Math Reference Tables

Fractions

- $a(b \pm c) = ab \pm ac$
- $a\left(\frac{b}{c}\right) = \frac{ab}{c}$
- $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cb}{bd}$
- $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$
- $\frac{a}{b/c} = \frac{ac}{b}$
- $\frac{a/c}{b} = \frac{a}{bc}$
- $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
- $\frac{a/b}{c/d} = \frac{ad}{bc}$

Inequalities & Absolute Values

- If $x < y$, then $x \pm z < y \pm z$.
- If $x < y$ and $z > 0$, then $xz < yz$ and $\frac{x}{z} < \frac{y}{z}$.
- If $x < y$ and $z < 0$, then $xz > yz$ and $\frac{x}{z} > \frac{y}{z}$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- $|x| \geq 0$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x| = a$ if and only if $x = a$ or $x = -a$
- $|x| < a$ if and only if $-a < x < a$
- $|x| > a$ if and only if $x > a$ or $x < -a$
- Triangle Inequality: $|x + y| \leq |x| + |y|$
- $|x - y| \geq ||x| - |y||$
- If $x = |y|$, then $x = y$ or $x = -y$.
- If $|x| < y$, then $-y < x < y$.
- If $|x| > y$, then $x > y$ or $x < -y$.

Exponential Functions

Assume throughout that $x, y > 0$.

- $x^0 = 1$
- $x^a x^b = x^{a+b}$
- $(x^a)^b = x^{ab}$
- $x^{-a} = \frac{1}{x^a}$
- $\frac{1}{x^{-a}} = x^a$
- $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $\left(\frac{x}{y}\right)^{-a} = \left(\frac{y}{x}\right)^a$
- $x^{\frac{a}{b}} = \left(x^{\frac{1}{b}}\right)^a = (x^a)^{\frac{1}{b}}$

Radicals

- $\sqrt[n]{x} = x^{\frac{1}{n}}$
- $\sqrt[n]{\sqrt[m]{x}} = \sqrt[nm]{x}$
- $\sqrt[n]{x^n} = (-1)^n x$
- $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$
- $x^{\frac{a}{b}} = \sqrt[b]{x^a}$
- $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$

Logarithms

Recall that $y = \log_b x$ if and only if $x = b^y$. We require $b > 0$ and the domain of $\log x$ is $x > 0$. Note in the Sciences and in Mathematics, log nearly always denotes ln, which is $\log_e$.

- $\log_b b = 1$
- $\log_b 1 = 0$
- $\log_b b^x = x$
- $b^{\log_b x} = x$
- $y \log_b x = \log_b x^y$
- $\log_b x + \log_b y = \log_b(xy)$
- $\log_b x - \log_b y = \log_b \left(\dfrac{x}{y}\right)$
- $\log_b x = \dfrac{\ln x}{\ln b}$

Factoring

- $x^2 - a^2 = (x+a)(x-a)$
- $x^2 \pm 2ax + a^2 = (x \pm a)^2$
- $x^2 + (a+b)x + ab = (x+a)(x+b)$
- $x^3 \pm a^3 = (x \pm a)(x^2 \mp ax + a^2)$
- $x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$
- $x^n \pm a^n = (x \pm a)(x^{n-1} \pm ax^{n-2} \mp \cdots a^{n-1})$; n odd

Expanding

- $(x \pm y)^2 = x^2 \pm 2xy + y^2$
- $(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$
- $(x \pm y)^4 = x^4 \pm 4x^3y + 6x^2y^2 \pm 4xy^3 + y^4$

Of course, this is generalized by the binomial formula:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

Solving Polynomial Equations

Fundamental Theorem of Algebra: A polynomial, $p(x)$, of degree n has exactly n (not necessarily distinct) roots in $\mathbb{C}$. None of the roots of $p(x)$ need be real. If $p(x)$ is a real polynomial and $a + bi$ is a complex number that is a root of $p(x)$ then the complex conjugate of $a + bi$, $\overline{a+bi}$, is also a root of $p(x)$, i.e. complex roots come in pairs. If $p(x)$ has odd degree, then $p(x)$ must have at least one real root. In fact, every real polynomial can be factored in a product of linear and quadratic polynomials.

Roots and Factors of Polynomials: If $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a real polynomial and $p(r) = 0$, then r is called a root (or zero) of the polynomial. A real r is a zero of $p(x)$ if and only if $x - r$ is a factor of $p(x)$.

Multiple Roots: If $p(x)$ is a real polynomial, then r is a multiple root if and only if r is a root of $p'(x)$.

Sum/Product of Roots: If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a real polynomial, then the sum of the roots of $p(x)$ is $-\frac{a_{n-1}}{a_n}$ and the product of the roots of $p(x)$ is $(-1)^n \frac{a_0}{a_n}$.

Descartes' Rule of Signs: If $p(x)$ is a real polynomial, then the number of positive roots (not necessarily distinct) is equal to (or less than by an even number) the number of sign changes in $p(x)$ starting from the leading coefficient. The number of negative roots (not necessarily distinct) of $p(x)$ is equal to (or less than by an even number) the number of sign changes in $p(-x)$ starting from the leading coefficient.

Rational Roots Theorem: If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficient, then r is a rational root of $p(x)$ if and only if $r = \frac{m}{n}$, where m is a divisor of a_0 and n is a divisor of a_n.

Eisenstein Criterion: If $p(x)$ is a polynomial with integer coefficients and there is a prime number p such that

(i) p does not divide the leading coefficient

(ii) p divides every non-leading coefficient

(iii) p^2 does not divide the constant term

then $p(x)$ cannot be factored into products of polynomials with rational coefficients.

The Quadratic Equation: If $ax^2 + bx + c = 0$ with $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $D = b^2 - 4ac$ is called the discriminant and it gives information about the roots:

- If $D > 0$, there are two unique real solutions.
- If $D = 0$, there a double real root.
- If $D < 0$, there are two complex solutions.

Moreover, if the roots of $ax^2 + bx + c$ are r_1 and r_2, then $r_1 + r_2 = \frac{-b}{a}$ and $r_1 r_2 = \frac{c}{a}$.

Completing the Square: To solve $ax^2 + bx + c = 0$ by completing the square,

1. Divide by the leading coefficient.
2. Move the constant term to the other side.
3. Take half the "middle term" (coefficient of x), square it, and add the result to both sides.
4. Factor the left side as a perfect square.
5. Take the square root of both sides.
6. Solve for x.

Cubic Equation: $x^3 + a_1x^2 + a_2x + a_3 = 0$. If the coefficient of x^3 is not 1, divide by this coefficient. If a_1, a_2, a_3 are real, then the discriminate, D, is given by $D = Q^3 + R^2$, where

$$Q = \frac{3a_2 - a_1^2}{9}$$
$$R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}$$
$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}$$
$$T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

and if $\begin{cases} D > 0, & \text{one real root, two complex conjugate roots} \\ D = 0, & \text{all real roots, at least two equal} \\ D < 0, & \text{all roots real and unequal} \end{cases}$

Take note that $ST = -Q$.

Combination, Permutations, Binomial Theorem

- $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
- $\binom{n}{k} = \binom{n}{n-k}$
- $P(n,r) = \frac{n!}{(n-r)!}$
- $C(n,r) = \frac{n!}{r!\,(n-r)!}$
- $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$
- $\sum_{i=0}^{n}\binom{n}{i} = 2^n$
- Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$
- Pascal's Formula: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
- Chairperson Identity: $k\binom{n}{k} = n\binom{n-1}{k-1}$
- Summation Identity: $\sum_{i=0}^{n}\binom{i}{k} = \binom{n+1}{k+1}$
- Vandermonde's Identity: $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$
- $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n-k+r-1}{r} = \binom{r-1}{n-1}$
- Multinomial Formula: $\binom{n}{k_1, k_2, \cdots, k_t} = \frac{n!}{k_1!k_2!\cdots k_t!}$

Trig Functions

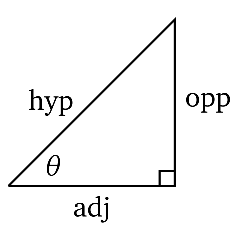
- $\sin\theta, \csc\theta$ are positive in Quadrants I and II, $\cos\theta, \sec\theta$ are positive in Quadrant I and IV, and $\tan\theta, \cot\theta$ are positive in Quadrant I and III.

Function	Domain	Range	Period
$\sin\theta$	$\mathbb{R}$	$[-1,1]$	2π
$\cos\theta$	$\mathbb{R}$	$[-1,1]$	2π
$\tan\theta$	$\mathbb{R}\setminus\{\frac{\pi}{2}\pm n\pi\}_{n\in\mathbb{Z}}$	$\mathbb{R}$	π
$\csc\theta$	$\mathbb{R}\setminus\{\pm n\pi\}$	$(-\infty,-1]\cup[1,\infty)$	2π
$\sec\theta$	$\mathbb{R}\setminus\{\frac{\pi}{2}\pm n\pi\}$	$(-\infty,-1]\cup[1,\infty)$	2π
$\cot\theta$	$\mathbb{R}\setminus\{\pm n\pi\}$	$\mathbb{R}$	π
$\sin^{-1}x$	$[-1,1]$	$[\pi/2,\pi/2]$	2π
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$	2π
$\tan^{-1}x$	$\mathbb{R}$	$(-\pi/2,\pi/2)$	π
$\csc^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[-\pi/2,0)\cup(0,\pi/2]$	π
$\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi/2)\cup(\pi/2,\pi]$	π
$\cot^{-1}x$	$\mathbb{R}$	$(0,\pi)$	π

Degrees & Radians: Note that 2π radians is 360°, i.e. π radians is 180°. This gives $\frac{d}{180}=\frac{r}{\pi}$, where d is the angle measure in degrees and r is the angle measure in radians. This allows conversions between the two.

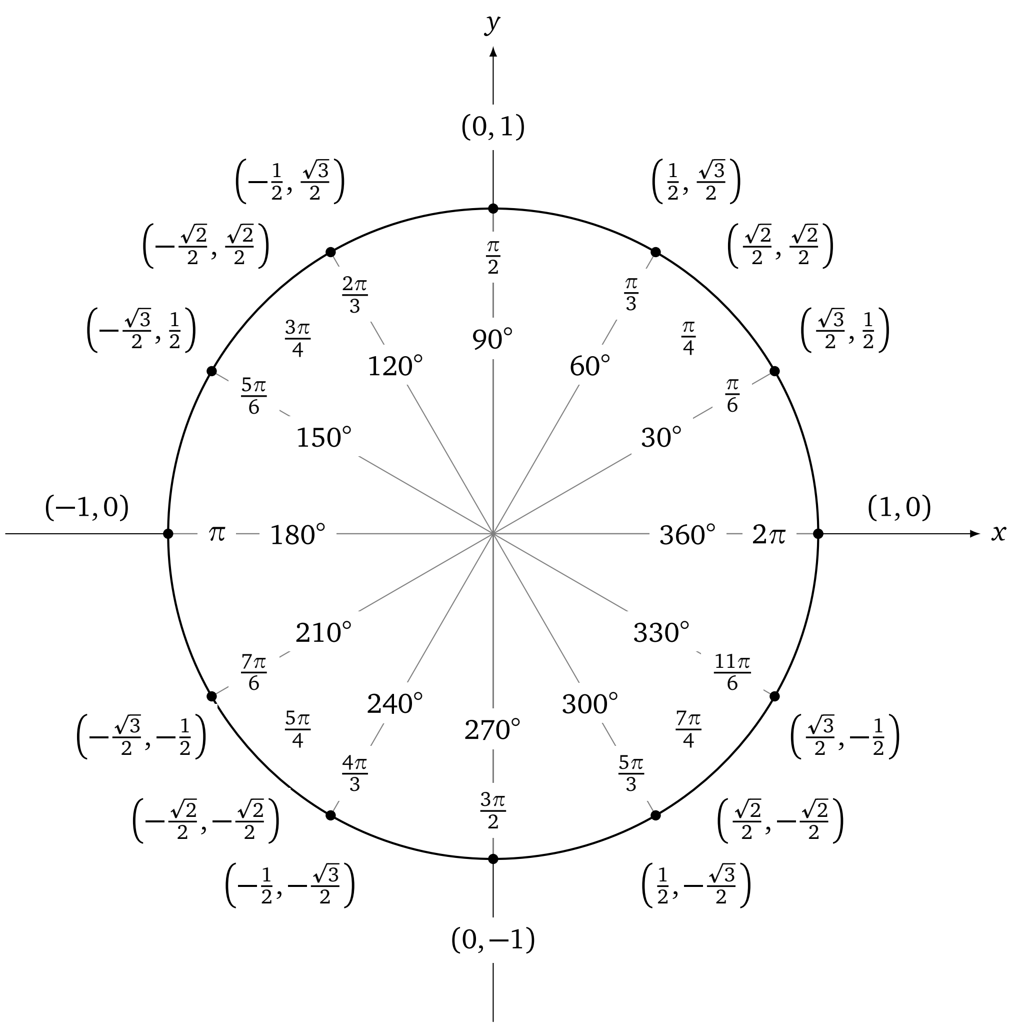
Right Triangle Trig:

- $\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}}$
- $\cos\theta=\frac{\text{adjacent}}{\text{hypotenuse}}$
- $\tan\theta=\frac{\text{opposite}}{\text{adjacent}}$
- $\csc\theta=\frac{\text{hypotenuse}}{\text{opposite}}$
- $\sec\theta=\frac{\text{hypotenuse}}{\text{adjacent}}$
- $\cot\theta=\frac{\text{adjacent}}{\text{opposite}}$
- $\csc\theta=\frac{1}{\sin\theta}$
- $\sec\theta=\frac{1}{\cos\theta}$
- $\cot\theta=\frac{1}{\tan\theta}=\frac{\cos\theta}{\sin\theta}$
- $\sin(-\theta)=-\sin\theta$
- $\cos(-\theta)=\cos\theta$

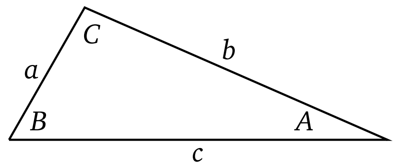


Function/Angle	$0°$	$30°, \frac{\pi}{6}$	$45°, \frac{\pi}{4}$	$60°, \frac{\pi}{3}$	$90°, \frac{\pi}{2}$	$180°, \pi$	$270°, \frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\frac{1}{2}$	$\pm\infty$	0	$\pm\infty$

Unit Circle:



- $\sin^2\theta + \cos^2\theta = 1$
- $\sec^2\theta - \tan^2\theta = 1$
- $\csc^2\theta - \cot^2\theta = 1$
- $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$
- $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$
- $\tan(\theta \pm \phi) = \dfrac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\cos 2\theta = 1 - 2\sin^2\theta$
- $\cos 2\theta = 2\cos^2\theta - 1$
- $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
- $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$
- $\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
- $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{\sin\theta}{1+\cos\theta}$
- $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos\theta}{\sin\theta}$
- $\tan\left(\dfrac{\theta}{2}\right) = \csc\theta - \cot\theta$
- $\tan\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$
- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
- $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$
- $\sin\theta \pm \sin\phi = 2\sin\left(\dfrac{\theta \pm \phi}{2}\right)\cos\left(\dfrac{\theta \mp \phi}{2}\right)$
- $\cos\theta + \cos\phi = 2\cos\left(\dfrac{\theta+\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)$
- $\cos\theta - \cos\phi = 2\sin\left(\dfrac{\theta+\phi}{2}\right)\sin\left(\dfrac{\theta-\phi}{2}\right)$
- $\sin\theta\sin\phi = \dfrac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$
- $\cos\theta\cos\phi = \dfrac{\cos(\theta+\phi)+\cos(\theta-\phi)}{2}$
- $\sin\theta\cos\phi = \dfrac{\sin(\theta+\phi)+\sin(\theta-\phi)}{2}$
- $\sin^{-1}(-\theta) = -\sin^{-1}\theta$
- $\cos^{-1}(-\theta) = \pi - \cos^{-1}\theta$
- $\csc^{-1}\theta = \sin^{-1}(1/\theta)$
- $\sec^{-1}\theta = \cos^{-1}(1/\theta)$
- $\cot^{-1}\theta = \tan^{-1}(1/\theta)$
- $\sin^{-1}\theta + \cos^{-1}\theta = \dfrac{\pi}{2}$
- $\tan^{-1}\theta + \cot^{-1}\theta = \dfrac{\pi}{2}$
- $\sec^{-1}\theta + \csc^{-1}\theta = \dfrac{\pi}{2}$
- $\sin(\pi/2 - \theta) = \cos\theta$
- $\cos(\pi/2 - \theta) = \sin\theta$
- $\tan(\pi/2 - \theta) = \cot\theta$



Other Trig Laws:

- Pythagorean Theorem: If C is a right angle, then $a^2 + b^2 = c^2$.
- Law of Sines: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$
- Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$
- Law of Tangents: $\dfrac{a-b}{a+b} = \dfrac{\tan\left(\dfrac{A-B}{2}\right)}{\tan\left(\dfrac{A+B}{2}\right)}$
- $\sin A = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter: $s = \dfrac{a+b+c}{2}$
- Heron's Area Formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter: $s = \dfrac{a+b+c}{2}$.
- Mollweide's Formulas: $\dfrac{a+b}{c} = \dfrac{\cos\left(\dfrac{A-B}{2}\right)}{\sin\left(\dfrac{C}{2}\right)}$ and $\dfrac{a-b}{c} = \dfrac{\sin\left(\dfrac{A-B}{2}\right)}{\cos\left(\dfrac{C}{2}\right)}$
- Law of Cotangents: $\dfrac{1}{r} = \dfrac{\cot\left(\dfrac{A}{2}\right)}{s-a} = \dfrac{\cot\left(\dfrac{B}{2}\right)}{s-b} = \dfrac{\cot\left(\dfrac{C}{2}\right)}{s-c}$, where $s = \dfrac{a+b+c}{2}$ and r is the in-radius: $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$.

Hyperbolic Functions

- sinh is 2π periodic, $\cosh x$ is 2π periodic, and $\tanh x$ is π periodic.
- $\sinh x = \dfrac{e^x - e^{-x}}{2}$
- $\cosh x = \dfrac{e^x + e^{-x}}{2}$
- $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
- $\tanh x = \dfrac{\sinh x}{\cosh x}$
- $\sinh(-x) = -\sinh x$
- $\cosh(-x) = \cosh x$
- $\cosh^2 x - \sinh^2 x = 1$
- $\operatorname{sech}^2 x + \tanh^2 x = 1$

- $\coth^2 x - \operatorname{csch}^2 x = 1$
- $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
- $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
- $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
- $\sinh 2x = 2\sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
- $\cosh 2x = 2\cosh^2 x - 1$
- $\cosh 2x = 1 + 2\sinh^2 x$
- $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
- $\sinh \dfrac{x}{2} = \operatorname{sgn}(x)\sqrt{\dfrac{\cosh x - 1}{2}}$
- $\cosh \dfrac{x}{2} = \sqrt{\dfrac{\cosh x + 1}{2}}$
- $\tanh \dfrac{x}{2} = \operatorname{sgn}(x)\sqrt{\dfrac{\cosh x - 1}{\cosh x + 1}}$
- $\tanh \dfrac{x}{2} = \dfrac{\sinh x}{\cosh x + 1}$
- $\tanh \dfrac{x}{2} = \dfrac{\cosh x - 1}{\sinh x}$
- $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
- $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
- $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$
- $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$
- $\tanh^{-1} x = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$
- $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \sqrt{1 + \dfrac{1}{x^2}}\right)$
- $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1}{x} + \sqrt{\dfrac{1}{x^2} - 1}\right)$
- $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$
- $\operatorname{csch}^{-1} x = \sinh^{-1}\left(\dfrac{1}{x}\right)$
- $\operatorname{sech}^{-1} x = \cosh^{-1}\left(\dfrac{1}{x}\right)$
- $\coth^{-1} x = \tanh^{-1}\left(\dfrac{1}{x}\right)$

Graphs

- Constant Function: $f(x) = c$
- Linear Function:
  $f(x) = mx + b$,
  $f(x) = y_0 + m(x - x_0)$,
  $Ax + By = C$
- Quadratic Equation (Parabola):
  $f(x) = ax^2 + bx + c$,
  $f(x) = a(x - h)^2 + k$
- Circle: $(x - x_0)^2 + (y - y_0)^2 = r^2$
- Ellipse: $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} = r^2$
- Hyperbola: $\dfrac{(y - y_0)^2}{b^2} - \dfrac{x - x_0)^2}{a^2} = 1$
- Line: $\dfrac{x - x_0}{A} = \dfrac{y - y_0}{B} = \dfrac{z - z_0}{C}$ or
  $l(t) = (A, B, C)t + (x_0, y_0, z_0)$
- Plane: $Ax + By + Cz + D = 0$
- Sphere: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$
- Ellipsoid:
  $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} + \dfrac{(z - z_0)^2}{c^2} = 1$
- Cone: $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

- (Elliptic) Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- Cylinder: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (in $\mathbb{R}^3$)
- (Parabolic) Cylinder: $y = ax^2 + bx + c$ (in $\mathbb{R}^3$)
- Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperboloid of Two Sheets: $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Hyperbolic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
- Lemniscate: $r^2 = a^2 \cos 2\theta$ or $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. The area of one loop is a^2
- Cardioid: $r = 2a(1 + \cos\theta)$ with area $6\pi a^2$
- $2n$-Leaved Rose: $r = a\cos(n\theta)$
- $2n$-Leaved Rose: $r = a\sin(n\theta)$

A Few Points: The slope of a linear function, m, is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, where $(x_1, y_1), (x_2, y_2)$ are two distinct points on the line. The y-intercept on the line is b while the equation has x-intercept $x = -\frac{b}{m}$. Lines of the form $x = c$ are vertical lines and have no y-intercept (unless $x = 0$). The parabola $f(x) = ax^2 + bx + c$ opens upwards if $a > 0$ and downwards if $a < 0$. The parabola has vertex (also called the turning point), which is the maximum/minimum of the function, at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. The hyperbola has center (x_0, y_0) with asymptotes passing through this center with slope $\pm b/a$ (where a is always matched with x and b is always matched with y).

Distance between Two Points: The distance between two points, (x_1, y_1), (x_2, y_2), in the plane is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Generally, the distance between two points, $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, in $\mathbb{R}^n$ is given by

$$d = \sqrt{\sum_{n=1}^{n} (x_i - y_i)^2}$$

Graphical Transformations: Compared to the graph of $y = f(x)$, the graph of...

- $f(x) + c$ is shifted upward/downward by c.
- $f(x - c)$ is shifted left/right by c.
- $cf(x)$ is stretched/shrunk vertically by a factor of c.
- $f(cx)$ is stretched/shrunk horizontally by a factor of c.
- $-f(x)$ is reflected across the x-axis.
- $f(-x)$ is reflected across the y-axis.

Polar/Cylindrical/Cartesian Coordinates: If there is no z, e.g. if you are working in the plane, then this is simply polar coordinates.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z \end{cases}$$

Spherical/Cartesian Coordinates: ρ is the radius from the origin, θ is the angle in the plane, and ϕ is the angle off of the z-axis.

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \\ \phi = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) \end{cases}$$

Spherical/Cylindrical:

$$\begin{cases} \rho = \sqrt{r^2 + z^2} \\ \theta = \theta \\ \phi = \tan^{-1}\left(\dfrac{r}{z}\right) \end{cases} \qquad \begin{cases} r = \rho \sin\phi \\ \theta = \theta \\ z = \rho \cos\phi \end{cases}$$

Area & Volumes

- **Triangle**:
  Area $A = \dfrac{bh}{2}$
- **Equilateral Triangle**:
  Area $A = \dfrac{s^2\sqrt{3}}{4}$
- **Parallelogram**:
  Area $A = bh$
- **Trapezoid**:
  Area $A = h\left(\dfrac{\text{Base 1 + Base 2}}{2}\right)$
- **Circle**:
  Area $A = \pi r^2$
  Sector Area $A = \dfrac{\theta r^2}{2}$
  Arc Length $s = r\theta$
  Circumference $C = 2\pi r = 2d$
- **Ellipse**:
  Area $A = \pi ab$
- **Regular Cone**:
  Volume $V = h\,\dfrac{\text{Base Area}}{3}$
- **Right Circular Cone**:
  Volume $V = \dfrac{\pi r^2 h}{3}$
  Surface Area $S = \pi r\sqrt{h^2 + r^2} + \pi r^2$
- **Frustum of Right Circular Cone (Conical Frustum)**:
  Volume $V = \pi h\left(\dfrac{r^2 + rR + R^2}{3}\right)$
  Surface Area $S = \pi(r^2 + R^2) + \pi(r + R)\sqrt{h^2 + (R - r)^2}$
- **Cylinder**:
  Volume $V = \pi r^2 h$
  Surface Area $S = 2\pi rh + 2\pi r^2$
- **Sphere**:
  Volume $V = \dfrac{4}{3}\pi r^3$
  Surface Area $S = 4\pi r^2$
- **Torus**:
  Volume $V = \dfrac{\pi}{2}(R^2 - r^2) \cdot \dfrac{\pi}{2}(R - r)$
  Surface Area $S = \pi^2(R^2 - r^2)$
- **Ellipsoid**:
  Volume $V = \dfrac{4}{3}\pi abc$

Limits

Limit Properties: Suppose $\lim_{x\to a} f(x), \lim_{x\to a} g(x)$ exist, then...

- $\lim\limits_{x\to a} cf(x) = c \lim\limits_{x\to a} f(x)$
- $\lim\limits_{x\to a} f(x) \pm g(x) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} f(x)\, g(x) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$
- $\lim\limits_{x\to a} f(x)^n = \left(\lim\limits_{x\to a} f(x)\right)^n$
- $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

Special Limits:

- $\lim\limits_{n\to\infty} x^n = 0$, if $|x| < 1$
- $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$, if $p > 0$
- $\lim\limits_{n\to\infty} \sqrt[n]{x} = 1$, if $x > 0$
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim\limits_{n\to\infty} \dfrac{n^a}{(1+p)^n} = 0$, if $p > 0$
- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$
- $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$
- $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{x} = 1$
- $\lim\limits_{n\to 0} \left(\dfrac{x^n - 1}{n}\right) = \ln x$, if $x > 0$
- $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x = e$
- $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$

Limit of Rational Functions: If $p(x), q(x)$ are polynomials, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & \deg q(x) > \deg p(x) \\ \text{sign leading coefficient } p(x) \cdot \infty, & \deg p(x) > \deg q(x) \\ \dfrac{\text{leading coefficient } p(x)}{\text{leading coefficient } q(x)}, & \deg q(x) = \deg p(x) \end{cases}$$

L'Hôpital's Rule: If $f(x), g(x)$ are differentiable functions and $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \pm\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

Indeterminate Forms: $\dfrac{0}{0}$, $\pm\dfrac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0.

Derivatives

- $\frac{d}{dx}$ (constant) $= 0$
- $\frac{d}{dx}|x| = \frac{x}{|x|} = \operatorname{sgn} x,\ x \neq 0$
- $\frac{d}{dx}x^n = n\,x^{n-1}$
- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\csc x = -\csc x \cot x$
- $\frac{d}{dx}\sec x = \sec x \tan x$
- $\frac{d}{dx}\cot x = -\csc^2 x$
- $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
- $\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$
- $\frac{d}{dx}\ln x = \frac{1}{x}$
- $\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$
- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}a^x = a^x \ln a$
- $\frac{d}{dx}\sinh x = \cosh x$
- $\frac{d}{dx}\cosh x = \sinh x$
- $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$
- $\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$
- $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$
- $\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$
- $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$
- $\frac{d}{dx}\cosh^{-1} x = \frac{\operatorname{sgn}\cosh^{-1} x}{x^2-1}$
- $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$
- $\frac{d}{dx}\operatorname{csch}^{-1} x = \frac{-1}{|x|\sqrt{1+x^2}}$
- $\frac{d}{dx}\operatorname{sech}^{-1} x = \frac{-\operatorname{sgn}\operatorname{sech}^{-1} x}{x\sqrt{1-x^2}}$
- $\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2}$

Constant Rule: $\frac{d}{dx}(cf(x)) = cf'(x)$

Sum Rule: $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

Product Rule: $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$

Leibniz Rule: Let $f^{(n)}$ denote the nth derivative of f, then

$$(fg)^{(n)} = \sum_{m=0}^{n} \binom{n}{m} f^{(m)} g^{(n-m)}$$

Mean Value Theorem: If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a point $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

Newton's Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Clairaut's Theorem: If f_{xy} and f_{yx} are continuous on an open set, then $f_{xy} = f_{yx}$.

Chain Rule: If $z = f(x,y)$ is differentiable and $x = x(t)$, $y = y(t)$ are differentiable, then $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

Total Differential: If $z = f(x,y)$, then

$$dz = df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Directional Derivative: The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of the unit vector $\mathbf{u} = \langle u_x, u_y \rangle$ is

$$D_{\mathbf{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_x, y_0 + hu_y) - f(x_0,y_0)}{h}$$

Since $D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$, the maximum value of the directional derivative is $|\nabla f(x)|$, and occurs when $\mathbf{u}$ and ∇f have the same direction.

Tangent Plane: $f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$, i.e. $\nabla f(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0}) = 0$

Normal Line: $\dfrac{x-x_0}{f_x(x_0,y_0,z_0)} = \dfrac{y-y_0}{f_y(x_0,y_0,z_0)} = \dfrac{z-z_0}{f_z(x_0,y_0,z_0)}$

Numerical Differentiation

Central-Difference Order, $O(h^2)$:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$
$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$
$$f^{(3)}(x) \approx \frac{f(x+2h) - f(x+h) + 2f(x-h) - f(x-2h)}{2h^3}$$
$$f^{(4)}(x) \approx \frac{f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)}{h^4}$$

Central-Difference Order, $O(h^4)$:

$$f'(x) \approx \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$$
$$f''(x) \approx \frac{-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)}{12h^2}$$

Forward/Backward Difference Order, $O(h^2)$:

$$f'(x) \approx \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$$
$$f'(x) \approx \frac{3f(x)-4f(x-h)+f(x-2h)}{2h}$$
$$f''(x) \approx \frac{2f(x)-5f(x+h)+4f(x+2h)-f(x+3h)}{h^2}$$
$$f''(x) \approx \frac{2f(x)-5f(x-h)+4f(x-2h)-f(x-3h)}{h^2}$$

Integrals

- $\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \ n \neq -1$
- $\int \frac{1}{x} \, dx = \ln|x|$
- $\int \sin x \, dx = -\cos x$
- $\int \cos x \, dx = \sin x$
- $\int \tan x \, dx = \ln|\sec x|$
- $\int \csc x \, dx = \ln|\csc x - \cot x|$
- $\int \sec x \, dx = \ln|\sec x + \tan x|$
- $\int \cot x \, dx = \ln|\sin x|$
- $\int \csc^2 x \, dx = -\cot x$
- $\int \sec^2 x \, dx = \tan x$
- $\int \sec x \tan x \, dx = \sec x$
- $\int \csc x \cot x \, dx = -\csc x$
- $\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$
- $\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$
- $\int e^x \, dx = e^x$
- $\int a^x \, dx = \frac{a^x}{\ln a}$
- $\int \ln x = x \ln x - x$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$

- $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x$
- $\int \sinh x \, dx = \cosh x$
- $\int \cosh x \, dx = \sinh x$
- $\int \tanh x \, dx = \ln \cosh x$
- $\int \sqrt{x}\, e^{ax} \, dx = \frac{1}{a}\sqrt{x} e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}(i\sqrt{ax})$
- $\int e^{ax^2} \, dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(ix\sqrt{a})$
- $\int e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}(x\sqrt{a})$
- $\int x^2 e^{-ax^2} \, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \operatorname{erf}(x\sqrt{a}) - \frac{x}{2a} e^{-ax^2}$

Note that all of the above integrals are missing the integration constant, C. This has been intentionally omitted to save space.

Integral Properties:

- $\int_a^a f(x) \, dx = 0$
- $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$
- $\int cf(x) \, dx = c \int f(x) \, dx$
- $\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
- $\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx$ if $a \le b \le c$
- $\int_a^b f(x) \, dx \ge 0$ if $f(x) \ge 0$ on $[a, b]$
- $m(b-a) \le \int_a^b f(x) \, dx \le M(b-a)$ if $m \le f(x) \le M$ on $[a, b]$
- $\left|\int_a^b f(x) \, dx\right| \le \int_a^b |f(x)| \, dx$

u-Substitution: $\int_{g(a)}^{g(b)} f(x) \, dx = \int_a^b f(g(x))\, g'(x) \, dx$

Integration by parts: $\int u \, dv = uv - \int v \, du$ or $\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$

Average Value: The average value of $f(x)$ on $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Polar Integrals: The area formed by a curve $f(\theta)$ is $\displaystyle\int_\alpha^\beta \frac{1}{2} f(\theta)^2\,d\theta$.

Mean Value Theorem for Integrals: If f is continuous on $[a, b]$, then there is a $c \in (a, b)$ such that $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a)$.

Generalized Mean Value Theorem for Integrals: If f, g are continuous on $[a, b]$ and $\operatorname{sgn} g$ is constant on $[a, b]$, then there is a $c \in (a, b)$ such that $\displaystyle\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

First Fundamental Theorem of Calculus: If F is a continuously differentiable function on $[a, b]$ such that $F'(x) = f(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Second Fundamental Theorem of Calculus: If $f(x)$ is a continuous function on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$.

Numerical Integration

Left Hand Sum, L_n:

$$h\left(f_1 + f_2 + f_3 + \cdots + f_{N-2} + f_{N-1}\right)$$

Right Hand Sum, R_n:

$$h\left(f_1 + f_2 + f_3 + \cdots + f_{N-2} + f_{N-1} + f_N\right)$$

Midpoint Rule, M_n:

$$h\left(f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + \cdots + f\left(\frac{x_{N-1} + x_N}{2}\right)\right)$$

Trapezoidal Rule, T_n:

$$\frac{h}{2}\left(f_1 + 2f_2 + 2f_3 + 2f_4 + \cdots + 2f_{N-2} + 2f_{N-1} + f_N\right)$$

Simpson's Rule, S_{2n}:

$$\frac{h}{3}\left(f_1 + 4f_2 + 2f_3 + 4f_4 + \cdots + 2f_{2N-2} + 4f_{2N-1} + f_{2N}\right)$$

Simpson's $\frac{3}{8}$-Rule:

$$\frac{3h}{8}\left(f_1 + 3f_2 + 3f_3 + \cdots + 3f_{N-1} + f_N\right)$$

Composite Boole's Rule:

$$\frac{2h}{45}\left(7f_1 + 32f_2 + 12f_3 + 32f_4 + 7f_5 + 32f_6 + 12f_7 + 32f_8 + \cdots + 7f_{4N-3} + 32f_{4N-2} + 12f_{4N-1} + 32f_{4N}\right)$$

Note: $T_n = \dfrac{L_n + R_n}{2}$ and $S_{2n} = \dfrac{2M_n + T_n}{3}$.

Series

Arithmetic Series: $a+(a+d)+(a+2d)+\cdots+(a+(n-1)d)=\dfrac{n(2a+(n-1)d)}{2}=n\cdot\dfrac{a+\text{last term}}{2}$

Geometric Series: $a+ar+ar^2+\cdots+ar^n=\dfrac{a(1-r^{n+1})}{1-r}$; that is, $\displaystyle\sum_{n=0}^{\infty}ar^n=\frac{a}{1-r} \quad |r|<1$

A Few Special Series:

- $1+3+5+\cdots+(2n-1)=n^2$
- $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$
- $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$
- $1^3+2^3+3^3+\cdots+n^3=\left(\dfrac{n(n+1)}{2}\right)^2$
- $\ln 2=1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{5}-\cdots$
- $\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\dfrac{1}{9}-\cdots$
- $\dfrac{\pi^2}{6}=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\cdots$
- $\dfrac{\pi^4}{90}=1+\dfrac{1}{2^4}+\dfrac{1}{3^4}+\dfrac{1}{4^4}+\cdots$
- $\dfrac{\pi^2}{12}=1-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\cdots$
- $\dfrac{\pi^2}{8}=1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\cdots$

To find higher sums of powers, $S_n(m)=\displaystyle\sum_{i=1}^{m}i^n$, use

$$S_n(m)=\sum_{i=1}^{m}i^n=\frac{(m+1)^{n+1}}{n+1}-\frac{1}{n+1}\sum_{j=0}^{n-1}\binom{n+1}{j}S_j(m)$$

Taylor's Theorem: If $f(x)$ is a n-times differentiable function at $x=a$, then

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x),$$

where $\lim_{x\to a}h_n(x)=0$. The function $R_n(x)$ is called the (Peano) remainder term. This can be given explicitly if f is $(n+1)$-times differentiable and $f^{(n)}$ is continuous on a interval (n,m) containing a as, for some c between a and x, we have

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Maclaurin Series:

- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1$
- $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$
- $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} \quad |x| \leq 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad |x| < 1$
- $\dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{2n+1} \quad |x| < 1$
- $\sinh x = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$
- $\cosh x = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$

Fourier Series: If $f(x)$ is defined on $[c, c+2L]$, $f(x)$, $f'(x)$ are piecewise continuous, and $f(x)$ has the property that $f(x) = f(x+2L)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

where a_n and b_n are given by

$$a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$$

$$b_n = \frac{1}{L} \int_c^{c+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

for each x where $f(x)$ is continuous. If x is a point of discontinuity of x, then the series converges to $\dfrac{\lim_{x\to 0+} f(x) + \lim_{x\to 0-} f(x)}{2}$.

Arc Length, Volumes, Surfaces of Revolution, & Surface Area

Arc Length: If a curve, C, is parametrized by $(x(t), y(t))$ for $a \leq t \leq b$, then $l = \int_C ds = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} \, dt$. If the curve C is given by $y = f(x)$, where $f(x)$ is differentiable, then $\int_a^b \sqrt{1 + f'(x)^2} \, dx$. The arc length of a polar curve $f(\theta)$ is $L = \int_a^b \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} \, d\theta$.

Washer Method: The volume of revolution resulting from revolving a region with outer radius R and inner radius r about the x-axis is $V = \pi \int_a^b R(x)^2 - r(x)^2 \, dx$. If $r = 0$ then this is the Disk Method.

Shell Method: The volume of revolution resulting from revolving a region formed by a function $f(x)$ and the x-axis about the y-axis is $V = 2\pi \int_a^b x f(x) \, dx$.

Surface Area: The surface area resulting from rotating the region formed by a function $f(x)$ and the x-axis about the y-axis is $2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$.

Dot & Cross Products

- $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$
- $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| \, |\mathbf{y}| \cos\theta$, where θ is the angle between $\mathbf{x}, \mathbf{y}$ and $0 \le \theta \le \pi$
- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
- $\mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2$
- $\mathbf{x} \cdot \mathbf{y} = 0$ if and only if $\mathbf{x}, \mathbf{y}$ are orthogonal.
- $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$
- $\mathbf{proj}_{\mathbf{y}} \, \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|^2} \, \mathbf{y}$
- $\mathbf{x} \times \mathbf{y} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$; $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$
- $|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}| \, |\mathbf{y}| \sin\theta$ This is the area of the parallelogram spanned by $\mathbf{x}, \mathbf{y}$
- $\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x})$
- $\mathbf{x} \times \mathbf{y} = \mathbf{0}$ if and only if $\mathbf{x}, \mathbf{y}$ are parallel.
- $\mathbf{x} \times \mathbf{y}$ is orthogonal to $\mathbf{x}, \mathbf{y}$.
- $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$
- $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$
- $|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$ = volume parallelepiped spanned by $\mathbf{x}, \mathbf{y}, \mathbf{z}$
- $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{y}(\mathbf{x} \cdot \mathbf{z}) - \mathbf{z}(\mathbf{x} \cdot \mathbf{y})$
- $(\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{z} \times \mathbf{w}) = (\mathbf{x} \cdot \mathbf{z})(\mathbf{y} \cdot \mathbf{w}) - (\mathbf{x} \cdot \mathbf{w})(\mathbf{y} \cdot \mathbf{z})$
- $\nabla \cdot (\mathbf{x} + \mathbf{y}) = \nabla \cdot \mathbf{x} + \nabla \cdot \mathbf{y}$
- $\nabla \times (\mathbf{x} + \mathbf{y}) = \nabla \times \mathbf{x} + \nabla \times \mathbf{y}$
- $\nabla \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\nabla \times \mathbf{x}) - \mathbf{x} \cdot (\nabla \times \mathbf{y})$
- Laplacian: $\nabla^2 = \nabla \cdot (\nabla f)$ $= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
- $\nabla \times (\nabla f) = 0$
- $\nabla \cdot (\nabla \times f) = 0$
- $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f$

Vector Calculus

Arclength: $L = \int_C ds = \int_a^b \|\mathbf{r}'(t)\|\, dt$

Unit Tangent Vector: $\mathbf{T} = \dfrac{\mathbf{r}(t)}{\|\mathbf{r}'(t)\|}$

Curvature: $\kappa(t) = \dfrac{\|d\mathbf{T}/dt\|}{ds/dt} = \left\|\dfrac{d\mathbf{T}}{ds}\right\|$

Principle Normal Vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$

Binormal Vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, note that we have $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$, where τ is the torsion of the path.

Note: If s' denotes ds/dt, then $\mathbf{v}(t) = s'\mathbf{T}$, $\mathbf{a}(t) = s''\mathbf{T} + \kappa s'^2\mathbf{N}$, and $\kappa = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$. Furthermore, if the function is given by $f(x)$, then we have $\kappa(x) = \dfrac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$.

Further Note: ∇f is the gradient of f (also written grad f) $\nabla \cdot f$ is the divergence of f (also written div f), and $\nabla \times f$ is the curl of f (also written curl f). A continuously differentiable function f such that $\nabla^2 f = 0$ is called a harmonic function.

Hessian: The Hessian of a function in $\mathbb{R}^n$ is

$$Hf = \begin{pmatrix} f_{x_1x_2} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

Then if f is twice continuously differentiable at $\mathbf{a}$ is a critical point on an open region where f is defined, we have

$$\begin{cases} Hf(\mathbf{a}) \text{ positive definite,} & \mathbf{a} \text{ local min} \\ Hf(\mathbf{a}) \text{ negative definite,} & \mathbf{a} \text{ local max} \\ \det Hf(\mathbf{a}) \neq 0 \text{ neither of above,} & \mathbf{a} \text{ saddle} \end{cases}$$

Higher Taylor Series: If f is $k+1$-times continuously differentiable, then the kth order Taylor Polynomial at $x = \mathbf{a}$ is

$$\begin{aligned} T_k(\mathbf{x}) =& f(\mathbf{a}) + \sum_{i=1}^n f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2}\sum_{i,j}^n f_{x_ix_j}(\mathbf{a})(x_i - a_i)(x_j - a_j) \\ &+ \cdots + \frac{1}{k!}\sum_{i_1,\cdots,i_k=1}^n f_{x_{i_1}\cdots x_{i_k}}(\mathbf{a})(x_{i_1} - a_{i_1})\cdots(x_{i_k} - a_{i_k}) \end{aligned}$$

with remainder term

$$R_k(\mathbf{x}, \mathbf{a}) = \frac{1}{(k+1)!}\sum_{i_1,\cdots,i_{k+1}}^n f_{x_{i_1}\cdots x_{i_{k+1}}}(\mathbf{z})h_{i_1}h_{i_2}\cdots h_{i_{k+1}}$$

for a suitable point $\mathbf{z}$ on the line segment joining $\mathbf{a}$ and $\mathbf{x} = \mathbf{a} + \mathbf{h}$.

Lagrange Multiplier: If X is open in $\mathbb{R}^n$ and $f, g_1, g_2, \cdots, g_k : X \to \mathbb{R}$ are continuously differentiable functions with $k < n$. Let $S = \{\mathbf{x} \in X \mid g_1(\mathbf{x}) = c_1, \cdots, g_k(\mathbf{x}) = c_k\}$. If $f|_S$ has an extremum at $\mathbf{x}_0$, where $\nabla g_1(\mathbf{x}_0), \cdots, \nabla g_k(\mathbf{x}_0)$ are linearly independent vectors, then there exist scalars $\lambda_1, \cdots, \lambda_k$ such that

$$\nabla f(\mathbf{x}_0) = \sum_{i=1}^{k} \lambda_i \nabla g_i(\mathbf{x}_0)$$

In the single constraint case, this is $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$.

Jacobian & Change of Variables: The Jacobian of a transformation $\mathbf{T}(u, v) = (x(u, v), y(u, v))$ is $D\mathbf{T}(u, v)$, i.e.

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Then if D, D^* are elementary regions in the xy-plane and uv-plane, $\mathbf{T}$ is a continuously differentiable transformation taking D^* bijectively to D and f is an integrable function on D, then

$$\iint_D f(x, y)\, dA = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA.$$

For Cartesian to Cylindrical Coordinates, this is $dA = r\ dr\ d\theta\ dz$ and for Cartesian to Spherical Coordinates, this is $dA = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$.

Vector/Scalar Line Integral: If $\mathbf{r}$ is a continuously differentiable path, $\mathbf{F}$ is a vector field, and f is a function. Then the scalar line integral is given by $\displaystyle\int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\|\ dt$ and the vector line integral is $\displaystyle\int_r \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt$. Note that if $-\mathbf{r}(t)$ denotes the path $\mathbf{r}(t)$ traversed in reverse, then we have $\displaystyle\int_{-\mathbf{r}} = -\int_{\mathbf{r}}$.

Path Independence: A continuous vector field $\mathbf{F}$ has path-independent line integrals if the line integral beginning at one point and ending at another is independent of the (simple, piecewise continuously differentiable) path connecting them for all pairs of points in $\mathbf{F}$. If $\mathbf{F}$ is a continuos vector field, then $\mathbf{F}$ has path-independent line integrals if and only if $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for all simple, closed, piecewise continuously differentiable curves in $\mathbf{F}$. If $\mathbf{F}$ is continuous on a connected open region of $\mathbb{R}^n$, then $\mathbf{F} = \nabla f$, where f is some continuously differentiable function, if and only if $\mathbf{F}$ has path-independent line integrals. If $\mathbf{F}$ is a vector field that is continuously differentiable and defined on a simply connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\mathbf{F} = \nabla f$, where f is twice-continuously differentiable, if and only if $\nabla \times \mathbf{F} = \mathbf{0}$ for all points in the region. The function f is called the potential function for $\mathbf{F}$ and call $\mathbf{F}$ a conservative vector field.

Surface Area: If $X(s,t)$ is a continuously differentiable map, then the standard normal vector is $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$. Then the surface area is given by $\displaystyle\iint_D \|\mathbf{T}_s \times \mathbf{T}_t\| \, dA$. This can also be given by

$$\iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2} \, ds \, dt$$

If the surface is given by $z = f(x,y)$, then this is $\iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA$.

Surface Integrals: If $\mathbf{X}(s,t)$ is a smooth parametrized surface and f is a continuous function, then the scalar surface integral is $\displaystyle\iint_{\mathbf{X}} f \, dS = \iint_D f(\mathbf{X}(s,t)) \|\mathbf{T}_s \times \mathbf{T}_t\| \, ds \, dt = \iint_D f(\mathbf{X}(s,t)) \|\mathbf{N}(s,t)\| \, ds \, dt$. The vector surface integral is $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t) \, ds \, dt$.

Green's Theorem: Let D be a closed, bounded region in $\mathbb{R}^2$ whose boundary $C = \partial D$ consists of finitely many simple, closed curves. Orient the curves of C so that D is on the left as one traverses C. Let $\mathbf{F}(x,y) = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ be a vector field of class C^1 throughout D. Then

$$\oint_C M \, dx + N \, dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Equivalently, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot k \, dA$$

Gauss' Theorem: Let D be a bounded solid region in $\mathbb{R}^3$ whose boundary ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals that point away from D. Let $\mathbf{F}$ be a vector field of class C^1 whose domain includes D. Then

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

Stokes' Theorem: Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$. Suppose that ∂S consists of finitely many piecewise C^1, simple, closed curves each of which is oriented consistently with S. Let $\mathbf{F}$ be a vector field of class C^1 whose domain includes S. Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Green's Formulas If f, g are twice-continuously differentiable scalar fields and D has a piecewise smooth boundary, S, oriented as in Gauss' Theorem, and $\mathbf{r}$ is inside D, then

$$\iiint_D \nabla f \cdot \nabla g \, dV + \iiint_D f \nabla^2 g \, dV = \oiint_S f \nabla g \cdot d\mathbf{S}$$

$$\iiint_D (f \nabla^2 g - g \nabla^2 f) \, dV = \oiint_S (f \nabla g - g \nabla f) \cdot d\mathbf{S}$$

$$f(\mathbf{r}) = \frac{-1}{4\pi}\iiint_D \frac{\nabla^2 f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\,dV + \frac{1}{4\pi}\oiint_S \left(-f(\mathbf{x})\nabla\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) + \frac{\nabla f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\right)\cdot d\mathbf{S}$$

Generalized Stokes' Theorem: Let $D \subseteq \mathbb{R}^k$ be a closed, bounded, connected region, and let $M = \mathbf{X}(D)$ be an oriented, parametrized k-manifold in $\mathbb{R}^n$. If $\partial M \neq \emptyset$, let ∂M be given the orientation induced from that of M. Let ω denote a $(k-1)$-form defined on an open set in $\mathbb{R}^n$ that contains M. Then

$$\int_M d\omega = \int_{\partial M} \omega$$

If $\partial M = \emptyset$, then we take $\int_{\partial M} \omega$ to be 0.

Complex Variables

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$
- $(x + yi) \pm (a + bi) = (x \pm a) + (y \pm b)i$
- $(x + yi)(a + bi) = (xa - yb) + (xb + ya)i$
- $\dfrac{a+ib}{c+id} = \dfrac{ac+bd}{c^2+d^2} + i\dfrac{bc-ad}{c^2+d^2}$
- $(x + yi)(x - yi) = x^2 + y^2$
- Complex Conjugate: $\overline{x + yi} = x - yi$
- Modulus: $|x + iy| = \sqrt{x^2 + y^2}$
- $|x + iy|^2 = (x + yi)(\overline{x - yi}) = x^2 + y^2$
- Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$
- $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$
- De Moivre's Theorem: $\left(re^{i\theta}\right)^n = r^n e^{in\theta}$
- $\left(re^{i\theta}\right)^{1/n} = r^{1/n} e^{i(\theta+2k\pi)/n}$
- $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$
- $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$
- $\tan\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)}$
- $\sin(ix) = i\sinh x$
- $\cos(ix) = \cosh x$
- $\tan(ix) = i\tanh x$
- $\ln\left(\left(re^{i\theta}\right) = \ln|r| + i(\theta + 2k\pi)\right)$

Special Functions

Gamma Function:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt \quad n > 0$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

To evaluate $\Gamma(n)$ for $n < 0$, use the relation $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$. Note that for $n = 0, 1, 2, \ldots$, we have $\Gamma(n+1) = n!$

Riemann Zeta Function:

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{u^{x-1}}{e^{u-1}}\, du \quad x > 1$$

$$\zeta(1-x) = 2^{1-x} \pi^{-x} \Gamma(x) \cos(\pi x/2) \zeta(x)$$

Beta Function: $B(m,n) = \displaystyle\int_0^1 t^{m-1}(1-t)^{n-1}\, dt; \quad m, n > 0$

$B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Error Function: $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-u^2}\, du$

Exponential Integral: $\operatorname{Ei}(x) = -\displaystyle\int_{-x}^{\infty} \frac{e^{-u}}{u}\, du$

Sine Integral: $\operatorname{Si}(x) = \displaystyle\int_0^x \frac{\sin u}{u}\, du$

Fresnel Sine Integral: $S(x) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^x \sin u^2\, du$

Legendre Polynomials: $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}(x^2-1)^n$

Hermite Polynomials: $H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n}\left(e^{-x^2}\right)$

Laguerre Polynomials: $L_n(x) = e^x \dfrac{d^n}{dx^n}\left(x^n e^{-x}\right)$

Chebyshev Polynomials of the First Kind: $T_n(x) = \cos(n \cos^{-1} x) = x^n - \binom{n}{2} x^{n-2}(1-x^2) + \binom{n}{4} x^{n-4}(1-x^2)^2 - \cdots$

Laplace Transform: $\mathscr{L}\{F(t)\} = \displaystyle\int_0^{\infty} e^{-st} F(t)\, dt = f(s)$

Fourier Transfrom: $\mathscr{F}\{f(x)\} = F(\alpha) = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx$

Incomplete Elliptic Integral of the First Kind: $F(k,\phi) = \displaystyle\int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^x \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$

Complete Elliptic Integral of the First Kind: $F(k,\pi/2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$

Incomplete Elliptic Integral of the Second Kind: $E(k,\phi) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^x \sqrt{\frac{1-k^2v^2}{1-v^2}}\, dv$

Complete Integral of the Second Kind: $E(k,\pi/2) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^1 \sqrt{\frac{1-k^2v^2}{1-v^2}}\, dv$

Incomplete Elliptic Integral of the Third Kind:
$\Pi(k,n,\phi) = \int_0^{\phi} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \int_0^x \frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}$

Complete Elliptic Integral of the Third Kind: $\Pi(k,n,\pi/2) = \int_0^{\pi/2} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}$

Inequalities

Triangle Inequality: $||x|-|y|| \le |x+y| \le |x|+|y|$

Cauchy-Schwarz Inequality: $\left|\sum_{i=1}^n x_i\overline{y}_i\right|^2 \le \sum_{j=}^n |x_j|^2 \sum_{k=1}^n |y_k|^2$ with equality if and only if **x**, **y** are linearly dependent. In integral form, this inequality is

$$\left(\int_a^b f(x)g(x)\, dx\right)^2 \le \left(\int_a^b f(x)^2\, dx\right)\left(\int_a^b g(x)^2\, dx\right)$$

with equality if and only if $f(x), g(x)$ are linearly dependent.

Arithmetic, Geometric, Harmonic Means:

$$\frac{1}{n}\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right) \le \sqrt[n]{x_1x_2\cdots x_n} \le \frac{x_1+x_2+\cdots+x_n}{n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Hölder's Inequality: $\sum_{i=1}^n |x_ky_k| \le \left(\sum_{j=1}^n |x_j|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}$, where $\frac{1}{p}+\frac{1}{q} = 1$ and $p,q > 1$. Equality holds if and only if $\frac{|x_1|^{p-1}}{|y_1|} = \frac{|x_2|^{p-1}}{|y_2|} = \cdots = \frac{|x_n|^{p-1}}{|y_n|}$. In integral form, this inequality is

$$\int_a^b |f(x)g(x)|\, dx \le \left(\int_a^b |f(x)|^p\, dx\right)^{1/p} \left(\int_a^b |g(x)|^q\, dx\right)^{1/q}$$

Equality holds if and only if $|f|, |g|$ are linearly dependent.

Chebyshev's Inequality: $x_n \le x_{n-1} \le \cdots \le x_1$ and $y_n \le y_{n-1} \le \cdots \le y_1$

$$\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right) \le \frac{x_1 y_1 + x_2 y_2 + \cdots + x_n y_n}{n}$$

Equivalently, we can write this as

$$(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) \le n\,(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

Minkowski's Inequality: If $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are positive and $p > 1$ then

$$\left(\sum_{i=1}^{n} |x_k + y_k|^p\right)^{1/p} \le \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}$$

with equality if and only if **x**, **y** are positively dependent. In integral form, this inequality is

$$\left(\int_a^b |f(x) + g(x)|^p \, dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p \, dx\right)^{1/p} + \left(\int_a^b |g(x)|^p \, dx\right)^{1/p}$$

Equality holds if and only if f, g are positively dependent.

Greek Alphabet

Greek Name	Greek Letter
Alpha	α, A
Beta	β, B
Gamma	γ, Γ
Delta	δ, Δ
Epsilon	ϵ, E
Zeta	ζ, Z
Eta	η, H
Theta	θ, Θ
Iota	ι, I
Kappa	κ, K
Lambda	λ, Λ
Mu	μ, M

Greek Name	Greek Letter
Nu	ν, N
Xi	ξ, Ξ
Omicron	o, O
Pi	π, Π
Rho	ρ, P
Sigma	$\sigma, \sum$
Tau	τ, T
Upsilon	υ, Υ
Phi	ϕ, Φ
Chi	χ, X
Psi	ψ, Ψ
Omega	ω, Ω

Constants

- Pi $\pi = 3.1415\ 9265\ 3589$
- Euler's Number $e = 2.7\ 1828\ 1828\ 4590$
- Euler's Constant $\gamma = 0.5772\ 1566\ 4901$
- 1 radian $= \frac{180^\circ}{\pi} \approx 57.2958^\circ$
- $1^\circ = \frac{\pi}{180^\circ} \approx 0.01745$ radians
- Speed of light: $c = 2.9979\ 2458 \cdot 10^8$ m/s
- Electron Charge: $e = 1.6021\ 7653 \cdot 10^{-19}$ C
- Electron Mass: $m = 9.1093\ 823 \cdot 10^{-31}$ kg
- Proton Mass: $m = 1.6726\ 2171 \cdot 10^{-27}$ kg
- Neutron Mass: $m - 1.6749\ 2728 \cdot 10^{-27}$ kg
- Gravitational Acceleration (Earth): $g = 9.80665\ \text{m/s}^2$
- Gravitational Constant: $G = 6.6742 \cdot 10^{-11}$ Nm^2/kg^2
- Planck's Constant: $h = 6.6260\ 693 \cdot 10^{-34}$ Js
- Boltzmann's Constant: $k = 1.3806\ 505 \cdot 10^{-23}$ J/K
- Avogadro's Number: $N = 6.0221\ 415 \cdot 10^{23}$ molecules/mole
- Gas constant: $R = 8.3144\ 72$ J/mol· K
- Volume of Ideal Gas: 22.4139 96 L/mol
- Permeability of Free Space: $\mu_0 = 4\pi \cdot 10^{-7}$ Wb/A· m
- Permittivity of Free Space: $\epsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541\ 8781\ \text{C}^2/\text{N}{\cdot}\text{m}^2$

Body	Mass, kg	Radius, m	Orbit Radius, m	Orbit Period
Sun	$1.99 \cdot 10^{30}$	$6.96 \cdot 10^8$	—	—
Moon	$7.35 \cdot 10^{22}$	$1.71 \cdot 10^6$	$3.84 \cdot 10^8$	27.3 days
Earth	$5.97 \cdot 10^{24}$	$6.38 \cdot 10^6$	$1.50 \cdot 10^{11}$	365.3 days
Mars	$6.42 \cdot 10^{23}$	$3.40 \cdot 10^6$	$2.28 \cdot 10^{11}$	687.0 days
Jupiter	$1.90 \cdot 10^{27}$	$6.91 \cdot 10^7$	$7.78 \cdot 10^{11}$	11.86 years

Conversions

- 1 inch = 2.540 cm
- 1 cm = 0.3937 in
- 1in = 2.540 cm
- 1 yard = 91.44 cm
- 1 km = 0.6214 miles
- 1 mile = 1.609 km
- 1 mile = 5280 feet
- Å$= 10^{-10}$ m
- 1 $\text{m}^2 = 10.76\ \text{ft}^2$
- 1 L = 1000 cm^3 = 61.02 in^3
- 1 gallon = 3.788 liters
- 1 year = 365.24 days

- 1 km/hr = 0.6214 mi/hr
- 1 mile/hour = 1.609 km/h
- 1 $\text{m/s}^2 = 3.281\ \text{ft/s}^2$
- 1 $\text{ft/s}^2 = 0.3048\ \text{m/s}^2$
- 1 mile/hour $\cdot$ s = 1.467 ft/s^2
- 1 kg = 2.2046 lb = 0.06852 slugs
- 1 newton = 0.2248 lb
- 1 Pa = 1 $\text{N/m}^2 = 0.2089\ \text{lb/ft}^2$
- 1lb/ft^2 = 47.88 Pa
- 1 bar = 10^5 Pa
- 1 mmHg = 133.3 Pa
- 1 atm = $1.013 \cdot 10^5\ \text{N/m}^2$ = 760 mmHg
- 1 J = 0.239 calorie
- 1ft·lb = 1.356 J
- 1 calorie = 4.186 J
- 1 electron volt = $1.602 \cdot 10^{-19}$ J
- 1 watt = 1 J/s = 0.2389 cal/sec
- 1 Btu/h = 0.293 W

References

[1] John D'Angelo and Douglas West, *Mathematical Thinking: Problem-solving and Proofs*. Upper Saddle River, NJ: Prentice Hall, 2000.

[2] Supreme Aryal, *Unit Circle*, 16 March 2010. http://www.texample.net/tikz/examples/unit-circle/ (Accessed 6 May 2016). License: Creative Commons BY-2.5.

[3] Susan Jane Colley, *Vector Calculus*. Third Edition. Pearson Education Inc., Upper Saddle River, NJ, 2006.

[4] Walter Rudin, *Principles of Mathematical Analysis*. International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1964.

[5] Murray Spiegel and Seymour Lipschutz and John Liu, *Mathematical Handbook of Formulas and Tables*. Third Edition. McGraw Hill, New York, 2009.

[6] James Stewart, *Essential Calculus: Early Transcendentals*. Second Edition. Brooks/Cole, Belmont, CA, 2013.

[7] Hugh Young and Roger Freedman, *University Physics*. Twelfth Edition. Pearson Education, San Francisco, CA, 2008.