{\sqrt{x^2+y^2}}\,dA$, where R is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

Ans: Sides $x=1$ (or $r=1/\cos\theta$) and $y=1$ (or $r=1/\sin\theta$). Break up over $\theta=\pi/4$. $\displaystyle\int_0^{\pi/4}\int_0^{1/\cos\theta}\frac{r\cos\theta}{r}\cdot r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{1/\sin\theta} r\cos\theta\,dr\,d\theta = \frac{1}{2}(\ln(1+\sqrt{2})+\sqrt{2}-1)$

31. Evaluate $\iint_R \sqrt{x^2+y^2}\,dA$, where R is the region give by $0\le r\le 1+\cos\theta$ for $0\le\theta\le 2\pi$.

Ans: $\displaystyle\int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta$

32. Evaluate $\iint_R \sin(x^2+y^2)\,dA$, where R is the circle centered at the origin with radius 2.

Ans: $\int_0^{2\pi}\int_0^2 r\sin r^2\,dr\,d\theta = \pi(1-\cos 4)$

Cylindrical Integrals

33. Evaluate $\iiint_R (x^2+y^2+z^2)\,dV$, where R is the region inside $x^2+y^2\le 4$, bounded above by $z=5$ and below by $z=-3$.

Ans: $800\pi/3$

34. Evaluate $\displaystyle\iiint_R (x^2+y^2+2z^2)\,dV$, where R is the solid cylinder defined by $x^2+y^2 \le 4$, $-1 \le z \le 2$.

Ans: $\displaystyle\int_{-1}^{2}\int_0^{2\pi}\int_0^2 r(r^2+2z^2)\,dr\,d\theta\,dz = 48\pi$

35. Evaluate $\displaystyle\iiint_R z^2\sqrt{x^2+y^2}\,dV$, where R is the solid cylinder formed by $x^2+y^2 \le 4$, $z=-1$, and $z=3$.

Ans: $\displaystyle\int_0^{2\pi}\int_{-1}^{3}\int_0^2 z^2r^2\,dr\,dz\,d\theta = \frac{448\pi}{9}$

36. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{2xy} x^2+y^2\,dz\,dy\,dx$.

Ans: $\displaystyle\int_0^{\pi/2}\int_0^1 2r\cos\theta\sin\theta r^3\,dr\,d\theta = 1/6$

37. Find the volume of the solid bounded above by the sphere $x^2+y^2+z^2=9$ and below by the xy–plane and laterally by $x^2+y^2=4$.

Ans: $\displaystyle\int_0^{2\pi|}\int_0^2\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = \frac{2\pi}{3}(27-5^{3/2})$

38. Find $\displaystyle\iiint_R y\,dV$, where R is the region below $z=x+2$, above $z=0$, and between $x^2+y^2=1$ and $x^2+y^2=4$.

Ans: $\displaystyle\int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r^2\sin\theta\,dz\,dr\,d\theta = 0$

39. Evaluate $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$.

Ans: $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^1\int_{r^2}^{r} r^3\sin\theta\cos\theta z\,dz\,dr\,d\theta = 0$

40. Find the volume under the plane $y=z$, above $z=0$, and within $x^2+y^2=1$.

Ans: $\displaystyle\int_0^{\pi}\int_0^1\int_0^{r\sin\theta} r\,dz\,dr\,d\theta = 2/3$

Spherical Integrals

41. Evaluate $\displaystyle\iiint_R z\,dV$, where R is the upper half of the unit sphere.

Ans: $\pi/4$

42. Find the volume of the region bounded above by $x^2+y^2+z^2=8$ and below by $z^2=x^2+y^2$.

Ans: $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{32\pi}{3}(\sqrt{2}-1)$

43. Evaluate $\displaystyle\iiint_R \frac{dV}{\sqrt{x^2+y^2+z^2+3}}$, where R is the sphere of radius 2 centered at the origin.

Ans: $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \frac{\rho^2\sin\phi}{\sqrt{\rho^2+3}}\,d\rho\,d\phi\,d\phi = \pi(4\sqrt{7}-6\ln(2+\sqrt{7})+3\ln 3)$

44. Evaluate $\displaystyle\iiint_R 4z\,dV$, where R is the upper half unit sphere.

Ans: $\displaystyle\int_0^{\pi/2}\int_0^{2\pi}\int_0^{1} \rho^2\sin\phi(4\rho\cos\phi)\,d\rho\,d\theta\,d\phi = \pi$

45. Evaluate $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2\,dz\,dx\,dy.$

Ans: $\displaystyle\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{3\sqrt{2}} \rho^4\sin\phi\,d\rho\,d\theta\,d\phi = \frac{486\pi}{5}(\sqrt{2}-1)$

46. Find the volume of the solid above the cone $z^2=x^2+y^2$ and below $z=1$.

Ans: $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \pi/3$

47. Evaluate $\displaystyle\iiint_R e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where $R=\{(x,y,z)\colon x^2+y^2+z^2\le 1, x,y,z\ge 0\}$.

Ans: $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2 e^{\rho^3}\sin\phi\,d\rho\,d\phi\,d\theta = \dfrac{\pi}{6}(e-1)$

48. Evaluate $\displaystyle\int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$

Ans: $\displaystyle\int_0^{\pi}\int_0^{\pi/2}\int_0^{2} \rho^2\sin\phi\sin\theta\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{16\pi}{5}$

Chapter 4

Line Integrals

4.1 Line Integrals

Line Integral: We want to build a new type of integral. Imagine a giant electric field (vector field) and a charged particle moving haphazardly through it. Because the particle has a charge, the field pushes the particle in different directions depending on the charge of the particle and the field strength at that point. If the field was constant and the path straight, we could find the work easily with $W = F \cdot D$. If the field changes, we could find the work via $W = \int F\, dx$. However, if the path and field vary in time, the situation is more complicated (and realistic). Imagine keeping track of where the particle is at any given moment. If we plot where the particle is every 3 seconds, we can form line segments approximating the path – like connect the dots. If we choose a field strength sometime over that 3 second interval, we can estimate the work by assuming the field strength is constant over that line segment and use $W = F \cdot D$, then add all these up, i.e. $\sum_{k=1}^n f(\mathbf{x}(t_k^*))\Delta s_k$, where $\Delta s_k = \int_{t_{k-1}}^{t_k} \|\mathbf{x}'(t)\|\, dt$. The smaller we make these intervals, the tighter we hug the curve (meaning a better approximation to the path) and the closer we are to the field value there. Hence, we should actually calculate the total work. Formally: : $[\mathbf{a}, \mathbf{b}] \to \mathbb{R}^3$ a path of C^1 and $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ a continuous function whose domain X contains the image of $\mathbf{x}$. Partition $[a, b]$ into $a = t_0 < t_1 < \cdots < t_k < \cdots < t_n = b$. Let t_k^* be a point in the kth subinterval $[t_{k-1}, t_k]$. Consider the sum $\sum_{k=1}^n f(\mathbf{x}(t_k^*))\Delta s_k$, where $\Delta s_k = \int_{t_{k-1}}^{t_k} \|\mathbf{x}'(t)\|\, dt$. Then we have

$$W = \lim_{\text{all } \Delta s_k \to 0} \sum_{k=1}^n f(\mathbf{x}(t_k^*))\, \Delta s_k = \lim_{\text{all } \Delta t_k \to 0} \sum_{k=1}^n f(\mathbf{x}(t_k^*))\, \Delta s_k$$

since $\mathbf{x}$ is of class C^1. The Mean Value Theorem gives a number t_k^{**} in $[t_{k-1}, t_k$ such that

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\mathbf{x}'(t)\|\, dt = (t_k - t_{k-1})\, \|\mathbf{x}'(t_k^{**})\| = \|\mathbf{x}'(t_k^{**})\|\, \Delta t_k$$

Since t_k^* is arbitrary, take it to be t_k^{**}. Then we have

$$W = \lim_{\text{all } \Delta t_k \to 0} \sum_{k=1}^n f(\mathbf{x}(t_k^{**}))\, \|\mathbf{x}'(t_k^{**})\| \Delta t_k = \int_a^b f(\mathbf{x}(t))\, \|\mathbf{x}'(t)\|\, dt$$

and we obtain the scalar line integral. We will often denote this $\int_C F\, ds$ or $\int_C F\, d\mathbf{r}$. Note that we have $f(\mathbf{x}(t))$ because the particle isn't just anywhere, it's on the path. So the only forces acting on it, are precisely the forces at that point.

To obtain the vector line integral, continue the work interpretation. Work along the kth segment is $\mathbf{F}(\mathbf{x}(t_k^*)) \cdot \Delta \mathbf{x}_k$. We have $\mathbf{x}'(t_k^*) = \dfrac{\Delta \mathbf{x}_k}{\Delta t_k}$ so that

$$W = \lim_{\Delta t_k \to 0} \sum_{k=1}^n \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t_k^*) \Delta t_k = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_C \mathbf{F} \cdot d\mathbf{s}$$

A common special case (commonly referred to as the differential form of the integral) is

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C M(x,y,z)\, dx + N(x,y,z)\, dy + P(x,y,z)\, dz$$

4.1 | Exercises

1. Evaluate $\int_C \cos \pi y \, dx$, where C is the line segment from $(0,0)$ to $(1,1)$.
Ans: 0

2. Evaluate $\int_C e^x \, dy$, where C is the curve (t^2, t^2) for $0 \le t \le 1$.
Ans: $e - 1$

3. Evaluate $\int_C x^2 y^3 \, dx$, where C is the curve (t^2, t) for $0 \le t \le 1$.
Ans: $2/9$

4. Evaluate $\int_C 2xy \, dx$, where C is the curve $y = x^2 + 9$ from $(0,9)$ to $(3,18)$.
Ans: $243/2$

5. Calculate $\int_C f \, ds$, where $f(x,y) = 2x + y$ and C is the line segment from $(-1,1)$ to $(2,-3)$.
Ans: 0

6. Calculate $\int_C f \, ds$, where $f(x,y) = xy - x + y$ and C is the line segment from $(3,3)$ to $(3,1)$.
Ans: 10

7. Calculate $\int_C f \, ds$, where $f(x,y) = x\sqrt{y}$ and C is the line segment from $(-1,0)$ to $(2,3)$.
Ans: $\dfrac{8\sqrt{6}}{5}$

8. Calculate $\int_C f \, ds$, where $f(x,y) = xy^4$ and C is the upper half of the circle $x^2 + y^2 = 9$.
Ans: 0

9. Evaluate $\int_C f \, ds$, where $f(x,y,z) = xyz$ and C is the path $\mathbf{x}(t) = (t, 2t, 3t)$, $0 \le t \le 2$.
Ans: $24\sqrt{14}$

10. Evaluate $\int_C f \, ds$, where $f(x,y,z) = \dfrac{x+z}{y+z}$ and C is the curve $\mathbf{x}(t) = (t, t, t^{3/2})$, where $1 \le t \le 3$.
Ans: $\dfrac{35\sqrt{35} - 17\sqrt{17}}{27}$

11. Evaluate $\int_C f \, ds$, where $f(x,y,z) = x + y + z$ and C is the straight line segments from $(-1,5,0)$ to $(1,6,4)$ then to $(0,1,1)$.

Ans: $\dfrac{15\sqrt{21}+13\sqrt{35}}{2}$

12. Evaluate $\displaystyle\int_C (2x+9z)\,ds$, where $x(t)=t$, $y(t)=t^2$, $z(t)=t^3$ for $0\le t\le 1$.

Ans: $\dfrac{1}{6}(14^{3/2}-1)$

13. Evaluate $\displaystyle\int_C f\,ds$, where $\mathbf{x}(t)=(\cos t,\sin t,t)$, $\mathbf{x}:[0,2\pi]\to\mathbb{R}^3$, and $f(x,y,z)=xy+z$.

Ans: $2\sqrt{2}\pi^2$

14. Evaluate $\displaystyle\int_C f\,ds$, where

$$\mathbf{x}(t)=\begin{cases}(2t,t), & 0\le t\le 1\\ (t+1,5-4t), & 1<t\le 3\end{cases}$$

Ans: $-\dfrac{\sqrt{5}}{2}-12\sqrt{17}$

15. Evaluate the integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}=\langle z,y,-x\rangle$ and C is the path $(t,\sin t,\cos t)$, $0\le t\le\pi$.

Ans: π

16. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s}$, where C is the curve $\mathbf{x}(t,3t^2,2t^3)$ and $\mathbf{F}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$.

Ans: 7

17. Find $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}=\langle x,y,z\rangle$ and C is the path $\mathbf{x}(t)=(2t+1,t,3t-1)$, $0\le t\le 1$.

Ans: 6

18. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}=(y+2)\,\mathbf{i}+x\,\mathbf{j}$ and C is the path $\mathbf{x}(t)=(\sin t,-\cos t)$, $0\le t\le\pi/2$.

Ans: 2

19. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}=\langle y\cos z, x\sin z, xy\sin z^2\rangle$ and C is the path $\mathbf{x}(t)=(t,t^2,t^3)$, $0\le t\le 1$.

Ans: $\dfrac{7-7\cos 1+2\sin 1}{6}$

20. Evaluate $\displaystyle\int_C x\,dy-y\,dx$, where C is the curve $\mathbf{x}(t)=(\cos 3t,\sin 3t)$, $0\le t\le\pi$.

Ans: 3π

21. Evaluate $\displaystyle\int_C \frac{x\,dx+y\,dy}{(x^2+y^2)^{3/2}}$, where C is curve $\mathbf{x}(t)=(e^{2t},\cos 3t,e^{2t}\sin 3t)$, $0\le t\le 2\pi$.

Ans: $1-e^{-4\pi}$

22. Evaluate $\displaystyle\int_C (x^2 - y)\,dx + (x - y^2)\,dy$, where C is the line segment from $(1,1)$ to $(3,5)$.

Ans: $-92/3$

23. Evaluate $\displaystyle\int_C x^2 y\,dx - (x+y)\,dy$, where C is the trapezoid with vertices $(0,0)$, $(3,0)$, $(3,1)$, and $(1,1)$, oriented counterclockwise.

Ans: $0 - 7/2 - 26/3 + 3/4 = -137/12$

24. Evaluate $\displaystyle\int_C x^2 y\,dx - xy\,dy$, where C is the curve with $y^2 = x^3$ from $(1,-1)$ to $(1,1)$.

Ans: $4/9$

25. Evaluate $\displaystyle\int_C yz\,dx - xz\,dy + xy\,dz$, where C is the line segment from $(1,1,2)$ to $(5,3,1)$.

Ans: $-11/3$

26. Evaluate $\displaystyle\int_C z\,dx + x\,dy + y\,dz$, where C is the curve obtained by intersecting $z = x^2$ and $x^2 + y^2 = 4$ and oriented counterclockwise around the z–axis.

Ans: $x = 2\cos t$, $y = 2\sin t$, then $z = 4\cos^2 t$, $0 \le t \le 2\pi$. Integral 4π.

27. Find the work done by the force $\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + (x + y + z)\,\mathbf{k}$ on a particle moving along the parabola $y = 3x^2, z = 0$ from the origin to the point $(2, 12, 0)$.

Ans: -70

4.2 The Fundamental Theorem for Line Integrals

Fundamental Theorem of Calculus: If f is continuous on $[a,b]$ and F is the indefinite integral of f on $[a,b]$ (that is, $F'=f$ on $[a,b]$), then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We want a similar theorem for line integrals. Notice the situation is we have the integral of a function $f(x)$ which is the derivative of some 'unknown' function $F(x)$. The integral depends only on the value of the function $F(x)$ at the ends of the interval, $x=b$ and $x=a$.

Fundamental Theorem for Line Integrals: If C is a smooth curve given by $\mathbf{r}(t)$ for $t \in [a,b]$ and F is such that ∇F is continuous on C, then

$$\int_C \nabla F \cdot d\mathbf{r} = F(\mathbf{r}(b)) - F(\mathbf{r}(a))$$

Notice how this exactly parallels the ordinary Fundamental Theorem of Calculus. Moreover, this means the integral depends only on the value the field F takes at the start/end points and *not* on the path taken – no matter how wild (or ordinary). This is path dependence. However, how can we tell when the integral of a function f is such that $f = \nabla F$ for some F? This will have great relation to the geometry of the space.

Open Set: A set is open if given any point in the set, there is a small ball around the point that is entirely contained in the set. [Give the students a few pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$.]

Closed Set: A set is closed if for any point that has the property that any open ball intersects the set, then this point is in the set. [Give the students a few pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$.]

(Path) Connected: A set is path connected if any two points in the set can be connected by a path in the set. [Give the students a few pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$.]

Domain: A domain is an open (path) connected subset of a space.

Simple Curve: A curve is simple if has no self intersections. [Give the students a few pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$.]

Closed Curve: A curve is closed if the start/end points of the curve are the same. [Give the students a few pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$.]

Simply Connected: A simply connected region is a domain which can be continuously deformed to a point. That is, any simple closed curve in the domain encloses *only* points in D. [Give the students a few pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$.]

Path Independence: The integral $\displaystyle\int_C F \cdot d\mathbf{r}$ is independent of the path if $\displaystyle\int_{C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r}$ for any two paths in a domain with the same initial and terminal points. Note that these curves can be used to form a closed loop. But then we obtain that $\int_C F \cdot d\mathbf{r}$ is independent of the path in the domain D if and only if $\displaystyle\int_C F \cdot d\mathbf{r}$ for every closed path in D.

Conservative: A vector field f is conservative if there is a vector field F such that $f = \nabla F$. [We also say that f is a gradient field. Mention the connection to Physics, especially in terminology.] By the Fundamental Theorem of Line Integrals, this means that $\displaystyle\int_C f \cdot d\mathbf{r}$ is path independent. So we know conservative vector fields have the path independence property. But what about the other way around? If a vector field has the path independence property, is the vector field conservative?

This does not work both ways unless the geometry of the space is 'nice'. If f is a vector field that is continuous on a domain D and $\displaystyle\int_C f \cdot d\mathbf{r}$ is independent of path in D, then f is conservative; that is, there is a function F so that $f = \nabla F$. But this is nearly impossible to check – we would have to compute the line integrals for *any* path between *any* two points. We need something better.

If $F = \langle M, N \rangle$ is a conservative vector field and M and N have continuous first–order partials on a domain D, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

We have a partial converse to this. If the domain is simply connected and M, N have continuous first–order partials on the simply connected domain D with

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

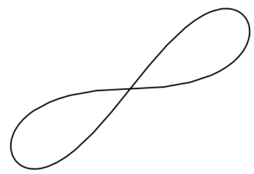
on D, then F is conservative. This is a condition we can check!

This has several important uses. We can check when a field is conservative. This will also give us a possible way to compute integrals quickly, especially when they are complicated, in the case where the field is conservative. The only barrier will be to find the potential field.

4.2 | Exercises

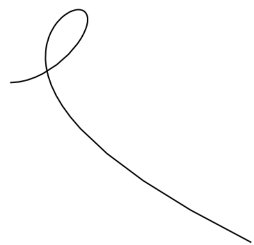
Curves & Regions

1. Is the curve below closed? Is the curve simple?



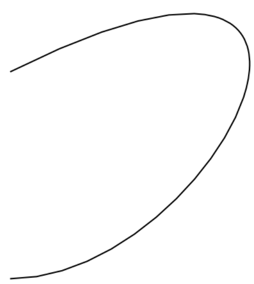
Ans: Closed, not simple.

2. Is the curve below closed? Is the curve simple?



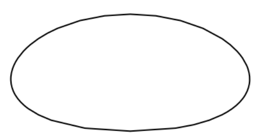
Ans: Not closed or simple.

3. Is the curve below closed? Is the curve simple?



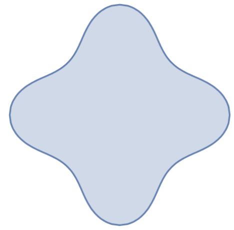
Ans: Simple but not closed.

4. Is the curve below closed? Is the curve simple?



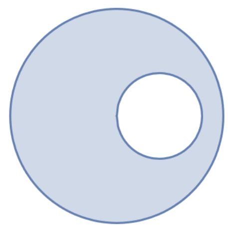
Ans: Closed and simple.

5. Is the region below simply connected? Is it open? Is it path connected? Explain why or why not.



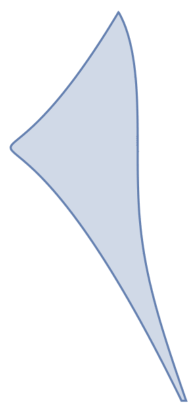
Ans: Simply connected. Open? Path connected.

6. Is the region below simply connected? Is it open? Is it path connected? Explain why or why not.



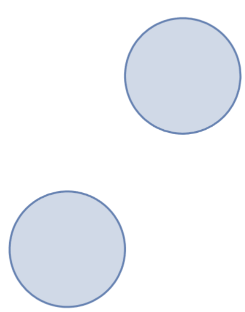
Ans: Not simply connected. Open? Path connected.

7. Is the region below simply connected? Is it open? Is it path connected? Explain why or why not.



Ans: Simply connected. Open? Path connected.

8. Is the region below simply connected? Is it open? Is it path connected? Explain why or why not.



Ans: Not simply connected. Open? Not path connected.

Conservative Vector Fields

9. Determine if the vector field $\mathbf{F} = \mathbf{i} + \mathbf{j}$ is conservative. If it is, find a potential function for $\mathbf{F}$.

Ans: $f(x, y) = x + y + C$

10. Determine if the vector field $\mathbf{F} = \langle 2xy - 1, x^2 + 1\rangle$ is conservative. If it is, find a potential function for $\mathbf{F}$.

Ans: $f(x, y) = x^2y + y - x + C$

11. Determine if the vector field $\mathbf{F} = \dfrac{\mathbf{i}}{y+1} + \left(\dfrac{1}{y} - \dfrac{x+y}{y^2}\right)\mathbf{j}$ is conservative. If it is, find a potential function for $\mathbf{F}$.

Ans: The field is not conservative.

12. Determine if the vector field $\mathbf{F} = \langle y - \cos x, x + \dfrac{1}{y}\rangle$ is conservative. If it is, find a potential function for $\mathbf{F}$.

Ans: $f(x, y) = \ln y - \sin x + xy + C$

13. Determine if the vector field $\mathbf{F} = (\cos xy - xy \sin xy)\mathbf{i} - (x^2 \sin xy + 1)\mathbf{j}$ is conservative. If it is, find a potential function for $\mathbf{F}$.

Ans: The field is not conservative.

14. Determine if the vector field $\mathbf{F} = (y \cos xy \cos yz\mathbf{i} + (x \cos xy \cos yz - z \sin xy \sin yz + y^2)\mathbf{j} - y \sin xy \sin yz\mathbf{k}$ is conservative, if it is, find a potential function for $\mathbf{F}$.

Ans: The field is not conservative.

15. Determine if the vector field $\langle 2xz, z, 1 + x^2 + y\rangle$ is conservative. If it is, find a potential function for $\mathbf{F}$.

Ans: $f(x, y, z) = x^2z + yz + z + C$

16. Determine if the vector field $\langle 3x^2 + \sin y, \dfrac{1}{z} + x\cos y, -\dfrac{y}{z^2}\rangle$ is conservative. If it is, find a potential function for $\mathbf{F}$.

Ans: $f(x, y, z) = x^3 + \dfrac{y}{z} + x \sin y + C$

17. Determine if the vector field $\mathbf{F} = \log y \sin z\mathbf{i} + x \cos y \ln z\mathbf{j} + \dfrac{x \sin y}{z}\mathbf{k}$. If it is, find a potential function for $\mathbf{F}$.

Ans: The field is not conservative.

18. Determine if the vector field $\mathbf{F} = \dfrac{z}{2\sqrt{x}}\mathbf{i} + (e^y - \pi z \sec^2(\pi yz))\mathbf{j} + (\sqrt{x} - \pi y \sec^2(\pi yz))\mathbf{k}$. If it is, find a potential function for $\mathbf{F}$.

Ans: $f(x, y, z) - z\sqrt{x} - \tan(\pi yz) + e^y + C$

Evaluating Integrals

19. Show that the line integral $\int_C (3x-5y)\,dx+(7y-5x)\,dy$, where C is the line segment from $(1,3)$ to $(5,2)$ is path independent and evaluate the integral.
Ans: $f(x,y)=\dfrac{3x^2}{2}-5xy+\dfrac{7y^2}{2}$. Integral $-33/2$

20. Show that the line integral $\displaystyle\int_C \frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}$, where C is the semicircular arc of $x^2+y^2=4$ from $(2,0)$ to $(-2,0)$ is path independent and evaluate the integral.
Ans: $f(x,y)=\sqrt{x^2+y^2}$. Integral 0

21. Show that the line integral $\int_C (2y-3z)\,dx+(2x+z)\,dy+(y-3x)\,dz$, where C is the line segment from $(0,0,0)$ to $(0,1,1)$ then the line segment to $(1,2,3)$ is independent of path and evaluate the integral.
Ans: $f(x,y,z)=2xy-3xz+yz$. Integral 1

22. Let $\mathbf{F}=\langle 2x\frac{3}{2}\sqrt{y}\rangle$. Compute the integral $\int_C \mathbf{F}\cdot ds$, where $C:[0,1]\to\mathbb{R}^2$ is the upper quarter of the ellipse, oriented counterclockwise, centered at the origin with semimajor axis (along the x-axis) of length 4 and semiminor axis (the y-axis) of length 3.
Ans: $f(x,y)=x^2+y^{3/2}+C$. $\mathbf{x}(0)=(4,0)$, $\mathbf{x}(1)=(0,3)$. The integral is $18-(2+3\sqrt{3})=16-3\sqrt{3}$.

23. Let $\mathbf{F}=x\mathbf{i}-y^2\mathbf{j}$. Compute the integral $\int_C \mathbf{F}\cdot ds$, where $C:[0,1]\to\mathbb{R}^2$ is the curve given by $\mathbf{x}(t)=(t,e^{t^4})$, where $0\le t\le 1$.
Ans: $f(x,y)=\dfrac{x^2}{2}-\dfrac{y^3}{3}+C$. $\mathbf{x}(0)=(0,1)$, $\mathbf{x}(1)=(1,e)$. The integral is $\left(\dfrac{1}{2}-\dfrac{e^3}{3}\right)-\dfrac{-1}{3}=\dfrac{5}{6}-\dfrac{e^3}{4}$.

24. Let $\mathbf{F}=(2xy+1)\mathbf{i}+(x^2-1)\mathbf{j}$. Compute the integral $\int_C \mathbf{F}\cdot ds$, where C is the path $C:[0,1]\to\mathbb{R}^2$ given by

$$\mathbf{x}(t)=\left(e^{t^2-t}+\sin\left(\pi\cos\left(\frac{\pi}{2}t\right)\right)-2,\frac{1}{t^2+2t-4}-\sin(\pi t)\right)$$

Ans: $f(x,y)=x^2y-y+x+C$. $\mathbf{x}(0)=(-1,-1/4)$, $\mathbf{x}(1)=(-1,-1)$. The integral is $1-1=0$.

25. Let $\mathbf{F}=\dfrac{1-2xy}{y^2-1}\mathbf{i}+\dfrac{x^2-2xy+x^2y^2}{(y^2-1)^2}\mathbf{j}$. Compute the integral $\int_C \mathbf{F}\cdot ds$, where $C:[0,1]\to\mathbb{R}^2$ is the curve given by

$$\mathbf{x}(t)=\left(2^t+t^4+\sqrt{\frac{2t^2+t}{3}},(t+1)^t+\frac{1}{t+1}+\frac{t}{2}\right)$$

Ans: $f(x,y)=\dfrac{x^2y-x}{1-y^2}+C$. $\mathbf{x}(0)=(1,2)$, $\mathbf{x}(1)=(4,3)$. The integral is $\dfrac{-11}{2}-\dfrac{1}{3}=\dfrac{-31}{6}$.

26. Let $\mathbf{F} = \left\langle \sqrt[3]{y} - \frac{y}{3\sqrt[3]{x}}, \frac{x}{3\sqrt[3]{y^2}} - \sqrt[3]{x} \right\rangle$. Compute the integral $\int_C \mathbf{F} \cdot ds$, where $C : [0,1] \to \mathbb{R}^2$ is the curve given by

$$\mathbf{x}(t) = \left((t+1)^{(t+1)^t} + t^2 + \arctan^2(t^2 - t) + 1, \frac{1}{t^2+1} \ln(t^2 - t + 1) + \sin^4(\pi t \cos^3(\pi t)) + \frac{t}{2} \right)$$

Ans: $f(x,y) = xy^{1/3} - x^{1/3}y$. $\mathbf{x}(0) = (2,1)$, $\mathbf{x}(1) = (6,1)$. The integral is $(6 - 6^{1/3}) - (2 - 2^{1/3}) = 4 + \sqrt[3]{2} - \sqrt[3]{6}$.

27. Let $\mathbf{F} = 2x\mathbf{i} + \cos y \cos z\mathbf{j} - \sin y \sin z\mathbf{k}$. Compute the integral $\int_C \mathbf{F} \cdot ds$, where C is the path from $(0,0,0)$ to $(1,3,1)$, $(1,3,1)$ to $(-4,5,6)$, and finally $(-4,5,6)$ to $(0,0)$.
Ans: $f(x,y,z) = x^2 + \sin y \cos z + C$. The integral is 0.

4.3 Green's Theorem

Green's Theorem: Let D be a closed, bounded region in $\mathbb{R}^2$ whose boundary $C = \partial D$ consist of finitely many simple, closed curves. Orient the curves of C so that D is on the left as one traverses C. Let $\mathbf{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ be a vector field of class C^1 throughout D. Then

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_{\partial D} \mathbf{F} \cdot ds = \iint_D \nabla \times \mathbf{F} \cdot \mathbf{k}\,dA$$

Note that if we consider $\mathbf{F}$ as a three–dimensional vector space via $\mathbf{F} = \langle M, N, P\rangle$, then the integrand is $\nabla \times \mathbf{F} \cdot \mathbf{k}$. This theorem is especially nice when there are many paths one would have to integrate over for the line integral but the region itself is 'nice'. Be sure to point out how to handle regions with holes (to which Green's Theorem still applies). That is, form a line from the 'boundary' to the hole and back – negating each other. Give physical interpretation, can either measure curl everywhere (but all microscopic curls cancel along their boundaries) or just along the boundary of the surface (the only region where cancelation does not occur).

4.3 | Exercises

1. Let $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ and D be the region bound by the curves $y = x$ and $y = x^2$ in the plane. Verify Green's Theorem for the integral $\displaystyle\oint_{\partial D} \mathbf{F} \cdot ds$.

Ans: $-1/12$

2. Evaluate the integral $\oint_C -y\,dx + x\,dy$, where C is a circle of radius a, oriented counterclockwise.

Ans: $2\pi a^2$

3. Calculate

$$\oint_C xy\,dx + x^2y^3\,dy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$, oriented counterclockwise.

Ans: $2/3$

4. Show that if D is any region to which Green's Theorem applies that then we have

$$\text{area } D = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy$$

Ans: Simple application of Green's Theorem.

5. Use the previous exercise to find the area of an ellipse with semimajor and semiminor axes of length a, b, respectively.

Ans: πab

6. Verify Green's Theorem for $D = \{(x, y) \colon x^2 + y^2 \leq 4\}$ and $\mathbf{F} = -x^2y\,\mathbf{i} + xy^2\,\mathbf{j}$.

Ans: 8π

7. Verify Green's Theorem for D the square centered at the origin with side length 2 and $\mathbf{F} = y\,\mathbf{i} + x^2\,\mathbf{j}$.

Ans: -4

8. Calculate

$$\oint_C y^2\,dx + x^2\,dy$$

where C is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, oriented counterclockwise.

Ans: $-14\sqrt{2}\,\pi$

9. Find the work done by the vector field $\mathbf{F} = (4y - 3x)\,\mathbf{i} + (x - 4y)\,\mathbf{j}$ on a particle moving counterclockwise twice around the ellipse $x^2 + 4y^2 = 4$.

Ans: $2 \cdot -6\pi = -12\pi$

10. Evaluate

$$\oint_C y^2\,dx + x^2\,dy$$

where C is the boundary of the triangle with vertices $(0,0)$, $(1,1)$ and $(1,0)$, oriented clockwise.
Ans: $-1/3$

11. Calculate

$$\oint_C \left(2y + \tan(\ln(x^2+1))\right)\, dx + \left(5x - e^{-y^2} + \sin^2 y^4\right)\, dy$$

where C is the circle of radius 3 centered at the origin.
Ans: Use Green's Theorem. $3 \cdot \text{area circle} = 3 \cdot (3^2\pi) = 27\pi$

12. Calculate

$$\oint_C \mathbf{F} \cdot ds$$

where $\mathbf{F} = \langle e^y, -\sin \pi x \rangle$ and C is the triangle with vertices $(1,0)$, $(0,1)$, and $(-1,0)$, oriented clockwise.
Ans: $2e + \dfrac{4}{\pi} - 4$

13. Calculate

$$\oint_C y^4\, dx + 2xy^3\, dy$$

where C is the ellipse $x^2 + 2y^2 = 2$.
Ans: Use Green's Theorem to reduce to $\iint_D -2y^3\, dA$, note that the region is symmetric about x and $-2y^3$ is odd with respect to y so the integral is 0.

14. If D is a region to which Green's Theorem applies and ∂D is oriented properly, show that

$$\text{area } D = \oint_{\partial D} x\, dy = -\oint_{\partial D} y\, dx$$

Ans: Simply apply Green's Theorem to the given line integrals.

15. Show that if C is the boundary of any rectangular region in $\mathbb{R}^2$, then

$$\oint_C (x^2y^3 - 3y)\, dx + x^3y^2\, dy$$

depends only on the area of the rectangle, not on the placement of the rectangle in $\mathbb{R}^2$.
Ans: -8area D. Simple application of Green's Theorem.

16. Show that if C is a simple closed curve forming a region D to which Green's Theorem applies, C being oriented properly, then

$$\oint_C -y^3\, dx + (x^3 + 2x + y)\, dy$$

is strictly positive.

Ans: Green's Theorem gives $\displaystyle\iint_D (3(x^2+y^2)+2)\, dA$

17. Let D be a region to which Green's Theorem applies and suppose $u(x, y)$ and $v(x, y)$ are two functions of class C^2 whose domain include D. Show that

$$\iint_D \frac{\partial(u,v)}{\partial(x,y)}\, dA = \oint_C (u\nabla v)\cdot ds$$

where $C = \partial D$ is oriented as in Green's Theorem.

Ans: $u\nabla = (uv_x, uv_y)$ then

$$\begin{aligned}
\oint_C (u\nabla v)\cdot ds &= \oint_C u\frac{\partial v}{\partial x}\, dx + u\frac{\partial v}{\partial y}\, dy \\
&= \iint_D \left(\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)\right) dA \\
&= \iint_D \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial x\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial v\partial x}\right) dA \\
&= \iint_D \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) dA \\
&= \iint_D \frac{\partial(u,v)}{\partial(x,y)}\, dA
\end{aligned}$$

18. Let $f(x, y)$ be a function of class C^2 such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

, i.e. f is harmonic. Show that if C is any closed curve to which Green's Theorem applies, then

$$\oint_C \frac{\partial f}{\partial y}\, dx - \frac{\partial f}{\partial x}\, dy = 0$$

Ans: If D is the area bound by the curve C, by Green's Theorem

$$\oint_C \frac{\partial f}{\partial y}\, dx - \frac{\partial f}{\partial x}\, dx = -\iint_D \left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2}\right) dA = -\iint_D 0\, dA = 0$$

19. Let D be a region to which Green's Theorem applies and $\mathbf{n}$ the outward unit normal vector to D. S tuppose $f(x, y)$ is a function of class C^2. Show that

$$\iint_D \nabla^2 f\, dA = \oint_{\partial D} \frac{\partial f}{\partial n}\, ds$$

where $\nabla^2 f$ denotes the Laplacian of f and $\partial f/\partial n$ denotes $\nabla f \cdot \mathbf{n}$.

Ans: We have $\oint_{\partial D} \frac{\partial f}{\partial n}\, ds = \oint_{\partial D} \nabla f \cdot \mathbf{n}\, ds = \oint_{\partial D} \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)\cdot \mathbf{n}\, ds$. Continue the calculation or note that this is the same computation as in the proof of the Divergence Theorem with $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$. Therefore, applying Green's Theorem

$$\oint_{\partial D} \frac{\partial f}{\partial n}\, ds = \oint_{\partial D} \frac{\partial f}{\partial y}\, dx + \frac{\partial f}{\partial x}\, dy = \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dA = \iint_D \nabla^2 f\, dA$$

4.4 Surface Integrals

Parametrization: A parametrization of a surface in $\mathbb{R}^3$ is a continuous function $X : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ that is one-to-one on D, except possible along ∂D. We refer to $X(D)$ as the underlying surface of X (or the surface parametrized by X) and denote it S.

Sphere: $D = [0, 2\pi) \times [0, \pi]$. $X(s, t) = (a \cos s \sin t, a \sin s \sin t, a \cos t)$

Cylinder: $0 \leq s \leq 2\pi$. $(a \cos s, a \sin s, t)$

Cone: $0 \leq t \leq 2\pi$. $(s \cos t, s \sin t, s)$

$z = f(x, y)$: $x = s$, $y = t$, $z = f(s, t)$, $(s, t) \in D$

Torus: $0 \leq s, t \leq 2\pi$, $a, b > 0$ with $a > b$. $((a + b \cos t) \cos s, (a + b \cos t) \sin s, b \sin t)$

Standard Normal Vector: The parametrized surface $S = \mathbf{X}(D)$ is smooth at $\mathbf{X}(s_0, t_0)$ if the map $\mathbf{X}$ is of class C^1 in a neighborhood of (s_0, t_0) and if the vector

$$\mathbf{N}(s_0, t_0) = \mathbf{T}_s(s_0, t_0) \times \mathbf{T}_t(s_0, t_0) \neq \mathbf{0}$$

If S is smooth at every point $\mathbf{X}(s_0, t_0) \in S$, then we refer to S as a smooth parametrized surface. If S is a smooth parametrized surface, we call $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$ the standard normal vector arising from the parametrization $\mathbf{X}$.

Surface Area:

$$S = \iint_R \|\mathbf{T}_s \times \mathbf{T}_t\| \, ds \, dt$$

or in scalar form

$$S = \iint_R \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2} \, ds \, dt$$

In the case where $z = f(x,y)$, taking the standard parametrization, we have

$$S = \iint_R \|\mathbf{T}_s \times \mathbf{T}_t\| \, ds \, dt = \iint_R \sqrt{f_s^2 + f_t^2 + 1} \, ds \, dt$$

Surface Integral Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset \mathbb{R}^2$ is a bounded region. Let f be a continuous function whose domain includes $S = \mathbf{X}(D)$, then the scalar surface integral of f along $\mathbf{X}$, denoted $\iint_X f \, dS$ is

$$\iint_X f \, dS = \iint_D f(X(s,t)) \, \|T_s \times T_t\| \, ds \, dt = \iint_D f(X(s,t)) \, \|N(s,t)\| \, ds \, dt$$

or in another form

$$S = \iint_R f(x(s,t), y(s,t), z(s,t)) \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2} \, ds \, dt$$

The vector form of surface integrals, denoted $\iint_X \mathbf{F} \cdot dS$ is

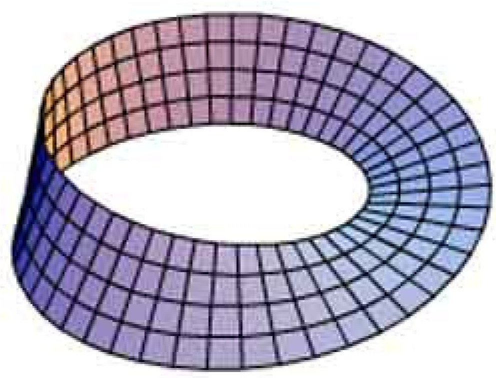
$$\iint_X \mathbf{F} \cdot dS = \iint_D F(X(s,t)) \cdot N(s,t) \, ds \, dt$$

where $N(s,t) = T_s \times T_t$.

Orientability: A smooth, connected surface S is orientable (or two–sided) if it is possible to define a single unit normal vector at each point of S so that the collection of these normal vectors varies continuously over S. Otherwise, S is said to be nonorientable or one–sided.

An example of a nonorientable surface is the Möbius Strip: $0 \le s \le 2\pi, -1/2 \le t \le 1/2$

$$\begin{cases} x = \left(1 + t\cos\dfrac{s}{2}\right)\cos s \\ y = \left(1 + t\cos\dfrac{s}{2}\right)\sin s \\ z = t\sin\dfrac{s}{2} \end{cases}$$



4.4 | Exercises

1. Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrized surface given by $X(s,t) = (s^2 - t^2, s+t, s^2+3t)$. Determine a normal vector to this surface at the point $X(2,-1) = (3,1,1)$. Find the equation of the tangent plane to the surface at this point. Give an equation for the surface of the form $z = f(x,y)$.

Ans: $T_S(2,-1) = (4,1,4)$, $T_t(2,-1) = (2,1,3)$. Normal $(-1,-4,2)$. Plane $-x-4y+2z+1=0$. Note $x = s^2 - t^2$, $y = s+t$ so $x = (s+t)(s-t) = y(s-t)$ then $x/y = s-t$. Solve for s,t and obtain $2s = y + x/y$ and $2t = y - x/y$ so $z = s^2 + 3t$ so $z = (y+x/y)^2/4 + 3(y-x/y)$.

2. Evaluate $\displaystyle\iint_S z^3\ dS$, where S is the sphere parametrized by $X : [0,2\pi] \times [0,\pi] \to \mathbb{R}^3$ where $X(s,t) = (a\cos s \sin t, a \sin s \sin t, a\cos t)$.

Ans: $|N(s,t)| = a^2 \sin t$. $\displaystyle\int_0^{\pi}\int_0^{2\pi} (a\cos t)^3 a^2 \sin t\ ds\ dt = 0$.

3. Let S be the closed cylinder of radius 3 with axis along the z–axis, the top face at $z = 15$ and bottom at $z = 0$. Find $\displaystyle\iint_S z\ dS$.

Ans: Lateral: $x = 3\cos s$, $y = 3\sin s$, $z = t$, $0 \le t \le 15$, $0 \le s \le 2\pi$. Bottom: $x = s\cos t$, $y = s\sin t$, $z = 0$, $0 \le s \le 3$, $0 \le t \le 2\pi$ Top $x = s\cos t$, $y = s\sin t$, $z = 15$, $0 \le s \le 3$, $0 \le t \le 2\pi$. Total Integral 810π

4. Find $\displaystyle\iint_S (4-z)\ dS$, where S is the surface given by $X(x,y) = (x,y,4-x^2-y^2)$.

Ans: $\dfrac{391\sqrt{17}+1}{60}\pi$

5. Find $\displaystyle\iint_X \mathbf{F}\cdot dS$, where $\mathbf{F} = \langle x, y, z-2y\rangle$ and $X(s,t) = (s\cos t, s\sin t, t)$, $0 \le s \le 1$ and $0 \le t \le 2\pi$.

Ans: $N(s,t) = \langle \sin t, -\cos t, s\rangle$. Integral π^2

6. Evaluate $\displaystyle\iint_S (x^3\,\mathbf{i} + y^3\,\mathbf{j})\cdot dS$, where S is the closed cylinder bound by $x^2+y^2 = 4$, $z = 0$, and $z = 5$.

Ans: 120π

7. Find $\displaystyle\iint_X (x^2+y^2+z^2)\ dS$, where $X(s,t) = (s, s+t, t)$, $0 \le s \le 1$, $0 \le t \le 2$.

Ans: $|N| = \sqrt{3}$, $\dfrac{26}{\sqrt{3}}$

8. Find $\displaystyle\iint_S x^2\ dS$, wehre S is the surface of the cube $[-2,2]\times[-2,2]\times[-2,2]$.

Ans: Total: $640/3$

9. Find $\displaystyle\iint_S y^2\ dS$. [Hint: $\displaystyle\iint_S (x^2+y^2+z^2)\ dS$ and use symmetry.]

Ans: $\displaystyle\iint_S (x^2+y^2+z^2)\ dS = a^2$ surface area $= 4\pi a^4$. Each is the same so $4\pi a^4/3$

10. Find $\iint_S (z-x^2-y^2)\,dS$, where S is the surface of the cylinder bound by $x^2+y^2=4$, $z=-2$, and $z=2$.

Ans: $|N|=2$. Total: -64π. Or $\iint_S (z-x^2-y^2)\,dS = \iint_S z\,dS - \iint_S (x^2+y^2)\,dS$. S symmetric about $z=0$ and $x^2+y^2=4$ on S. So $\iint_S z\,dS = 0$ and $-\iint_S (x^2+y^2)\,dS = -4\cdot$ surface area $= -4(4\pi\cdot 4) = -64\pi$

4.5 Divergence Theorem & Stokes' Theorem

Divergence Theorem: Divergence Theorem/Gauss' Theorem allows use to turn Surface Integrals into Triple Integrals.

Gauss' Theorem: Let D be a bounded solid region in $\mathbb{R}^3$ whose boundary ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals pointing outwardly from D. Then if $\mathbf{F}$ is a vector field of class C^1 whose domain includes D, then

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

That is, the "total divergence" of a vector field in a bounded region in space is equal to the flux of the vector field away from the region, i.e. the flux across the boundary surface. Think of $\mathbf{F}$ as a three–dimensional flow. We can think of $\div F$ as the source rate at (x, y, z) (negative rate means fluid is being removed). In this way, we have flux across S is the source rate for D, i.e. the net flux outward across S is the same as the rate at which fluid is being produced (or added to the flow) inside S.

Stokes' Theorem: Stokes' Theorem allows us to turn surface integrals into line integrals.

Stokes' Theorem: Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$. Suppose that ∂S consists of finitely many piecewise C^1, simple, closed curves each of which is oriented consistently with S. Let $\mathbf{F}$ be a vector field of class C^1 whose domain includes S. Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{S}$$

4.5 | Exercises

Divergence/Gauss' Theorem

1. Verify the Divergence Theorem for R the portion of the paraboloid $z = 9 - x^2 - y^2$ above the xy–plane and $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

Ans: $243\pi/2$

2. Verify the Divergence Theorem for R the unit cube and $\mathbf{F} = (y - x)\,\mathbf{i} + (y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$.

Ans: 0

3. Verify the Divergence Theorem for R the standard unit cube and $\mathbf{F} = y^2\,\mathbf{i} + (2xy + z^2)\,\mathbf{j} + 2yz\,\mathbf{k}$.

Ans: 2

4. Let S be the solid cylinder of radius a and height b centered along the z-axis with bottom at $z = 0$. Let $\mathbf{F} = \langle x, y, z\rangle$. Verify Gauss' Theorem for S and $\mathbf{F}$.

Ans: $3\pi a^2 b$

5. Find the flux of $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ across the sphere given by $x^2 + y^2 + z^2 = a^2$.

Ans: $\dfrac{12\pi a^5}{5}$

6. Let S be the sphere $(x - 2)^2 + (y + 5)^2 + (z - 1)^2 = 4$ along with its interior and $\mathbf{F} = 5x\,\mathbf{i} - 3\,\mathbf{j} - \mathbf{k}$. Calculate $\displaystyle\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S}$.

Ans: $\dfrac{32\pi}{3}$

7. Find the flux of $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ across the sphere given by $x^2 + y^2 + z^2 = a^2$.

Ans: $\dfrac{12\pi a^5}{5}$

8. Let $\mathbf{F} = e^y \cos z\,\mathbf{i} + \sqrt{x^3 + 1}\,\sin z\,\mathbf{j} + (x^2 + y^2 + 3)\,\mathbf{k}$ and S be $z = (1 - x^2 - y^2)e^{1 - x^2 - 3y^3}$ for $z \geq 0$, oriented outwards. Find $\displaystyle\oiint_{\partial D} \mathbf{F} \cdot dS$.

Ans: $7\pi/2$

9. Let S be the region formed by $z = x^2 + y^2$ and $z = 2$. Let $\mathbf{F} = y^{2/3}\,\mathbf{i} + \sin^3 x\,\mathbf{j} + z^2\,\mathbf{k}$. Find $\displaystyle\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S}$.

Ans: 8π

10. Let $S_1 = \langle (x, y, z) \colon z = 1 - x^2 - y^2, z \geq 0\}$, $S_2 = \{(x, y, z) \colon z = 0, x^2 + y^2 \leq 1\}$, and define S to be the surface created by putting S_1 and S_2 together, appropriately outwardly oriented. Define $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$. Find $\displaystyle\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S}$, $\displaystyle\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$, and $\displaystyle\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$. Show without direct calculation of the surface integrals that $\displaystyle\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$.

Ans: The total integral is 0. Since $\iint_{S_1+S_2} \mathbf{F}\cdot d\mathbf{S}$ is the difference of the two. Finally, $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} xy\,dx\,dy = 0.$

11. Let S be the boundary of the cube defined by $-2 \le x \le 2$, $-1 \le y \le 1$, and $-1 \le z \le 5$ and $\mathbf{F} = x^3y^3\,\mathbf{i} + 4yz\,\mathbf{j} - 3x^2y^3z\,\mathbf{k}$. Calculate $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

Ans: Ans: 384

12. Let R be the region formed by $x^2+y^2+z^2 \le 1$. Find $\iiint_R z^2\,dV$.

Ans: $\dfrac{4\pi}{15}$

13. Find

$$\iint_S \mathbf{F}\cdot dS$$

where S is the box with vertices $(\pm1, \pm2, \pm3)$ with outward normal and $\mathbf{F} = x^2y^3\,\mathbf{i} + y^2z^3\,\mathbf{j} + z^2x^3\,\mathbf{k}$.
Ans: 0

14. Let S be the surface given by $z^2 = x^2 + y^2$ and $0 \le z \le 1$. Define $\mathbf{F} = \langle x, 2y, 3z\rangle$. Calculate $\iint_{\partial S} \mathbf{F}\cdot d\mathbf{S}$.
Ans: 16π

15. Consider a fluid having density $\rho(\mathbf{r})$ and velocity $\mathbf{v}(\mathbf{r})$. Let V be a volume with no fluid sources or sinks bounded by a closed surface S. The mass flux is given by $\int_S \rho\,\mathbf{v}\cdot d\mathbf{S}$ so that

$$\int_S \rho\mathbf{v}\cdot d\mathbf{S} := \int_S \mathbf{J}\cdot d\mathbf{S}$$

where $\mathbf{J} = \rho\,\mathbf{v}$ is the mass current. Argue why

$$\int_S \mathbf{J}\cdot d\mathbf{S} = -\frac{\partial M}{\partial t}$$

Find any integral representation for the mass M in V. Use this and the Divergence Theorem to derive the Continuity Equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

Interpret the equation. How general is the result?

Ans: We have $M = \int_V \rho\,dV$. Then we have

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_S \mathbf{J}\cdot d\mathbf{S} = 0$$

Then the Divergence transforms the second integral so that we have

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) dV = 0$$

and as this holds for any V, we arrive at the continuity equation. This holds for *any* conserved quantity. If $\nabla \cdot \mathbf{J}(r) > 0$, then $\dfrac{\partial \rho}{\partial t} < 0$ and the mass density at $\mathbf{r}$ decreases and vice versa.

16. Let $\mathbf{F} = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k}$. Show that

$$\text{div } \mathbf{F}(P) = \lim_{V \to 0} \frac{1}{V} \oiint_S \mathbf{F} \cdot dS$$

where S is a piecewise smooth, orientable, closed surface S enclosing a region D of volume V oriented outwardly and the limit is taken to shrink D down to the point P.

Ans: Shrink D down to P. The volume decreases monotonically downwards. Let D_V be the shrunken version of D which is the solid of volume V and let $S_V = \partial D_V$ for $0 \le V \le$ the volume of D. Then by Gauss' Theorem,

$$\oiint_{S_V} \mathbf{F} \cdot dS = \iiint_{D_V} \nabla \cdot \mathbf{F}\, dV$$

By the MVT, there is a $Q_V \in D_V$ so that

$$\iiint_{D_V} \nabla \cdot \mathbf{F}\, dV = \iiint_{D_V} \nabla \cdot \mathbf{F}(Q_V)\, dV = \nabla \cdot \mathbf{F}(Q_V)(\text{volume of } D)$$

so that

$$\lim_{V \to 0} \frac{1}{V} \oiint_{S_V} \mathbf{F} \cdot dS = \lim_{V \to 0} \nabla \cdot \mathbf{F}(Q_V) = \nabla \cdot \mathbf{F}(P) = \text{div } \mathbf{F}(P)$$

Stokes' Theorem

17. Verify Stokes' Theorem for $R = \{(x, y, z) \colon z = \sqrt{1 - x^2 - y^2}, z \ge 0\}$ and $\mathbf{F} = \langle x, y, z \rangle$.

Ans: $\nabla \times \mathbf{F} = \mathbf{0}$ and tangent at $(x, y, 0)$ is $(-y, x, 0)$ – perpendicular to $\mathbf{F}$. So both integrals 0.

18. Verify Stokes' Theorem for $R = \{(x, y, z) \colon x = 0, -1 \le y, z \le 1\}$ and $\mathbf{F} = (2xz + 3y^2)\,\mathbf{j} + 4yz^2\,\mathbf{k}$.

Ans: 4/3

19. Verify Stokes' Theorem for $\mathbf{F} = x^2\,\mathbf{i} + 2x\,\mathbf{j} + z^2\,\mathbf{k}$ and S the surface given by $\{(x, y, z) \colon 4x^2 + y^2 \le 4, z = 0\}$.

Ans: 4π

20. Let C be the boundary of $2x + y + 2z = 2$ in the first octant, oriented counterclockwise viewed from above. Let $\mathbf{F} = \langle x + y^2, y + z^2, z + x^2 \rangle$. Find $\displaystyle\int_C \mathbf{F} \cdot ds$.

Ans: $X(s, t) = (s, t, 1/2(2 - 2s - t))$. $T_s = (1, 0, -1)$, $T_t = (0, 1, -1/2)$, $T_s \times T_t = (1, 1/2, 1)$. $-4/3$

21. Let $S = \{(x,y,z)\colon z \le 9 - x^2 - y^2, z \ge 5\}$ with normal vector pointing outwards. Let $\mathbf{F} = yz\,\mathbf{i} + x^2z\,\mathbf{j} + xy\,\mathbf{k}$. Find

$$\iint_S \nabla \times \mathbf{F} \cdot dS$$

Ans: Use $\mathbf{x}(t) = (2\cos t, 2\sin t, 5)$ and Stokes' Theorem. Reduce to $\int_0^{2\pi} -20\sin^2 t + 40\cos^3 t\ dt$. -20π.

22. Verify Stokes' Theorem for R the portion of the paraboloid above the xy–plane and $\mathbf{F} = (2z - y)\,\mathbf{i} + (x+z)\,\mathbf{j} + (3x - 2y)\,\mathbf{k}$.

Ans: 18π

23. Find

$$\iint_S \nabla \times \mathbf{F} \cdot dS$$

where S is the surface $S = \{(x,y,z)\colon 1 \le z \le 5 - x^2 - y^2\}$ with outward normal and $\mathbf{F} = \langle z^2, -3xy, x^3y^3\rangle$.

Ans: 0

24. Find

$$\oint_{\partial S} \mathbf{F} \cdot ds$$

where $\mathbf{F} = (x + 2y + 3z, x^2 + 2y^2 + 3z^2, x + y + z)$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

Ans: $z = 1 - x - y$. $X(x,y) = (x, y, 1 - x - y)$. $N(x,y) = (-z_x, -z_y, 1) = (1,1,1)$. $0 \le x + y \le 1$ so $0 \le x \le 1$, $0 \le y \le 1 - x$. Integral $-1/6$

25. Evaluate

$$\oint_{\partial S} \mathbf{F} \cdot ds$$

where ∂S is the path $C_1 : \mathbf{x} = (t, 0, 0)$, where $0 \le t \le 2$, followed by $C_2 : \mathbf{x}(t) = 2\cos t(1,0,0) + 2\sin t\frac{1}{\sqrt{2}}(0,1,1) = (2\cos t, \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}})$ for $0 \le t \le 2\pi$, and finally $C_3 : \mathbf{x}(t) = (0, 2 - t, 2 - t)$, where $0 \le t \le 2$ and $\mathbf{F} = (z - y)\,\mathbf{i} - (x + z)\,\mathbf{j} - (x + y)\,\mathbf{k}$.

Ans: Curve $C : x^2 + y^2 + z^2 = 4$ and $z = y$. $\nabla \times \mathbf{F} \cdot n\, dS = -2\, dx\, dy$. Integral $-4\sqrt{2}\pi$.

26. Find $\iint_S \nabla \times \mathbf{F} \cdot dS$, where $S = S_1 \cup S_2$, where $S_1 = \{(x,y,z)\colon x^2 + y^2 = 9, 0 \le z \le 8\}$ and $S_2 = \{(x,y,z)\colon x^2 + y^2 + (z-8)^2 = 9, z \ge 8\}$ and $\mathbf{F} = (x^3 + xz + yz^2)\,\mathbf{i} + (xyz^3 + y^7)\,\mathbf{j} + x^2z^5\,\mathbf{k}$.

Ans: 0

27. Calculate

$$\oint_S \nabla \times \mathbf{F} \cdot n\, dS$$

where $\mathbf{F} = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$ and S is the triangle with vertives $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$.

Ans: 4/3

28. Verify that Stokes' Theorem implies Green's Theorem.

Ans: Take $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$.

29. Find the word done by the vector field $\mathbf{F} = \langle x + z^2, y + x^2, z + y^2 \rangle$ on a particle moving around the edge of the sphere of radius 2 centered at the origin lying in the first octant, oriented outwards.

Ans: 16J

30. Evaluate

$$\oint_{\partial S} \mathbf{F} \cdot ds$$

where $\mathbf{F} = \left\langle e^{-x^2} + \sin\ln(x^2+1) - y + z, \sin y^2 - \sqrt{1+y^4} + 2x + z, x - y - e^z + \tan\sqrt[3]{x} \right\rangle$ and ∂S is the intersection of $x^2 + y^2 = 16$ and $z = 2x + 4y$, oriented counterclockwise viewed from above.

Ans: $n = \langle -2, -4, 1$ for plane $-2x - 4y + z = 0$. Curl $\langle -2, 0, 3 \rangle$. Integral $7\cdot$ area region $= 7 \cdot 16\pi = 112\pi$.

31. Show that $\mathbf{x}(t) = (\cos t, \sin t, \sin 2t)$ lines on the surface $z = 2xy$ and evaluate

$$\oint_S (y^3 + \cos x)\,dx + (\sin y + z^2)\,dy + x\,dz$$

where C is closed curve parametrized and oriented by the path $\mathbf{x}(t)$.

Ans: The first part is trivial. Take normal $\dfrac{-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$, then we have the integral $-3\pi/4$.

32. Calculate $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot dS$, where $\mathbf{F} = (e^{y+z} - 2y)\,\mathbf{i} + (xe^{y+z} + y)\,\mathbf{j} + e^{x+y}\,\mathbf{k}$ and S is the surface $z = e^{-(x^2+y^2)}$ and $z \geq 1/e$. [Hint: Stokes' Theorem works for *any* surface with appropriate boundary.]

Ans: Use the fact that Stokes' Theorem works for *any* orientable piecewise smooth surface with appropriate boundary. Choose $S' = \{(x,y,z)\colon x^2 + y^2 \leq 1, z = 1/e\}$. Then we obtain 2π