MAT 702: Functional Analysis

Hilbert Spaces

*All vector spaces are over $\mathbb{C}$ unless otherwise stated.

Inner product $\langle x,y\rangle$

lin in x, conj lin. in y

$\langle x,y\rangle = \overline{\langle y,x\rangle}$

$\langle x,x\rangle \geq 0$ & $=0$ iff $x=0$

Ex: $\langle x,y\rangle = \sum \underbrace{x(k)\overline{y(k)}}_{\text{our notation for } x_k,\, \overline{y_k}}$

Elements of $\mathbb{C}^n$ are functions from $\{1,\dots,n\}$ to $\mathbb{C}$.

Ex: ℓ^2: Functions from $\mathbb{N}$ to $\mathbb{C}$

inner product $\langle x,y\rangle = \sum x(k)\overline{y(k)}$

$\sum |x(k)|^2 < \infty$. Need check for convergence.

$a,b\in\mathbb{R} \to$ $\underbrace{|ab| \leq \frac{(a+b)^2}{2}}$

$a^2 \pm 2ab + b^2 = (a\pm b)^2 \geq 0$

So $|x(k)\overline{y}(k)| \leq \frac{|x|^2 + |y|^2}{2}$ (of k, of k). Done

Some important elements in ℓ^2: $e_n(k) = \begin{cases} 1, & k=n \\ 0, & k\neq n \end{cases}$

the standard basis vectors for $\mathbb{R}^\infty$.

These are pairwise orthogonal:

$\langle e_n(k), e_m(k)\rangle = 0$

$x(k) = 1/k \in \ell^2$ but

$x(k) = 1/\sqrt{k} \notin \ell^2$.

Why square? To control products?
$\updownarrow$
Why? To control sums?

Inner product gives a norm in the usual way. So then we have a metric.

Norm: $\|x\| = \sqrt{\langle x,x\rangle}$

Metric: $d(x,y) = \|x-y\|$

Then we can talk about convergence of sequences.

Def: $\mathcal{H}$ hilbert space if it is a complete inner product space.

Ex: C_{00} = functions from $\mathbb{N}$ to $\mathbb{C}$ with finite support. This is not complete: $x_n(k) = \begin{cases} 1/k, & k\leq n \\ 0, & k>n \end{cases}$

$x_n \in C_{00}$ but goes to harmonic which is not in C_{00}.

* If a Cauchy sequence doesn't converge it not the fault of the sequence. Fault of the space. It doesn't contain the point the sequence is working towards.

Thm: ℓ^2 is complete

Needs Fatou's Lemma & Dominated Conv. Thm

Thm: Normed space is complete iff ab. conv. series converge

Pf: $\Rightarrow$: HW1

$\Leftarrow$: Assume $\sum_1^\infty \|x_n\|$ conv. then $\sum_1^\infty x_n$ conv. Suppose $\{y_n\}$ is a Cauchy seq. Choose N_i so that $\|y_n - y_m\| < 1/2^i$. Gets a subsequence $\{y_{N_i}\}$, where
$\|y_{N_i} - y_{N_{i+1}}\| < 1/2^i$

Let $x_j = y_{N_j} - y_{N_{j+1}}$

Then $\sum_1^\infty \|x_j\| < \sum_1^\infty 1/2^j < \infty$

So $\sum x_j$ exists. So $\sum_1^\infty x_j$ has partial sums $y_{N_1} - y_{N_2} + y_{N_2} - y_{N_3} + \cdots$

So $\{y_{N_j}\}$ has a limit. But $\{y_n\}$ Cauchy w/ conv. subseq. so $\{y_n\}$ has a limit.

Orthogonality

If $\langle x,y\rangle = 0$, we say x & y are orthogonal, $x \perp y$,

For a set A in a hilbert space $\mathcal{H}$, let

$$A^\perp = \{x \mid \langle x,a\rangle = 0 \ \forall a \in A\}$$

Called the orthogonal complement of A. Note $0 \in A^\perp$ always.

Suppose $A = \mathcal{H}$. Then

$A^\perp = \{0\}$ as $x \neq 0 \rightarrow \langle x,x\rangle > 0$

If $x,y \in A^\perp$ then $x+y \in A^\perp$:

$$\langle x+y, a\rangle = \langle x,a\rangle + \langle y,a\rangle$$

Term: linear manifold is a subset of $\mathcal{H}$ that is a linear subspace in the algebraic sense.

linear subspace = closed linear manifold.

Claim $A^\perp$ closed so lin. subspace.:

take $x_n \in A^\perp$ with $x_n \to x$.

WTS $x \in A^\perp$, ie $\langle x,a\rangle = 0$

$\langle x,a\rangle = \langle x - x_n + x_n, a\rangle = \langle x - x_n, a\rangle + \overset{0}{\cancel{\langle x_n,a\rangle}}$

$\leq \|x - x_n\| \cdot \|a\| \to 0$

Cauchy Schwartz

Ex: Not every lin. manifold is closed.

C_{00} is a lin. manifold which is not closed in ℓ^2. In fact, $\overline{C_{00}} = \ell^2$ so dense in ℓ^2.

C_{00} clearly lin. manifold. Now take $x \in \ell^2$. Now $x_n(k) = \begin{cases} x(k), & k \le n \\ 0, & k > n \end{cases}$

$x_n \in C_{00}$: $\|x - x_n\|$ is measure of tail of x, which converges.

Pythagorean Thm:

If $x \perp y$ then $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$

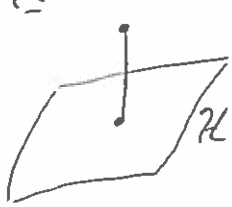
just look @ $\langle x+y, x+y \rangle$

Parallelogram Law:

$x, y \in \mathcal{H}$

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Orthogonal Projection onto a lin. subspace $\mathcal{H}$:

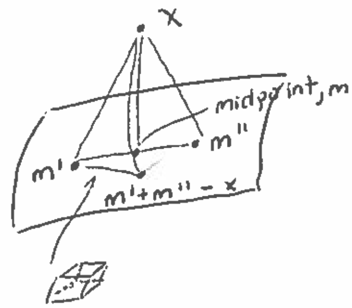


$P_{\mathcal{H}}x$ defined as $m \in \mathcal{H}$ such that $\|x - m\|$ smallest.

$d = \inf_{m \in \mathcal{H}} \|x - m\|$

$\forall n, \exists\, m_n \in \mathcal{H} \ni \|x - m_n\| < d + 1/n$

Need to show $m_n \to m$.



$m = \frac{1}{2}(m' + m'')$

$4d^2 \approx \underline{4\|x - m\|^2} + \|m' - m''\|^2$

$< 2\|x - m'\|^2 + 2\|x - m''\|^2$

$\approx 4d^2$

So $\|m' - m''\|$ 'small'

Cauchy so convergent $m_n \to m$.

$\ast$ $(M \cup N)^\perp \subseteq (M \cap N)^\perp$

$M^\perp \cup N^\perp \subseteq (M \cap N)^\perp$

$(M \cup N)^\perp = M^\perp \cap N^\perp$

Claim: $x - P_M x \in M^\perp$

Indeed, $\forall v \in M$ function

$\|x - P_M x + tv\|^2$; $t \in \mathbb{R}$

is a quadratic function

$\|x - P_M x\|^2 + 2t\,\mathrm{Re}\langle x - P_M x, v\rangle + t^2\|v\|^2$

has min at $t = 0$

Hence $\operatorname{Re}\langle x - P_M x, v\rangle = 0$

for all $v \in M$. Apply to

iv:

$$\operatorname{Re}\langle x - P_M x, iv\rangle = 0$$

then $\operatorname{Im}\langle x - P_M x, v\rangle = 0$

So $x - P_M x \perp v$

So $\forall x \in \mathcal{H}$, $\exists x' \in M, x'' \in M^\perp$

such that $x = x' + x''$

This element is unique: $x = y' + y''$

Then $\underbrace{x' - y'}_{\in M} = \underbrace{y'' - x''}_{\in M^\perp}$

must be 0.

So $P_M x$ is unique closed element of $M \ni x - P_M x \in M^\perp$

$P_M : \mathcal{H} \to M$ is linear:

$$P_M(x+y) \stackrel{?}{=} P_M x + P_M y$$

$$x + y - (P_M x + P_M y) \in M^\perp$$

&

$$P_M x + P_M y \in M$$

Hence it is in $P_M(x+y)$

To prove linear manifold is dense, suffices to show $A^\perp = \{0\}$. If A not dense:

$M = \bar{A}$, M closed lin. subspace not all of $\mathcal{H}$. Let $x \in \mathcal{H} \setminus M$

then $x - P_M x \in M^\perp$. Then

$x - P_M x \in A^\perp$, $\Rightarrow\!\Leftarrow$

Ex: e_i standard basis vector w/ $1 \in i^{th}$ spot

$S = \{e_i\}_{i \geq 1}$ has trivial $\perp$ but not dense.

$$L^2(X, \mu) = \{f \mid \int |f|^2 \, d\mu < \infty\} \;\; = \text{a.e.}$$

$$L^2([0,1]) = \{f : [0,1] \to \mathbb{C} \mid \int_0^1 |f|^2 < \infty\}$$

Lebesgue integral

Need for completeness

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\, dx$$

(well defined)

Examples of orthogonal functions:

$\{\sin nx, \cos nx\}_{n=1}$ are orthogonal in $\mathcal{L}^2([-\pi,\pi])$. Verify directly

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0$$

if $n \neq m$.

Legendre poly: $P_n(x)$

orth. on $\mathcal{L}^2([-1,1])$

$$\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0$$

$$M = \{ f \in \mathcal{L}^2([-1,1]) \mid f(x) = f(-x) \overset{\text{a.e.}}{} \}$$

lin. subspace is clear.

But why is M closed. Express cond. as...

$$\int_0^1 |f(x) - f(-x)| \, dx = 0$$

So $f_n \to f$ in $\mathcal{L}^2$.

$$\int_0^1 |f_n(x) - f_n(-x)| dx \to \int_0^1 |f(x) - f(-x)| dx$$

Try $f = f_n + g$; $\|g\|_{\mathcal{L}^2} < \epsilon$

What is orthogonal complement?

$\mathcal{O}$ $\{ f \in \mathcal{L}^2[-1,1] \mid f(x) = -f(-x) \text{ a.e.} \}$

f even, g odd $\rightarrow \int_{-1}^{1} fg = 0$

So $\mathcal{O} \subseteq M^\perp$.

$$\underbrace{P_M f}_{\text{Take } f \text{ and make 'it' even}} = \overbrace{\frac{f(x) + f(-x)}{2}}^{\text{even}}$$

$\checkmark$ $P_M f \in M$

$\checkmark$ $x - P_M f = f(x) - \dfrac{f(x) + f(-x)}{2}$

$$= \left.\frac{f(x) - f(-x)}{2}\right\} \text{odd}$$

$\in M^\perp$

So $M^\perp = \{ f(x) + f(-x) = 0 \text{ a.e.} \} =$ odd funct. a.e.

Generally,

$$M^\perp = \{ x \mid P_M x = 0 \}$$

because $\forall x \in \mathcal{H}$ $\exists!$

$$x = \underbrace{\underset{\in M}{x'}}_{P_M x} + \underset{\in M^\perp}{x''}$$

if $x \in M^\perp$; $x'' = x$, $x' = 0$ ($\leftarrow P_M x$)

Linear Functionals & Riesz Representations

$\mathcal{L}: \mathcal{H} \to \mathbb{C}$ lin. functional

If $\mathcal{L}(ax + by) = a\mathcal{L}(x) + b\mathcal{L}(y)$

$\forall x, y \in H$; $a, b \in \mathbb{C}$

Ex:

- $\mathcal{H} = \ell^2, \quad L(x) = x(1)$
- $\mathcal{H} = \ell^2, \quad L(x) = x(1) + x(2) + x(3)$
- $\mathcal{H}^2 = L^2([0,1]), \quad L(f) = \int_0^1 f(x)\,dx$
- $\mathcal{H}^2 = L^2([0,1]), \quad L(f) = f(1/2)$ is not a lin. funct. ~~First~~ Not well-defined. (Could be defined using Hamel basis) → Makes sense for cont. functions
- $\mathcal{H} = \ell^2, \quad L(x) = \sum_1 x(k)$ is a non-example (conv: take $x(k) = 1/k$). Makes sense on C_{00}.
- Lin. functionals may be discont.

Ex: $L : C_{00} \to \mathbb{C}$

$$L(x) = \sum x(k)$$

$$x_n = \begin{cases} 1/n, & k \le n \\ 0, & k > n \end{cases}$$

$x_n \to 0$ & $L(x_n) = 1$

$$\|x_n\|_2 = \sqrt{\tfrac{1}{n^2} \cdot n} = \tfrac{1}{\sqrt{n}} \to 0$$

$L(0) = 0$ so not cont.

Fact: Discont. functionals also exist on ∞-dim. Hilbert spaces but there are no explicit examples. (cannot be written down). (consistent with Z.F.)

$$\operatorname{Ker} L = \{x \mid L(x) = 0\}$$

is a linear manifold. If L is cont. $\rightsquigarrow$ $\operatorname{Ker} L$ is closed.

Thm: TFAE for lin. funct. $L : \mathcal{H} \to \mathbb{C}$

1) L cont.
2) L cont. at 0
3) $\exists\, C \ni |Lx| \le C\|x\| \quad \forall x \in \mathcal{H}$

Pf: $1 \to 2$: ...

$2 \to 3$ (contrapos): $\forall n, \exists\, x_n \ni |L(x)| > n\|x_n\|$. Let $y_n = \dfrac{x_n}{\|x_n\|} \cdot \dfrac{1}{n}$

$y_n \to 0$ as $\|y_n\| = \frac{1}{n}$

But $|L(y_n)| > 1$ so not cont. at 0.

$3 \to 1$: Clear by linearity:

$$|L(x_n) - L(x)| = |L(x_n - x)| \le C\underbrace{\|x_n - x\|}_{\to 0}$$

$\square$

* Then lin. funct. are either everywhere discont. or everywhere cont.

Smallest possible C in (3) is called the norm of L.

Note: $\|L\| = \sup_{\|x\|\le 1} |L(x)|$

$\underbrace{\|x\|\le 1}_{\text{Could be unit ball just as well}}$

Ex: $\mathcal{H} = \ell^2$

- $L(x) = x(1)$

  $|x(1)| \le C\|x\|$

  $C=1$ is the best

  $\|L\| = 1$

- $L(x) = x(1) + x(2) + x(3)$

  $\|L\| = \sqrt{3}$ (use Cauchy-Sch.)

- $L(f) = \int_0^1 f(x)\,dx$ on $L^2[0,1]$

  $$\left|\int_0^1 1\cdot f(x)\,dx\right| \le \sqrt{\int_0^1 1^2}\sqrt{\int_0^1 |f|^2} = \|f\|$$

  $\|L\| = 1$

* Bounded lin. functional means bounded in sense above, $\le C\underbrace{\|x\|}_{\text{extra piece}}$

So bounded in unit ball.

Riesz Rep

$\forall$ cont. $L: \mathcal{H} \to \mathbb{C}$

$\exists!\, x_0 \in \mathcal{H} \ni L(x) = \langle x, x_0\rangle$

for all $x \in \mathcal{H}$.

Rem: Clear above is lin. rep & bounded by Cauchy-Sch.

Uniqueness is clear.

$\langle x, x_0\rangle = \langle x, x_1\rangle \quad \forall x$

then $x_0 - x_1 \perp x \quad \forall x \in \mathcal{H}$

so $x_0 = x_1$. So only need existence.

Pf: Let $M = \ker L$

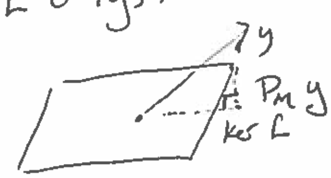
(if $L=0$, use $x_0 = 0$)

$\exists y \in \mathcal{H} \ni L(y) \neq 0$.

Then $\forall x \in \mathcal{H}$ $L\left(x - \frac{L(x)}{L(y)} y\right) = 0$

So x is in span of

$(\ker L \cup \{y\})$



Let $z = y - P_M y$. Note $L(z) = L(y) \neq 0$. $x_0 = \frac{\overline{L(z)}}{\|z\|^2} z$

Claim x_0 works.

Check: $L(x) = \langle x, x_0 \rangle$

Check for z:

$L(z) \overset{?}{=} \langle z, x_0 \rangle = \langle z, \frac{\bar{L}(z)}{\|z\|^2} z \rangle$

$= \frac{L(z)}{\|z\|^2} \langle z, z \rangle$

$= L(z)$

Agree on M: $x \in M$

$L(x) = 0 = \langle x, x_0 \rangle$

(orthogonality) $\Rightarrow x_0 \perp M$

done Every $x \in M$ lin. comb. of y & Ker L. ☒

Ex: $L(x) = x(1) + x(2) + x(3)$

$x_0 = (1,1,1,0,0,0,\dots)$

$L(f) = \int_0^1 f(x)\,dx$

$x_0 =$ const funct. 1

<u>Orthogonal Sets and Bases</u>

Uncond Conv: How to define $\sum_{i \in I} x_i$; $x_i \in \mathcal{H}$

I ← abstract set

This is the problem

* Aside:

vector $\leftrightarrow$ lin. functional

$x_0 \leftrightarrow L(x) = \langle x, x_0 \rangle$

We say $S = \sum_{i \in I} x_i$

if $\forall \epsilon > 0$, $\exists$ finite set $A \subset E$

such that $\forall$ finite $B \supseteq A$

$\|\sum_{i \in B} x_i - S\| < \epsilon$

Uncond. Conv.

no cond. on order of terms

except "large A" ↔ small ϵ

like your δ

B "suff large set"

Why sum of arb set? Maybe collection of vectors, want to add them. Have no part. order.

For $\sum_{n=1}^{\infty} x_n$, there are 3 notions of conv.

1) Absolute, $\sum \|x_n\| < \infty$

2) Uncond.

3) Ordinary (lim partial sums $\rightarrow$ #)

1) $\rightarrow$ 2) $\rightarrow$ 3) all strict

Ex $2 \not\rightarrow 1$

$x_n = \frac{1}{n} e_n$; $e_n = (0, 0, \dots, \underset{n^{th}\text{ spot}}{1}, 0, \dots)$

$\sum \|x_n\| = \sum \frac{1}{n} = \infty$ so not abs. But

$\sum x_n = (1, 1/2, 1/3, \dots) = S$

$\|\sum_{\text{fin } B} x_n - S\|_2^2 = \sum_{n \notin B} \frac{1}{n^2}$

Given $\epsilon > 0$, $\exists N \ni$

$\sum'_{n>N} \frac{1}{n^2} < \epsilon^2$. Let $A = \{1, \dots, N\}$

If $B \supseteq A$, then $\|\sum'_{n \in B} x_n - S\|_2^2 < \epsilon^2$.

as $n \notin B \to n > N$.

ON $\stackrel{\text{def}}{=}$ orthonormal

ON set: all elements have norm 1 and are $\perp$ to each other.

ON basis: (ONB): A max. orthonormal set. (max. wrt inclusion)

Ex: $\{e_n \mid n \in \mathbb{N}\}$ is an ONB in ℓ^2

Why max? If $x \perp e_n$ $\forall n \to$ then $x = 0$

Def: Hamel Basis: max. lin. indep. set

$\{e_n\}$ not max we can add $(1, 1/2, 1/3, 1/4, \dots)$ & still lin. indep. (fin. lin. comb.)

If B Hamel Basis then every $x \in \mathcal{H}$ is a fin. lin. comb. of elements of B.

What does ONB do? $\underset{\{v_i\}_{i \in I}}{\overset{}{B}}$ ONB

$\forall x \in \mathcal{H}$, $\exists!$ $c_i \in \mathbb{C} \ni$

$\sum' c_i v_i = x$ in the sense of uncond. conv.

Pf: Uniqueness is clear: enough to show

if $\sum' c_i v_i = 0 \to c_i = 0$

Fix i_0, $0 = \langle \underbrace{\sum' c_i v_i}, v_{i_0} \rangle$ All

$= \sum_i c_i \langle v_i, v_{i_0} \rangle = c_{i_0}$

Existence: Define.

$c_i = \langle x, v_i \rangle$. Claim $\forall$ fin. set

A, $\left(\sum_{i \in A} c_i v_i - x \right) \perp v_j$; $j \in A$

Let $M_A = \operatorname{span}\{v_j \mid j \in A\}$

Then $\sum'_{i \in A} c_i v_i = P_{M_A} x$

Know $B^\perp = 0$ (max. orth. set so no nonzero orth could add to B)

So lin. span dense: $\{v_i\}^\perp = 0$

So $\forall \epsilon > 0$, $\exists$ lin. comb. $\sum'_{i \in A} b_i v_i$

$\ni \|\sum'_{i \in A} b_i v_i - x\| < \epsilon$. Hence

$\|P_{M_A} x - x\| < \epsilon \leq \|\sum'_{i \in A} c_i v_i - x\| < \epsilon$

If $B \supseteq A \to M_B \supseteq M_A$ $\to$ $\text{dist}(x, M_B) \leq \text{dist}(x, M_A) < \epsilon$

($\pm$ dist. $\|\sum'_{i \in B} c_i v_i - x\| = $) $< \epsilon$

Bessel's Ineq:

$$\sum |c_i|^2 \leq \|x\|^2$$

for any ON set

Parseval's Thm

$$\sum |c_i|^2 = \|x\|^2$$

for ONB.

$\{v_n\}$ ON set

$\sum v_n$ not conv. ?

$$\left\| \sum_1^N v_n \right\|^2 = N$$

But $\sum_1^\infty \frac{v_n}{n}$ conv.

Rem: In a sep. Hilbert Space (countable dense set)

every ON is at most countable

PF: $\{e_i\}$ uncountable ON set,

then $\|e_i - e_j\| = \sqrt{2}$. So neigh

$N_{\sqrt{2}/2}(e_i)$ are disjoint

$1/\sqrt{2}$

So any dense set must be uncountable.

What is an iso between Hilbert spaces $\mathcal{H}$ & K? Surj lin. map $T: \mathcal{H} \to K$ such that

$$\underbrace{\langle Tx, Ty \rangle_K = \langle x, y \rangle_{\mathcal{H}}}$$

So inner product on one $\approx$ I.P on other

So properties in one $\approx$ prop. in other

In particular, $\|Tx\|_K = \|x\|_{\mathcal{H}}$

In f.d. space $\langle -, - \rangle_K = \langle -, - \rangle_{\mathcal{H}}$ is enough

Ex: Shift operator: $S: \ell^2 \to \ell^2$

$$S(x) = (0, x_1, x_2, x_3, \dots)$$

Clearly preserving inner product but is not surjective.

$M = \{x \in \ell^2 \mid x_1 = 0\}$ then

$S: \ell^2 \to M$ is an iso.

So entire space iso to subspace.

$\hookrightarrow$ á la Hilbert's Hotel.

* Every closed lin. subspace of Hilbert is also a Hilbert space.

To construct a $T: \mathcal{H} \to K$

Pick ONB $\{e_n\}$ in $\mathcal{H}$

ONB $\{f_n\}$ in K then

for $x \in \mathcal{H}$, $x = \sum \langle x, e_n \rangle e_n$

Let $T(x) = \sum \langle x, e_n \rangle f_n$

Norm clear (f_n ON so just get squares)

Surj ($T^{-1}(y) = \sum_i \langle y, f_n \rangle e_n$ for $y \in K$)

$T \circ T^{-1} = 1$

$\hookrightarrow T(\sum_i \langle y, f_n \rangle e_n)$

$= \sum \langle y, f_n \rangle f_n$

$= y$

$T^{-1} T = 1_{\mathcal{H}}$

When does this work? Need bijection.

Need to be able to index by 'same' set.

So same cardinality $\rightsquigarrow$ iso.

All sep. inf. dim. Hilbert spaces are iso. (countable ONB)

$\Downarrow$

So all iso to ℓ^2.

Why is ℓ^2 separable. Dense set is $\{x \in C_{00} \mid \forall k, \overset{Re, Im}{x(k)} \in \mathbb{Q}\}$

Countable clearly as well.

Iso. $L^2[0,1] \to \ell^2(\mathbb{Z})$

Fourier transform / series

$$\hat{f}(n) = \underbrace{\int_0^1 f(t) e^{-2\pi i t n}}_{\langle f, e^{2\pi i t n} \rangle} dt$$

minus from conjugation

ONB of $L^2[0,1]$

$f_n(t) = e^{2\pi i n t}$; $n \in \mathbb{Z}$

Construct iso. using ONB $\{f_n\}_{n \in \mathbb{Z}}$ in $L^2[0,1]$ and $\{e_n\}_{n \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$

$T(f) = (\hat{f}(n))_{n \in \mathbb{Z}}$

An iso. $U: \mathcal{H} \to \mathcal{H}$ is a unitary operator.

Ex: unitary operator on $L^2[0,1]$

take a measurable $\phi: [0,1] \to \mathbb{C}$ such that $|\phi(t)| = 1$ for all t

Let $\underbrace{\mathcal{U}(f)(t) = \phi(t) f(t)}_{\text{"multiplication operator"}}$

$$\langle \mathcal{U}f, \mathcal{U}g \rangle = \int_0^1 \phi(t) f(t) \overline{\phi(t) g(t)}\, dt$$

$$= \int_0^1 \underbrace{|\phi(t)|^2}_{=1} f(t) \overline{g(t)}\, dt$$

$$= \int_0^1 f(t) \bar{g}(t)\, dt$$

$$= \langle f, g \rangle$$

$* \; \mathcal{U}^{-1} f = \phi^{-1} f$

What if $|\phi|$ not 1?

So can find set pos measure so that $|\phi| < 1-\epsilon$ or $|\phi| > 1+\epsilon$

$$\|\phi\chi_E\|_{L^2} \neq \|\chi_E\|_{L^2}$$

mult. operator unitary $\Leftrightarrow |\phi(t)|=1$

$\{x_n\}^\infty$ lin indep. set in $\mathcal{H}$

$S = \sum_1 x_n$. Can $S=0$? Yes.

ℓ^2: $x_1 = (1, 0, \cdots)$

$x_2 = (-1, 1/2, 0, \cdots)$

$x_3 = (0, -1/2, 1/3, 0, \cdots)$

$x_4 = (0, 0, -1/3, 1/4, 0, \cdots)$

$\sum_1 x_n = 0$

$\hookrightarrow$ partial sums $\sum^N_1 = \frac{1}{N}$

$\sum c_k x_{n_k}$; $m = \max(n_k)$; $c_k \neq 0$

(n_k+1 component $\neq 0$ & nothing to cancel)

m^{th} component of sum is $\neq 0$.

(Aside) Schauder Basis:

$\{v_n\} \ni \forall x \ \exists! \ c_n \in \mathbb{C}$

$\ni x = \sum_1 c_n v_n$

1.6 Direct Sums of Hilbert Spaces

If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces

$$\mathcal{H} \oplus \mathcal{K} = \left\{ (h, k) \;\middle|\; \begin{matrix} h \in \mathcal{H} \\ k \in \mathcal{K} \end{matrix} \right\}$$

Adding simple. Mult. simple (scalar)

Inner product:

$\langle (h,k), (h',k') \rangle =$

$\langle h, h' \rangle + \langle k, k' \rangle$

Need to show complete:

$\{(h_n, k_n)\}$ Cauchy sequence

$\|(h_n, k_n) - (h_m, k_m)\| < \epsilon$

$\|$

$$\sqrt{\underbrace{\|h_n - h_m\|^2}_{\text{Cauchy}} + \underbrace{\|k_n - k_m\|^2}_{\text{Cauchy}}} < \epsilon$$

why?

$h_n \to k$
$k_n \to k$

Also, $\|(h_n, k_n) - (h, k)\| = \sqrt{\|h_n - h\|^2 + \|k_n - k\|^2} \to 0$

The direct sum contains copies of $\mathcal{H}$ & $\mathcal{K}$. In $\mathcal{H} \oplus \mathcal{K}$, $\mathcal{H}, \mathcal{K}$ orthogonal.

$\mathcal{H} \cong \{(h, 0)\}$

$\mathcal{K} \cong \{(0, k)\}$

Also, $\mathcal{H} \oplus \mathcal{K}$ complete $\rightarrow \mathcal{H}, \mathcal{K}$ complete

($\mathcal{H}, \mathcal{K}$ closed subspace complete space)

M, N orth. subspaces of $\mathcal{H}$
and span$(M \cup N) = \mathcal{H}$ then
$\mathcal{H} \cong M \oplus N$

Iso: $T(x) = (P_M x, P_N x)$

Really can just think $N = M^\perp$
$P_N x = x - P_M x$

$T^{-1}((m,n)) = m+n$

Why is inner product preserved?

$\langle (m,n), (m',n') \rangle_{M \oplus N} = \langle m, m' \rangle + \langle n, n' \rangle$

$\langle m+n, m'+n' \rangle = \langle m, m' \rangle + \langle n, n' \rangle$

by $\perp$. (see from 'FOIL') just get 'FO'

Ex: $L^2[0,1] \oplus L^2[1,2]$
$\cong L^2[0,2]$

$(f_1, f_2) \xrightarrow{T} f(x) = \begin{cases} f_1(x), & x \leq 1 \\ f_2(x), & x > 1 \end{cases}$

$\|(f_1, f_2)\|^2 = \int_0^1 |f_1|^2 + \int_1^2 |f_2|^2 = \int_0^2 |f|^2 = \|f\|^2$

Inf. Sums of Hilbert Spaces

$\oplus \mathcal{H}_n = \mathcal{H} = \{ (h_n)_{n=1}^\infty \mid h_n \in \mathcal{H}_n \text{ \& } \sum \|h_n\|^2_{\mathcal{H}_n} < \infty \}$

$\langle (h_n), (h_n)' \rangle = \sum_n \langle h_n, h_n' \rangle$ gives conv. by Cauchy Sch.

Ex: $\ell^2 = \bigoplus_{n=1}^\infty \mathbb{C}$

$L^2(\mathbb{R}) = \bigoplus_{-\infty}^\infty L^2[n, n+1]$

Uncountable? Many Hilbert Spaces

$\ell^2(\mathbb{R}) =$ funct. $\mathbb{R} \to \mathbb{C} \ni$
$\sum_{x \in \mathbb{R}} |f(x)|^2 < \infty$

Non-sep. Hilbert space
functions $= 0$ a.e. otherwise would have div. (notice $x \in \mathbb{R}$)

$ONB = \{ \chi_{\{a\}} : a \in \mathbb{R} \}$
characteristic function

May be useful for counterexamples

$\ell^2(\mathbb{R}) \cong \bigoplus_{x \in \mathbb{R}} \mathbb{C}$

Good exercise:
{all seq. on $\mathbb{R}$ or $\mathbb{C}$}
try to put nontriv. norm on it.

Exam Fri. Everything through Sect. 2.2

2.1 Linear Operators on Hilbert Spaces

- $T: H \to K$

lin: $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$

bounded if $\exists C \ni \|Tx\|_K \le C\|x\|_H$

smallest such C is the norm of T.

$B(H,K) :=$ all bounded lin. operators

$B(H) := B(H,H)$

$\hookrightarrow$ linear normed spaces but not Hilbert spaces.

Ex: $H = \mathbb{C}^2$

$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\|T_1\| = 1$

$T_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\|T_2\| = 1$

$T_1 \pm T_2 = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$

+: Id
−: reflection } isometries → norm = 1

$\|T_1 \pm T_2\| = 1$ → if Hilbert space

So $\|T_1 + T_2\|^2 + \|T_1 - T_2\|^2 = 2(\|T_1\|^2 + \|T_2\|^2)$

But $1 + 1 = 2(1+1)$ false so not.

Ex: $T \in B(\ell^2)$ $\forall$ bounded (b_n)

$Tx = (b_n x_n)$

See HW

$\|T\| = \sup|b_n|$

"infinite diagonal matrix"

(think what it does to sequence)

$\operatorname{Ker} T = \{x \in H \mid Tx = 0\}$ closed for all $T \in B(H,K)$

lin. subspace

im $T = \{Tx \mid x \in H\} \subset K$

lin manifold, not always closed

more often called range here

ran T

Ex: $Tx = \left(\frac{x_n}{n}\right)$ on ℓ^2

Why im not closed? Contains c_{00}. But c_{00} dense so $\overline{\text{im}} = \ell^2$. But im $\neq \ell^2$ as

$(1/n) \notin \operatorname{Im} T$

Good possibility to show if cont. dense the closure whole space. Show not so not closed.

$T \in B(\ell^2)$ can be described by "infinite matrix"

$\langle Te_n, e_m \rangle$

$$\begin{pmatrix} | & | & \\ Te_1 & Te_2 & \cdots \\ | & | & \end{pmatrix}$$

Shift operators:

$Sx = (0, x_1, x_2, \dots)$

$S^* x = (x_2, x_3, \dots)$

$\operatorname{Ker} S = 0 \quad \operatorname{im} S = \{e_1\}^\perp$

$\operatorname{Ker} S^* = \langle e_1 \rangle \quad \operatorname{im} S = \ell^2$

$S^* S = 1$

$\underbrace{S S^*}_{} \neq 1$

proj. onto complement of $e_1^\perp$, $P_{e_1^\perp}$

Ex: $T: L^2[0,1] \to L^2[0,1]$

$C_\phi : f \mapsto f \circ \phi$

$\phi: [0,1] \to [0,1]$

non-ex: $\phi(x) = x^2$

$f(x) = x^{-1/3} \in L^2$ on $[0,1]$

$\int_0^1 x^{-1/3} dx = 3$

$f\phi = x^{-2/3}$ but its square not int. $[0,1]$ so not in ℓ^2.

Claim: If $\exists\, \epsilon > 0 \ni$ $|\phi'(x)| \geq \epsilon \to C_\phi$ is bounded on $L^2[0,1]$ ← gives monotone

$\phi \nearrow$ or down $\searrow$ consider $\nearrow$.

$\frac{\phi'}{\epsilon} \geq 1$

$$\int_0^1 |f\phi(x)|^2 dx \leq \frac{1}{\epsilon} \int_0^1 f(\phi(x))^2 \phi'(x)\, dx$$

$$= \frac{1}{\epsilon} \int_{\phi(0)}^{\phi(1)} |f(u)|^2 du \leq \frac{1}{\epsilon} \|f\|^2_{\ell^2}$$

sub.

So $\|C_\phi\| \leq \dfrac{1}{\sqrt{\inf |\phi'|}}$

In fact, this is equality

Integral Operators

Has Kernel $K(x,y)$ which has nothing to do with 'ordinary kernel'.

$$Kf(x) = \int_0^1 K(x,y) f(y)\, dy$$

Thm: If all slices of $|K|$ have bounded integrals $\to$ $K: L^2[0,1] \to L^2[0,1]$

Slices: $\int_0^1 |K(x,y)| dy \leq C_1 \quad \forall x$

$\int_0^1 |K(x,y)| dx \leq C_2 \quad \forall y$

So like on a square

Hint for proof:

$$\int |K|^{1/2} |K|^{1/2} |f|\, dy \leq \sqrt{\int |K|\, dy}\ \sqrt{\int |K|\, |f|^2 dy}$$

+ C.S. like ineq.

Ex:

$$K(x,y) = \begin{cases} 1, & x > y \\ 0, & x < y \end{cases}$$

$$\underbrace{KF(x) = \int_0^x F(y)\,dy}_{}$$

Volterra Operator

(Indef. integral)

Strict ineq. for

$\|AB\| \leq \|A\|\,\|B\|$

$A = \mu_x$

$B = \mu_{1-x}$

$\left.\begin{array}{l} \|A\| = x \\ \|B\| = 1 \\ \|AB\| = 1/4 \end{array}\right\}$ max on $[0,1]$

Hilbert - Schmidt Operators $\subset B(H)$

Form hilbert space.

All norms on f.d. Hilbert spaces are bounded. ↑ domain

Image space f.d. is not sufficient.

2.2 Adjoint Operator

The adjoint of $A \in B(H, K)$ is the unique $B \in B(K, H)$ such that $\langle Ah, k\rangle = \langle h, Bk\rangle$ for $h \in H$, $k \in K$.

Unique: If B, B' are such that above holds $\rightsquigarrow$ $Bk - B'k$ is $\perp$ to H so $Bk = B'k$.

Existence: Based on rep. theory. Fix k. Consider $\langle Ah, k\rangle$ as lin func. in h. This is bounded so $\exists$ element which represents it.

$\underbrace{\phi(h)}_{} = \langle h, h_0\rangle$

$Bk = h_0$

Bounded:

$\|Bk\| = \|h_0\| = \|\phi\|$ ~~$\leq \|A\|$~~

$|\phi(h)| \leq \|Ah\|\,\|k\|$

$\leq \|A\|\,\|h\|\,\|k\|$

So $\|\phi\| \leq \|A\|\,\|k\|$

So $\|B\| \leq \|A\|$

Notation: A^*

$(A+B)^* = A^* + B^*$

$(\alpha A)^* = \bar{\alpha} A^*$

So conjugate lin. operation $\quad (AB)^* = B^* A^*$

$(A^*)^* = A$

But then $\|A\| \leq \|A^*\| \leq \|A\|$

so $\|A\| = \|A^*\|$

Ex: Mult. op. M_ϕ on L^2

$M^*_\phi = \langle \phi f, g \rangle \to \int \phi f \bar{g} = \int f (\overline{\quad}) $?

$= \langle f, \bar{\phi} g \rangle$

So $M^*_\phi = M_{\bar{\phi}}$

Note $M^*_\phi M_\phi = M_{|\phi|^2} = M_\phi M^*_\phi$

Ex: Shift Operator on ℓ^2

$Sx = (0, x_1, x_2, \dots)$

$\langle Sx, y \rangle = \langle x, S^* y \rangle$

$= \sum_{i=1}^{\infty} x_i \bar{y}_{i+1}$

$S^* y = (y_2, y_3, y_4, \dots)$

Backwards shift

S, S^* do not commute

Def: $A \in B(H)$ is hermitian or self-adjoint if $A = A^*$

Def: $A \in B(H)$ is normal if $A^* A = A A^*$

Ex: M_ϕ self adjoint $\iff$ ϕ real

$\forall A$, A^*A, AA^* is self adjoint.

Also, $\forall A \in B(H)$,

$\frac{1}{2}(A + A^*)$ is self adjoint

called $\operatorname{Re} A$, real part.

$\frac{1}{2i}(A - A^*)$ is self adjoint

called $\operatorname{Im} A$, imag. part

Exercise: A is normal $\iff$ $\operatorname{Re} A$, $\operatorname{Im} A$, commute.

Thm: $\ker A = (\operatorname{im} A^*)^\perp \quad \forall A \in B(H,K)$

Pf: $h \in \ker A \iff Ah = 0 \iff$

$\langle Ah, k \rangle = 0 \ \forall k \in K \iff \langle h, A^* k \rangle = 0$

So $h \perp \operatorname{im} A^* \iff h \in (\operatorname{im} A^*)^\perp$

$\text{im } A = (\ker A^*)^\perp$ False

Range does not have to be closed whereas right side doesn't always is.

But...

$\overline{\text{im} A} = (\ker A^*)^\perp$

is true.

$\overline{\text{im } A} = ((\text{im } A)^\perp)^\perp$

$= (\ker A^*)^\perp$

Special case: If A is self adjoint then $\ker A = (\text{im } A)^\perp$ so $H = \ker A \oplus \overline{\text{im} A}$

so nice when image closed.

Observations

$\langle A^* A x, x\rangle = \|Ax\|^2$

so $\ker A = \ker A^* A$

$\leq \|A^* A\| \, \|x\|^2$

So $\|Ax\| \leq \sqrt{\|A^* A\|} \, \|x\| \quad \forall x$

Then $\|A\|^2 \leq \|A^* A\| \leq \|A^*\| \, \|A\|$
$= \|A\|^2$

$\|A^* A\| = \|A\|$ $\quad \to \bar{z} z \leftarrow$ This order

Think of $A^* A$ as $|z|^2$ for $z \in \mathbb{C}$

Fact: A normal $\Leftrightarrow$

$\|Ax\| = \|A^* x\|$ for all x.

Pf:

$\Rightarrow$ (only):

$\|Ax\|^2 = \langle Ax, Ax\rangle$

$= \langle x, A^* A x\rangle$

$= \langle x, AA^* x\rangle$

$= \langle A^* x, A^* x\rangle$

$= \|A^* x\|^2$

So can use this to show something is not normal ?

$A \in B(H)$ normal $\to A^2$ normal

True for any poly. in A.

$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$

Conv. ab $\quad \left\|\frac{A^n}{n!}\right\| \leq \frac{\|A\|^n}{n!}$

A, B commute $\to B^* A^* = A^* B^*$ commute

Find adjoint of $Tf = f(x/2)$

$T: L^2[0,1] \to L^2[0,1]$.

Exercise: $\|T\| = \sqrt{2}$

$\langle Tf, g\rangle = \langle f, ?\rangle$

$$\int_0^1 f(x/2)\overline{g(x)}\,dx = \int_0^1 f(x)\ \overline{?}\ dx$$

$\{ u = x/2$

$$\int_0^{1/2} f(u)\,\overline{g(2u)}\,2\,du = \int_0^1 f(u)\,\overline{2g(2u)\chi_{[0,1/2]}}\,du$$

$$T^*g = \begin{cases} 2g(2x), & x \in [0, 1/2] \\ 0, & x > 1/2 \end{cases}$$

Generalize

C_ϕ^*? (Comp. operator - with ϕ)

$\phi : [0,1] \to [0,1]$, $\phi \in C^1$

$\inf |\phi'| > 0$

$C_\phi f = f \circ \phi$

$$\int_0^1 f(\phi(x))\,\overline{g(x)}\,dx = \int_0^1 f(x)\ \overline{?}\ dx$$

$u = \phi(x)$

$du = \phi'(x)\,dx$

$$\int_{\phi(0)}^{\phi(1)} f(u)\,\overline{g(\phi^{-1}(x))}\,\frac{1}{\cancel{\phi'(x)}\ \phi'(\phi^{-1}(x))}\,du$$

Inv. Function Thm

Notational ease: $\phi^{-1} = \psi$

$$\int_{\phi(0)}^{\phi(1)} f(u)\,\overline{g(\psi(u))\,\psi'(u)}\,du$$

$$\int_{\phi(0)}^{\phi(1)} f(u)\,\overline{g(\psi(u))}\,\overline{\psi}'(u)\,du$$

still real

$$C_\phi^* g = \begin{cases} (g\circ\psi)(\psi'), & [\phi(0), \phi(1)] \\ 0 \end{cases}$$

$H = \ell^2(\mathbb{N} \cup \{0\})$

a) $\alpha \in H \to \sum_1 \alpha_n z^n$ has $\overset{\text{radius conv.}}{\widetilde{R_o C}} \geq 1$

$1/R = \limsup |\alpha_n|^{1/n} \leq 1$

as $|\alpha_n| \leq 1$ for n suff. large

b) $|\lambda| < 1$

$$L(\alpha) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

compare

Find $\alpha_0 \ni L\alpha = \langle \alpha, \alpha_0\rangle$

$$= \sum_0^\infty \alpha(k)\,\overline{\alpha_0(k)}$$

$$\alpha_0 = (\overline{\lambda}^n)_{n=0}^{\infty}$$

or $\sum_1 n\alpha_n \lambda^{n-1} \longrightarrow$

$$\alpha_0 = (n\lambda^{n-1})_{n=0}^{\infty}$$

$\mathcal{L}^2[0,1]$

$M = \{ f \mid f' \text{ is cont.} \} \subset \mathcal{L}^2[0,1]$

$Lf = f'(t)$ fixed $t \in (0,1)$ Cannot be extended to bounded cont. on all of $\mathcal{L}^2$

Show not bounded. Need big deriv e 1 pt.

t

± pinch

$1, z, z^2, z^3$ (can normalize)

not a basis on $\mathcal{L}^2(\lambda)$

eg $\bar{z} \perp$ to all of them

$$\iint z^m \bar{z}^n \, d\lambda \quad ; z = re^{it}$$

$$= \int_0^1 r\,dr \int_0^{2\pi} \underbrace{r^{m+n} e^{i(m-n)t}}_{=0} dt$$

$M_\phi : \mathcal{L}^2 \to \mathcal{L}^2$

Claim: $\|M_\phi\| = \text{ess sup} |\phi| = \inf \{ c \mid |\phi| \le c \text{ a.e.} \}$

$:= \|\phi\|_\infty$

$\|M_\phi\| \le \|\phi\|_\infty$ immediate

Take $\epsilon > 0$

$E = \{ x \mid |\phi(x)| \ge \|\phi\|_\infty - \epsilon \}$

has positive measure

χ_E Char. function

$\|\chi_E\| = \sqrt{\mu(E)}$

$\|\phi \chi_E\| \ge (\|\phi\|_\infty - \epsilon) \sqrt{\mu(E)}$

So that $\|M_\phi\| \ge \|\phi\|_\infty - \epsilon$

(If inf. measure, intersect with set of finite measure).

Recall in a ring with 1.

$a^2 = a \to (1-a)^2 = 1-a$

2.3: Projections & Idempotents

$E \in B(H)$ is idempotent if $E^2 = E$

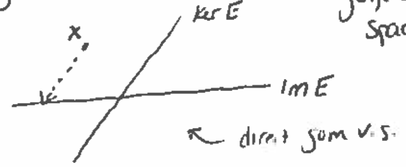
Now then $1-E$ idempotent (if and only if)

$\operatorname{im} E = \ker(1-E)$ so $\operatorname{im} E$ is closed.

$x \in \ker(1-E) \rightleftarrows x = Ex \rightleftarrows x \in \operatorname{im} E$

For $x \in H \rightsquigarrow x = \underset{x \in \operatorname{im} E}{Ex} + \underset{\ker E}{(x - Ex)}$ ← are disjoint

Why not direct sum: (as defined b/f) ← sense of Hilbert Spaces



Direct sum of Hilbert spaces:

$\to (x_1, x_2)$; $x_1 \in M$, $x_2 \in N$

$H = M \oplus N$

$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle_M + \langle x_2, y_2 \rangle_N$

$\langle (x,0), (0,y) \rangle = 0$ so $M \perp N$

"Could view not as by def. but by finding $M \perp N \to$ decomposing?" →Me

So need orth. for H.S. decomp.

So when dec. v.s. dec. as Hilbert spaces?

$H = \operatorname{im} E \oplus \ker E$ as H.S.

need $\operatorname{im} E \perp \ker E$

Thm: TFAE for idempotent:

1) $\operatorname{im} E \perp \ker E$
2) $E = E^*$ (normal)
3) $\|E\| = 1$

Def: Any of following above is called projection (self adjoint idempotent).

Pf:

$3 \to 1)$ $\operatorname{ran} E \not\perp \ker E$. $u \in \operatorname{im} E$

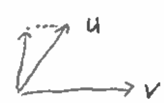
$v \in \ker E$ & $\langle u, v \rangle \neq 0$. WLOG,

$\langle u, v \rangle > 0$

Let $x = u - tv$ for $t > 0$

$\|x\|^2 = \|u\|^2 - 2t\langle u, v \rangle + t^2 \|v\|^2$

$< \|u\|^2$



← for t sufficiently small

But $Ex = u$ so $\|E\| > 1$

← $\|Ex\| \geq 1$ unless $E = 0$

as $Ex = x$

$1 \to 2)$ E^* idempotent.

$\ker E^* = (\operatorname{im} E)^\perp = \ker E$

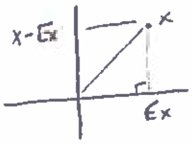
$\operatorname{im} E^* = (\ker E)^\perp = \operatorname{im} E$

← closed

So $E = E^*$ on $\operatorname{im} E$ (both identity)

$E = E^*$ agree on kernel (both 0)

so agree everywhere (they span space)



Pythag. Thm

So 1→3 simple $\nearrow$ $\|Ex\| \le \|x\|^2$

$x = \underbrace{Ex}_{} + \underbrace{x - Ex}_{\perp} \to \|x\| = \|Ex\|^2 + \|x - Ex\|^2$

2→1) immediate : $\ker E = \ker E^* = (\operatorname{im} E)^\perp$

$E^2 = E^* E = E$

$\|\underbrace{E^* E}_{=E}\| = \|E\|^2$

$\to = E^2 = E$

$\|E^*\| = \|E\|^2$

$\Leftrightarrow \|E\| = 0$ or $\|E\| = 1$

Let $P^\perp = 1 - P$ if P is a projection

$\operatorname{im} P^\perp = \ker P = (\operatorname{im} P)^\perp$

For any $A \in B(H)$, write

$A = (P + P^\perp) A (P + P^\perp)$

"Block matrix"
$$\begin{array}{ccc} & P & P^\perp \\ P & PAP & PAP^\perp \\ P^\perp & P^\perp AP & P^\perp AP^\perp \end{array} = A$$

Say A invariant subspace for A if $AM \subseteq M$. If $P = P_M$ then

$\Updownarrow$

$PAP = AP$

Indeed, $PAP = AP \Leftrightarrow \operatorname{im} AP \subseteq M$

$\Leftrightarrow AM \subseteq M$

In block matrices, this means

$P^\perp AP = 0$
$$\begin{array}{ccc} & M & M^\perp \\ M & * & * \\ M^\perp & 0 & * \end{array}$$

Would be better if upper right corner were also 0.

Def: M reducing subspace for A if both M & $M^\perp$ invariant.

In terms of $P = P_M$, means

$PAP = AP$

&

$P^\perp A P^\perp = AP^\perp$

$\Updownarrow$

$0 = PAP^\perp$

$\Leftrightarrow 0 = PA(1-P)$

$\Leftrightarrow PAP = PA$

So M reducing $\Leftrightarrow P^\perp AP = 0$ & $PAP^\perp = 0$

Claim: M reduces $A \Leftrightarrow A$ commutes with P_M

Pf: $\Rightarrow$: M reducing $\rightsquigarrow PAP = AP$ & $PAP = PA$

$\Leftarrow$: $AP = PA \to PAP = PA$ and $AP = PAP$

So M reduces. $\blacksquare$

M reducing for A, let A_M & $A_{M^\perp}$ be restrictions of A to $M, M^\perp$

Then $A = A_M \oplus A_{M^\perp}$ in sense

orth. decomp. $\longrightarrow H \cong M \oplus M^\perp$ & $A(x,y) = (\underset{M}{Ax}, \underset{M^\perp}{Ay})$

$PAP = PA$, adj
P self adj $\to \dots$

So A has block diagonal form.

If A self adj $\rightsquigarrow$ invariant = reducing

1 $\begin{cases} (E+E^*-I)(x)=0 \\ \text{Hint: Think about } 1-E \\ \text{and } E^* \end{cases}$

2 $\begin{cases} u+v=0 \to u=v=0 \\ u\in M \\ v\in N \end{cases}, M\cap N=0$

P, Q proj. : $\operatorname{ran} P \subseteq \operatorname{ran} Q$

$Q\underbrace{Px}_{\in \operatorname{im} P \subseteq \operatorname{im} Q} = Px$ & Q idempotent

$PQx = Px$

$(PQ)^* = Q^*P^* = QP = P = P^*$

so $PQ = P$ (take adjoints again)

2.4 Compact Operators

A set A is compact (in a metric space)

$\Leftrightarrow$ A complete & totally bounded

totally bounded: $\forall \epsilon>0$, $\exists\, x_1, \dots, x_n$ $\ni A \subset \bigcup_{i=1}^{n} N_\epsilon(x_i)$

In f.d. space. bounded $\Leftrightarrow$ totally bounded.

Ex: ℓ^2 unit ball $\{x \mid \|x\|\le 1\}$

not totally bounded contains $e_1, e_2, \dots$ & $\|e_n - e_m\| = \sqrt{2}$

Thm: Let $\{e_n\}$ be ONB in H

Let $M_n = \operatorname{span}\{e_1, \dots, e_n\}$, then

$A \subset H$ compact $\Leftrightarrow$ A closed bounded & $\forall \epsilon>0$ $\exists n \ni A \subset N_\epsilon(M_n)$ $\to$ $\sup \operatorname{dist}(a, M_n) \to 0$

up to ϵ A should be f.d.

Pf:

$\Rightarrow$: A compact. $\forall \epsilon>0$ $\exists\, x_1, \dots, x_m \ni$

$A \subset \bigcup_{i=1}^{m} N_{\epsilon/2}(x_i)$

$\forall i\ \exists\, y_i \ni \|x_i - y_i\| < \epsilon/2$ &

y_i is fin. lin. comb. of the ONB — lin. span of ONB dense

Hence, $\exists n \ni y_1, \dots, y_m \in M_n$.

Then $A \subset N_\epsilon(M_n)$. Any metric space compact $\to$ closed, bounded.

$\Leftarrow$: A closed in H.S., A complete. $\exists n \ni A \subset N_{\epsilon/2}(M_n)$.

$P_{M_n}(A)$ is bounded in this f.d. space so totally bounded. So

$P_{M_n} A \subset \bigcup_{i=1}^{N} N_{\epsilon/2}(x_i) \to$

$A \subset \bigcup_{i=1}^{N} N_\epsilon(x)$ ∎

Def: A "flat" if $\forall \epsilon>0, \exists n \ni$

$A \subset N_\epsilon(M_n)$

Totally bounded in H.S. $\Leftrightarrow$ bounded & flat

Compact $\Leftrightarrow$ closed, bounded & flat

Ex: Hilbert Cube:

$\{x \in \ell^2 \mid |x_n| \le 1/n\}$

is compact.

Def: $T \in B(H,K)$ compact if

$\overline{T(\{\|x\| \le 1\})}$ is compact; equiv.

$T(\{\|x\| \le 1\})$ is totally bounded

So I is not compact. (in ∞-dim. spaces)

Equiv. $T(\|x\|<1)$ Fict

$B_0(H,K) =$ compact operators

$B_0(H)$ is a two-sided algebra $B(H)$.

$B_{00}(H,K) = \{$op. of $\underbrace{\text{fin. rank}}_{\dim \operatorname{im} T < \infty}\}$

Thm: $B_0(H,K) = \overline{B_{00}(H,K)}$

Pf: $\subseteq$: T compact consider

$P_n T$ where P_n proj onto M_n.

Claim $\{P_n T\}$ conv. & $\|P_n T - T\| \to 0$

$\forall x, \|x\| \le 1 \ \exists n \ni$

$T(\{\|x\|<1\}) \subseteq N_\epsilon(M_n)$

$\to \|P_n T - T\| \le \epsilon$

$\supseteq$: $T_n \to T$ & every T_n fin rank then T compact as im. unit ball

dense: $\exists n \ni \|T_n - T\| < \epsilon$

$T(\|x\|<1)$ is cont. in ϵ-nhbd of $T_n(\{\|x\|<1\})$

H hil. space

$K \subset H$ compact

$M = \overline{\operatorname{span} K}$

Then M is a sep. subspace.

Compact Operators & their Eigenvalues

H inf. dim. If $AB = I \to$ neither A, B compact

True: $B_0(H)$ two-sided ideal.

I not compact as image $B_1 := \{\|x\| \le 1\}$ contains inf ONB (H inf. dim, keep choosing lin. indep. vectors & use Gram Schmidt)

Recall $\overline{B_{00}} = B_0$. Important fact ☟

Conseq.

1) $A \in B_{00} \to A^* \in B_{00}$

Pf: $P = P_{\operatorname{im} A}$. $A = PA$

then $A^* = A^* \underset{\text{f.d.}}{P}$

So after A^* f.d.

So $\operatorname{im} A^* \le \dim \operatorname{im} P = \dim \operatorname{im} A$

2) From (1), if $A \in B_0 \to A^* \in B_0$

$A = \lim A_n \to A^* = \lim A^*$

$\hookrightarrow A_n \in B_{00}$

3) B_{00} is an ideal $\to$ B_0 is an ideal!

$A \in B_0$, $B \in B(H)$

$\hookrightarrow \lim A_n$ $\to$ $AB = \lim A_n B$

$A_n \in B$ $\qquad \in \overline{B_{00}}$

Ex: On ℓ^2, mult. operator $x_n \xrightarrow{T} b_n x_n$

where $b_n \to 0$ compact.

Let T_n be mult. by $(b_1, \cdots, b_n, 0, 0, \cdots)$

$\|T - T_n\| = \sup_{k>n} |b_k| \to 0$ as $n \to \infty$

Lem: If $\exists$ inf. dim. manifold on which $\|Tx\| \geq C\|x\|$ then T not compact.

Pf: Take ON set $\{e_j\} \subset M$

Then $\underline{\|Te_j - Te_i\|} \geq C\sqrt{2}$ $\forall i \neq j$

So $T(B_1 \cap M)$ not compact.

Ex: Mult. $\{x_n\} \to \{b_n x_n\}$ compact

$\rightleftarrows$ $b_n \to 0$

Pf: $\Rightarrow$: $b_n \not\to 0$: $\exists$ $|b_{n_k}| \geq \epsilon$

Let M be span span$\{e_i$

Then $\|Tx\|$

λ eigenvalue for T if $\exists$ $x \neq 0$

$\ni Tx = \lambda x$

$\sigma_p(T) = \{$set of eigenvalues$\}$

σ for spectrum

point spectrum

$T(x_n) = (b_n x_n)$ has

$\sigma_p(t)$; $T(e_n) = b_n e_n$

σ_p may be empty

Thm: T compact. If $\lambda \neq 0$ is eigenvalue then eigenspace $M_\lambda = \{x \mid Tx = \lambda x\}$ is f.d.

Pf: $\|Tx\| = |\lambda|\|x\|$ on $M_\lambda \to$

T has lower bound on M_λ on inf. dim. so compact $\dim M_\lambda < \infty$

Thm: T compact & $T - \lambda I$ has no lower bound then $\lambda \in \sigma_p(T)$. $(\lambda \neq 0)$

Pf: $\exists$ x_n, $\|x_n\| = 1$ $\ni \|T(x_n) - \lambda x_n\| \to 0$

$\exists$ conv. subsequence $Tx_{n_k} \to h$

Note $\|h\| = \lim \|Tx_{n_k}\| = |\lambda|$

Check: $Th = \lambda h$

$Th = Tx_n + T(h - x_n)$

$\hookrightarrow \to h$

Thm 3: If $\lambda \neq 0$, $\lambda \notin \sigma_p(T)$

$\bar{\lambda} \notin \sigma_p(T^*)$ & T not compact $\to$

$T - \lambda I$ is inv.

Pf: By Thm 2, $T - \lambda I$ has lower bound; so we need to prove onto. Its range is closed

(If A has lower bound and $\{y_n\} \to y$; $y_n \in \operatorname{im} A$)

Write $y_n = A x_n$

$\underbrace{\| y_n - y_m \|}_{\text{Cauchy}} \geq C \| x_n - x_m \|$

So x_n's cauchy so conv.

So $x_n \to x$. Hence

$y = Ax$. Then im A is closed.)

$\bar{\lambda} \notin \sigma_p T^* \to \underbrace{(T - \lambda I)^*}_{T^* - \bar{\lambda} I}$ has lower bound

↳ Thm 2

So $\ker T - \lambda I = \{0\}$

↳ complement of range

$\operatorname{im} (T - \lambda I)^\perp = \{0\}$

so im dense. But also closed

So onto so invertible. ☒

[$\| Ax \| \geq C \| x \|$ so

lower bound Then inj. Otherwise

$0 = 0$

$\| A^{-1} y \| \leq C^{-1} \| y \|$]

λ e.v. for A

$Ax = \lambda x$

$AAx = A(\lambda x) = \lambda Ax = \lambda^2 x$

Works for poly. $p(\lambda)$ e.g. of $p(A)$

Spectral Thm for Self-Adj. Compact Op.

$\sigma(T) = \{ \lambda \in \mathbb{C} \mid T - \lambda I \text{ not inv.} \}$

↑ Spectrum

Fact

$\sigma(T) \neq \emptyset$

Think function of λ. Otherwise

$\lambda \mapsto (T - \lambda I)^{-1}$

operator on $\mathbb{C}$ goes to 0 at ∞, i.e. holomorphic function on $\mathbb{C}$ w/ limit 0 at ∞. Bounded + holo $\to$ constant, 0 @ ∞ so 0.

$\sigma(T)$ can be $\emptyset$ over $\mathbb{R}$

• $\sigma(T)$ is bounded (by $\|T\|$): If

$|\lambda| > \|T\| \to (T - \lambda I)^{-1} =$

$\frac{-1}{\lambda} (I - T/\lambda) = \frac{-1}{\lambda} \underbrace{\sum_0^\infty \frac{T^n}{\lambda^n}}_{\text{conv. ab. by ratio Test}}$

• $\sigma(T)$ is compact (only need closed):

$\forall$ inv. A, $\exists\, \epsilon > 0 \ni A - B$ inv. if $\|B\| < \epsilon$

$(A - B)^{-1} = (A(I - A^{-1}B))^{-1}$

$= (I - A^{-1}B) A^{-1}$

$= \left(\sum_{n=0}^\infty (A^{-1}B)^n \right) \cdot A^{-1}$

Conv. if $\|A^{-1}B\| < 1$

so $\epsilon = 1/\|A^{-1}\|$

Ex: Find $\sigma(T)$ for $Tx = (b_n x_n)$

$\{b_n\}$ bounded & $T: \ell^2 \to \ell^2$

Recall $\sigma_p(T) = \{b_n\}$

What is $T - \lambda I$?

$(x_n) \mapsto ((b_n - \lambda) x_n)$

Inverse has to be $y_n \mapsto \left(\frac{y_n}{b_n - \lambda}\right)$

When is this bounded? Iff $\left\{\frac{1}{b_n - \lambda}\right\}$ bounded $\Leftrightarrow \lambda \notin \overline{\{b_n\}}$

So $\sigma(T) = \overline{\{b_n\}} = \underbrace{\{b_n\}}_{\sigma_p} \cup \underbrace{\{\text{rest}\}}_{\sigma_c}$

- In f.d., σ is just set of eigenvalues.

Generally, $\sigma(T)$ consists of 3 parts:

- $\sigma_p(T)$, point spectrum: eigenvalues ($\operatorname{Ker} T - \lambda I \neq 0$)
- $\sigma_c(T)$: continuous spectrum
  $T - \lambda I$ inj. & range dense but not all of $\mathcal{H}$.
- σ_r: residual spect, $T - \lambda I$ inj. but range is not dense

These are all disjoint but form spectrum.

Ex: $\sigma(S)$; $Sx = (0, x_1, x_2, \ldots)$

$0 \in \sigma_r(S)$

$S - 0I = S$ its range is a proper subset

im $S = \{e_1\}^\perp$ not dense

Thm (Spectral Thm) If T compact & self adjoint $\to$ T can be written as $T = \sum_i \lambda_n P_n$, λ_n eigenvalue & P_n projection onto eigenspaces. Sum conv in $B(H)$ uncond.

Also, $\lambda_n \in \mathbb{R}$ and eigenspaces are orthogonal.

How use it? If $\lambda_n > 0$ $\forall n$, we can let $\sqrt{T} = \sum \sqrt{\lambda_n} P_n$

$$\sqrt{T}\sqrt{T} = \left(\sum_i \sqrt{\lambda_n} P\right)^2 = \sum_i \lambda_n P_n$$

proj. mutually orth.

Gen: replace self-adj $\to$ normal

Still works but $\lambda_n \in \mathbb{C}$

drop compact must replace $\sum_i$ by $\int$ (wrt $\sigma(T)$).

Lem!: Diff. eigenspaces of a self adjoint op. are orth. & eigenvalues are real.

Goal: Prove Spectral Thm

Self adjoint: T is compact $S\text{-}a \rightarrow T = \sum_i \lambda_n P_n$

λ_n = eigenvalues ; P_n = proj to eigenspaces

PF:

$$\lambda \langle x, x\rangle = \langle \lambda x, x\rangle$$
$$= \langle Ax, x\rangle$$
$$= \langle x, A^* x\rangle$$
$$= \langle x, Ax\rangle$$
$$= \langle x, \lambda x\rangle$$
$$= \bar{\lambda}\langle x, x\rangle$$

If $Ax = \lambda x$, $Ay = \mu y$

$\lambda \neq \mu$

$$\lambda\langle x,y\rangle = \langle \lambda x, y\rangle = \langle Ax, y\rangle$$
$$= \langle x, Ay\rangle \text{ Know real}$$
$$= \mu\langle x,y\rangle$$

So $\langle x,y\rangle = 0$ so $x \perp y$

A unitary $\rightarrow |\lambda| = 1$

$Ax = \lambda x \sim \|Ax\| = \|\lambda x\|$

$\Downarrow$

$|\lambda| = 1$

actually $\sigma(A) \subset \{|z| = 1\}$

A unitary $|\lambda| \neq 1 \rightarrow$

$A - \lambda I$ inv.

Lem 2: T normal (self adj sufficient) & λ eigenvalue then $\bar{\lambda}$ is e.v. for T^*.

PF: Recall A normal

$$\underbrace{\ker A = \ker A^* A}_{\text{always true}} = \ker AA^* \underbrace{= \ker A^*}_{\text{always true}}$$

(normal)

$$\ker(T - \lambda I) = \ker(T^* - \bar{\lambda} I)$$

Cor: If T self adjoint $\rightarrow \ker(T - \lambda I)$ reducing subspace (inv. & for adjoint)

So $T = PTP + P^\perp T P^\perp$

where P proj. on $\ker(T - \lambda I)$

$$\begin{array}{c|cc} & P & P^\perp \\ \hline P & \lambda I & 0 \\ P^\perp & 0 & * \end{array}$$

Lem 3: If T is self adj. then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Pf: $4\,\mathrm{Re}\langle Tx,y\rangle = \langle T(x+y),x+y\rangle - \langle T(x-y),x-y\rangle$

just simply expand & use fact that T is s.a.

So if $M = \sup_{\|x\|=1} |\langle Tx,x\rangle|$ then

$|\mathrm{Re}\langle Tx,y\rangle| \le M$. Why? For unit vectors

$$|\mathrm{Re}\langle Tx,y\rangle| \le \frac14 |\langle T(x+y),x+y\rangle| + \frac14 |\langle T(x-y),x-y\rangle|$$

$$\le \frac M4 \|x+y\|^2 + \frac M4 \|x-y\|^2$$

$$\le \frac M2 (\|x\|^2 + \|y\|^2)$$

So for unit vectors have the inequality

Then for unimodular (modulus 1) c,

$|\mathrm{Re}\langle Tcx,y\rangle| \le M$ move

So using c, can rotate this real part. So bounded on whole unit circle. Then

$|\langle Tx,y\rangle| \le M \to \|Tx\| \le M$

then $\|T\| \le M$. Equality as

$|\langle Tx,x\rangle| \le \|Tx\|\cdot\|x\|$ ☑

Lemma 4: If T is compact and self adj then $\|T\|$ or $-\|T\|$ is an eigenvalue.

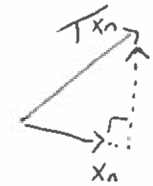
Pf: We have unit $\{x_n\}$ such that $|\langle Tx_n, x_n\rangle| \to \|T\|$ (Lem 3). Take a subseq. conv. to $\|T\|$ or $-\|T\|$

WLOG consider $\to \|T\|$

Let $\lambda = \|T\|$ for ease of notation.

$$Tx_n = \langle Tx_n, x_n\rangle x_n + P_{\{x_n\}^\perp}(Tx_n)$$

Proj "form" note unit vector



$$\|Tx_n\|^2 = |\langle Tx_n,x_n\rangle|^2 + \overbrace{\|Tx_n - \langle Tx_n,x_n\rangle x_n\|}^{\to\lambda}$$

Pythag Thm

$$\|T\|^2 = \|T\|^2 \quad \text{so} \quad 0$$

That is, $Tx_n - \lambda x_n \to 0$

By old lemma, $\inf \|T - \lambda I\|$ over unit vectors is 0, λ is eigenvalues. ☑

↳ Needed $\lambda \ne 0$. But this case trivial. $T=0 \to \lambda = 0$ eigenvalue.

Pf: (Spectral Thm)

We have $\lambda_1 \in \{\pm \|T\|\}$ lem 4

By lem 2, $\ker(T - \lambda_1 I)$, we can write

$$T = P_1 T P_1 + P_1^\perp T P_1^\perp$$

where P proj onto $\ker(T - \lambda_1 I)$.

$$= \lambda_1 P_1 + \underbrace{P_1^\perp T P_1^\perp}_{\text{compact, self adj. } \lambda_1 \text{ not its ev.}}$$

$P_1^\perp T P_1^\perp x = \lambda_1 x$

Then $x \in \operatorname{ran} P_1^\perp$; $Tx = \lambda_1 x$ so $x \in \operatorname{ran} P_1$

Contradiction. Cont. to get

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n + \cdots$$

If we get 0 operator, process stops

If inf seq of eigenvalues

Norms $\longrightarrow$ $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$

If eigenvalues don't $\to 0$, pick unit eigenvector e_n for each

$\{Te_n\} = \{\lambda_n e_n\}$ orth, don't go to 0

Then no conv. subseq $\Rightarrow \Leftarrow$ (compactness)

So we get $T = \lambda_1 P_1 + T_2$

$$= \lambda_1 P_1 + \lambda_2 P_2 + T_3$$

$$= \cdots$$

Since $|\lambda_n| = \|T_n\|$

We have $\|T_n\| \to 0$

So $\|T - \sum_{k=1}^{n} \lambda_k P_k\| \to 0$

If we choose ONB for each eigenspace, we get a ONS $\{e_n\}$ such that

HW $\longrightarrow$ $Tx = \sum_n \mu_n \langle x, e_n \rangle e_n$

μ_n are eigenvalues with repetition.

Note: $\ker T = \{e_n\}^\perp$

A s.a. Compact. $\lambda_n \geq 0$

$$\langle Ax, x \rangle \geq 0$$

Pf: $Ax = \sum_1 \mu_n \langle x, e_n \rangle e_n$

$$x = \sum_1 \langle x, e_n \rangle e_n + \underbrace{P_{\ker A} x}_{\text{orth. to everything in } Ax}$$

$$\langle Ax, x \rangle = \sum_1 \mu_n |\langle x, e_n \rangle|^2 \geq 0$$

Banach Spaces

Def: X is a Banach Space if it is a vector space with a norm and it is complete.

Banach = complete normed

Hilbert $\subset$ Banach

Ex:
$C[0,1]$: Cont. (complex) Functions on $[0,1]$

Norm $\|f\| = \max_{x \in [0,1]} |f(x)|$ ← cont. on $[0,1]$ so attains sup

$[0,1]$ can be replaced with any top. space K.

Rem: Why a Banach Space?
Key is why complete.
$\{f_k\}$ Cauchy in X. $\forall x$
$\{f_k(x)\}$ is Cauchy of $\mathbb{C}$'s.
$|f_m(x) - f_n(x)| \le \|f_n - f_m\| < \epsilon$
So it has a limit. Call this lim $f(x)$. Need show $f(x)$ cont.
Need $f_n \to f$
show this. Then get f cont.

$\exists \; f_{n_k} \ni \|f_{n_k} - f_{n_{k+1}}\| \le 2^{-k}$

$$\underbrace{f = f_{n_1}}_{\text{cont.}} + \sum_{k=1}^{\infty} \underbrace{f_{n_{k+1}} - f_{n_k}}_{\sup|f_{n_k} - f_{n_k}| \le 2^{-k}} \; ; \text{ pointwise}$$

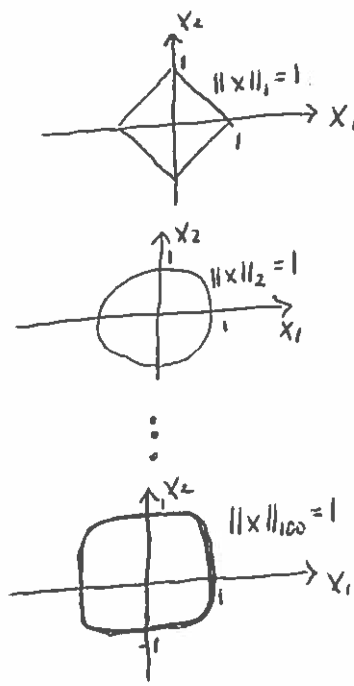
$\sum 2^{-k}$ conv. Weierstraß M-test
So $\sum f_{n_{k+1}} - f_{n_k}$ conv. uniformly
So conv. to f uniformly. So f cont. & $f_{n_k} \to f$ in $C[0,1]$.
So $f_n \to f$ as f_n cauchy.

Ex:
$\ell^p = \{x = (x_n) \mid \sum |x_n|^p < \infty\}$

$\|x\|_p = \left(\sum |x_n|^p\right)^{1/p}$

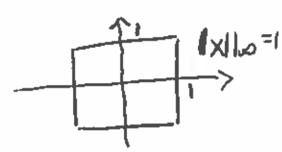
Norm of $1 \le p < \infty$

2D picture $\|x\|_p = 1$



$\ell^\infty = \{x \mid \sup |x_n| < \infty\}$

$\|x\|_\infty = \sup |x_n|$



$L^p[0,1] = \{f \mid \int_0^1 |f(x)|^p dx < \infty\}$

$\|f\|_p = \left(\int_0^1 |f|^p\right)^{1/p}$

$L^\infty[0,1] = \{f \mid \text{ess sup } |f| < \infty\}$

$\|f\|_\infty = \text{ess sup } |f| = \text{smallest } M \ni$
$|f| \le M$ a.e.

Nested

$\ell^p \subseteq \ell^q$ if $p \le q$

$(1, 1/2, 1/3, \dots) \in \ell^2 \setminus \ell^1$

[0,1] ↳ $L^p \supseteq L^q$ if $p \le q$

No relation between $L^p(\mathbb{R})$ & $L^q(\mathbb{R})$

Set theoretic, not as subspaces.

Operators

$T: X \to Y$ is bounded

$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} < \infty$

Ex: Shift on ℓ^p. Mult. on ℓ^p
L^p, $C[0,1]$.

Subspace

A closed linear subspace.
(a Banach space of its own).

Closed in complete → complete

C_{00} is not closed in ℓ^p

↳ Fin. many exceptions to $x_n = 0$

In fact, C_{00} dense in ℓ^p for
$1 \le p < \infty$ but not for $p = \infty$

Choose $x \in \ell^p$, $\forall \epsilon, \exists N \ni$

$\left(\sum_{N+1}^\infty |x_n|^p\right)^{1/p} < \epsilon$

Take $x' = (x_1, \dots, x_N, 0, \dots) \in C_{00}$

$\|x - x'\|_p < \epsilon$

But ℓ^∞ is different.

$x = (1, 1, 1, \dots)$

cannot be approx by anything in C_{00}.

$\|x - x'\|_\infty \ge 1 \quad \forall x' \in C_{00}$

Claim: $\overline{C_{00}} = C_0$ ({set of all seq. in $\ell^\infty \ni \lim x_n = 0$})

a) if $x \in C_0 \rightsquigarrow \forall \epsilon, \exists N \ni$
$|x_n| < \epsilon$ for all $n \ge N$ so
$\|x - (x_1, \dots, x_N, 0, \dots)\| < \epsilon$

So C_{00} is dense in C_0. Now just show C_0 closed...

b) C_0 closed: Prove complement open.
$x \in \ell^\infty \setminus C_0$. Then $\limsup_{n\to\infty} |x_n| > 0$

if $y \in C_0 \to \limsup |(x_n - y_n)| = \limsup |x_n| > 0$

So $\|x - y\| \ge \limsup |x_n|$, as desired.

$r = -$

& B_r disjoint from C_0

Def: $T: X \to Y$ is an iso. if linear bijection & T^{-1} bounded

Isometric iso: $\|Tx\|_Y = \|x\|_X$

T^{-1} may be unbounded when T is bounded bijection:

$T: C_{00} \to C_{00}$

$Tx = (x_n/n)$

$\underbrace{T^{-1}y}_{\text{not bounded}} = (ny_n)$

Such examples do not exist in Banach spaces.

3.2: Linear Operators between normed spaces

Def: $\|\cdot\|$ & $|||\cdot|||$ are equivalent if the identity map is an isomorphism: $\exists\, C_1, C_2 \ni$

$$\|x\| \le C_1 |||x||| \qquad \forall x$$
$$|||x||| \le C_2 \|x\|$$

So equivalent to id : $(X, \|\cdot\|) \to (X, |||\cdot|||)$ being a homeo.

$C^1[0,1]$ can be given norm

$$\left.\begin{array}{l} \|f\| = |f(0)| + \sup_{[0,1]} |f'| \\ \text{or} \\ \sup|f| + \sup|f'| \\ \text{or} \\ \max(|f(0)|, \sup|f'|) \end{array}\right\} \text{All equivalent}$$

How to tell X, Y Banach spaces are isomorphic? This generally hard

Ex: ℓ^p & ℓ^q with $p \neq q$ are not isomorphic.

We'll prove ℓ^p not iso to ℓ^∞.

Recall X is separable if $\exists$ countable dense subset. Isomorphisms preserve separability.

Ex: ℓ^p is separable

$C = \{x \in C_{00} \mid x_n \in \mathbb{Q}\}$

Countable clear. Dense: for $x \in \ell^2$, $\forall \epsilon > 0$ $\exists N \ni \left(\sum_{N+1}^{\infty} |x_n|^p\right)^{1/p} < \epsilon/2^p$. Can approximate $x_1, \dots, x_n$ by $y_i \in \mathbb{Q} \ni$

$$|x_i - y_i| < \frac{\epsilon}{2N^{p^{1/p}}}$$

$$\|x - (y_1, \dots, y_N, 0, \dots)\|_p < \left(\frac{\epsilon^p}{2^p N} + \frac{\epsilon^p}{2}\right)^{1/p}$$

However, ℓ^∞ is not separable.

$\{0,1 \text{ sequences}\}$ is uncountable.

distance between any two is $1 = \sup_n |x_n - y_n|$

Ex: If X & Y are homeo. spaces (both compact) then $C(X)$ & $C(Y)$ are isometrically iso.

PF: $\exists$ homeo. $\phi: X \to Y$ so $T: C(Y) \to C(X)$

$f \mapsto f \circ \phi$: $T^{-1}: C(X) \to C(Y)$

$f \mapsto f \circ \phi^{-1}$

The converse is also true

Banach - Stone Theorem

(proved in VI.2)

Ex: $L^p[0,1]$ & $L^p[0,2]$ are isometrically isomorphic

Change of variables

$x = u/2$; $u \in [0,2]$

$x = \phi(u)$

$$\int_0^2 |f(\phi(u))|^p \phi'(u) = \int_0^1 |f(x)|^p dx$$

$$f \in L^p[0,1] \longrightarrow \underbrace{2^{-1/p} f(x/2)}_{Tf} \in L^p[0,2]$$

Check $\|Tf\|_p = \|f\|_p$

Can even work for other non-obvious change of variables.

Mult. Operator

$\phi \in L^r$ is cont. from L^p to L^q provided $r = \frac{pq}{p-q}$ (Need fin. p, q, r)

Notice $p > q$

Pf: Hölder's Inequality: $\int |fg| \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^{p'}\right)^{1/p'}$ where $1/p' + 1/p = 1$ (p' conj. exp. to p).

Take $f \in L^p$

$$\int |\phi f|^q \leq \left(\int |f|^{\underbrace{q \frac{p}{q}}_{p}}\right)^{q/p} \left(\int |\phi|^{\underbrace{q/1-\frac{q}{p}}_{pq/p-q}}\right)^{1-\frac{q}{p}}$$

Now just recall L^p spaces are nested.

$TF(x) = f(x)^2$

Iso $L^2 \to L^1$?

No: Not linear + more

- $T2F = 4TF$
- $T1 = T(-1)$
- $\|TF\|_1 = \int |F|^2 = \|F\|_2^2 + \|F\|_2$

Fin. dim. Normed Spaces

Thm: Any two norms on n-dim. spaces are equivalent.

PF: Choose a basis $e_1, \dots, e_n$

For all $x \in V$, $x = \sum_1^n c_k e_k$. Let $\|x\|_\infty = \max_k |c_k|$. Need to show $\|\cdot\|$ is equiv. to $\|\cdot\|_\infty$.

$$\|x\| = \|\sum c_k e_k\| \le \sum \|c_k e_k\| \le \|x\|_\infty \underbrace{\sum_1^n \|e_k\|}_{C}$$

So $\|x\| \le C\|x\|_\infty$

Suppose $\nexists\, C > 0 \ni \|x\| \ge C\|x\|_\infty$. So $\exists$ x_j such that $\|x_j\| \to 0$ and $\|x_j\|_\infty = 1$

$x_j = \sum_1^n c_{jk} e_k$, where $\max_k |c_{jk}| = 1$ for all j.

Passing to a subsequence, we get $c_{jk} \to b_k$ as $j \to \infty$ for all $k = 1, \dots, n$. (Bounded so conv. subseq.). Then $\max_k |b_k| = 1$

Claim $x_j \to \sum_1^n b_k e_k$ in ∞-norm

$\|x_j - \sum b_k e_k\|_\infty = \max_k |c_{jk} - b_k| \to 0$

But $\|x\| \le C\|x\|_\infty$ so also in $\|\cdot\|$-norm

So $\underbrace{\|\sum_1^n b_k e_k\|}_{\to = 0} \le \|x_j - \sum_1^n b_k e_k\| + \|x_j\| \to 0$

So $b_k = 0$ for all k. $\Rightarrow\Leftarrow$ $\max_k |b_k| = 1$ ∎

Cor: Every n-dim. normed space is complete. ∎

PF: Must be iso to $\mathbb{R}^n$ which is complete.

So these are Banach Spaces.

Cor: Every Fin. dim. linear manifold M in a normed space is closed.

PF: Complete so closed.

Quantitative Study of Norms

[$\exists$ no norm for which C_{00} is complete]

↳ Hint: Baire Category Thm.

Sketch: $M_n \subset C_{00}$ be $\{x \mid x_k = 0 \text{ for } k > n\}$

Then M_n closed as fin. dim. M_n has empty interior $(x + \epsilon e_{n+1} \notin M_n)$.

$C_{00} = \bigcup_1^\infty M_n$. Baire Cat. Thm: a nonempty complete metric space is not countable union of closed empty interior sets.

↳ So in inf dim. Banach Spaces, every Hamel basis is uncountable.

Banach-Mazur Distance:

$$d_{BM}(X,Y) = \log \inf \{ \|T\| \|T^{-1}\| \mid T: X \xrightarrow{\text{iso}} Y \}$$

{n-dim. normed spaces} with d_{BM} form a compact metric space.

* 1 iff iso.

Claim:

$$d(\ell_n^1, \ell_n^2) = \log\sqrt{n}$$

Id: $Tx = x$

$\|T\| = 1$ as $\|x\|_2 \le \|x\|_1$

$\|T^{-1}\| = \sqrt{n}$ as $\|x\|_1 = \sum_i |x_i| \cdot 1$

$\le \sqrt{\sum_i |x_i|^2}\,\sqrt{n}$

Cauchy Sch. $= \sqrt{n}\,\|x\|_2$

So $d_{BM} \le \log\sqrt{n}$

Goes to ∞ as $n \to \infty$. So no isomorphism between them.

Est. from below:

$$\left\| \sum_1^n \pm e_k \right\|_1 = n$$

$$\frac{1}{2^n} \sum_{\pm} \left\| \sum_k \pm T e_k \right\|_2^2$$

$$= \sum_k \|T e_k\|_2^2$$

$$\le n \|T\|^2$$

$$\left\| \sum \pm T e_k \right\| \ge \frac{n}{\|T^{-1}\|}$$

So this avg $\ge \frac{n^2}{\|T^{-1}\|^2}$ so

$$\frac{n^2}{\|T^{-1}\|} \le n\|T\|^2 \text{ so } \|T\|\|T^{-1}\| \ge \sqrt{n}$$

Then $d_{BM}(\ell_n^1, \ell_n^2) = \log\sqrt{n}$

*This is as far as n dim. spaces can be from each other

Same proof gives $\ell^1 \not\cong \ell^2$

Boundedness

$\|x\| + \|Tx\|$ norm

$\underbrace{\|x\| + \|Tx\|}_{\text{all norms on f.d. equivalent}} \le C\|x\|$

§3.4 Products & Quotients of Normed Spaces

$$X \oplus Y = \{ (x,y) \mid x \in X, y \in Y \}$$

$$\|(x,y)\| \overset{?}{=} \begin{cases} \|x\| + \|y\| \\ \text{or} \\ \sqrt{\|x\|^2 + \|y\|^2} \\ \text{or} \\ \max\{\|x\|, \|y\|\} \end{cases}$$

Generally, introduce

$$\|(x,y)\|_p = \left(\|x\|^p + \|y\|^p\right)^{1/p}$$

where if $p = \infty$ means max, ie bottom one above.

Infinite Products:

$$\bigoplus_{n=1}^{\infty}{}_p X_n = \left\{ (x_1, x_2, \dots) \;\middle|\; \begin{array}{l} x_k \in X_k \\ \sum \|x_k\|^p < \infty \end{array} \right\}$$

If $p = \infty$, means $\sup \|x_k\| < \infty$

$$\|x\|_p = \left(\sum \|x_k\|^p\right)^{1/p}$$

$$\bigoplus_{n=1}^{\infty}{}_0 X_n = \{ (x_1, \dots, x_n, \dots) \mid \lim \|x_n\| = 0 \}$$

$$\bigoplus_{n=1}^{\infty}{}_\infty X_n$$

If each X_n is separable then

$\bigoplus_p$ is sep $1 \le p < \infty$ &

$\bigoplus_0$ is sep.

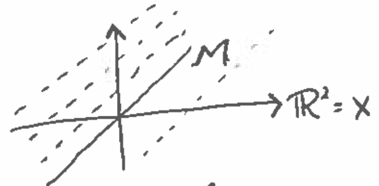
$\bigoplus_\infty$ is not separable.

$$\ell^p = \bigoplus_{n=1}^{\infty}{}_p \mathbb{C}$$

Quotients

If M is a closed linear subspace of X

Consider $X/M = \{x + M \mid x \in X\}$



all lines // to M

$$\|x + M\| = \inf_{y \in M} \|x + y\|$$
$$= \inf_{y \in M} \|x - y\|$$
$$= \operatorname{dist}(x, M)$$

Closed: Takes care of ≥ 0 & $= 0$ iff $= 0$.

Claim: If X complete, so is X/M

Suppose $\sum_{n=1}^{\infty} \|x_n + M\| < \infty$ (ab. conv. series)

Choose $y_n \in x_n + M \ni \|y_n\| < \|x_n + M\| + 1/2^n$

then $\sum \|y_n\| < \infty$. But space complete so $y = \sum y_n$ exists. Hence $\sum (x_n + M) = y + M$

$$\left\| \sum_{n=1}^{N} (x_n + M) - (y + M) \right\| \leq \|y_1 + \cdots + y_N - y\| \to 0 \quad \blacksquare$$

Note: $x + M$ also denoted $\bar{x}$ or $[x]$. But these don't make M explicit.

What if $\{y_n\}$ has no conv. subsequence? inf dim. spaces

Ex: $X = \ell^1$

$$M = \left\{ x \;\middle|\; \sum \tfrac{n}{n+1} x_n = 0 \right\}$$

$\frac{1}{2}x_1 + \frac{2}{3}x_2 + \frac{3}{4}x_3 + \cdots$

Claim: $\operatorname{dist}(e_1, M) = \frac{1}{2}$ not attained

$(1, 0, 0, \ldots)$ close to thing with sum 0.

dist 1 $(0, 0, \ldots)$

dist $3/4$ $(1, -3/4, 0, 0, \ldots)$

$$e_1 - \frac{n+1}{2n} e_n \in M$$

because $\frac{1}{2} \cdot 1 + \frac{n}{n+1}\left(-\frac{n+1}{2n}\right) = 0$

$$\left\|e_1 - \left(e_1 - \frac{n+1}{2n} e_n\right)\right\| \to 1/2$$

$$\operatorname{dist}(e_1, M) \leq 1/2$$

remains to show $\nexists\, x \in M \ni$

$$\|e_1 - x\| \leq 1/2$$

$$y = e_1 - x$$

$$\sum \frac{n}{n+1} y_n = 1/2$$

$$\underbrace{\sum \frac{n}{n+1} \text{"}e_1\text{"}}_{= 1/2} + \underbrace{\sum \frac{n}{n+1} x(n)}_{= 0}$$

Hence $\sum_{n=1}^{\infty} |y_n| > \sum_{1}^{\infty} \frac{n}{n+1} |y_n|$

$$\geq \sum \frac{n}{n+1} y_n = 1/2$$

$\|x + M\|$ is attained $\forall$ subspaces ℓ^p; $1 < p < \infty$ but for ℓ^∞?

(Can't recall)

Quotient Map

$X \to X/M$

$x \mapsto x+M$

lin. operator : Q

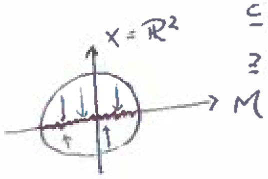
$\|Q\| \le 1$ In fact,

$\|Q\| = 1$ unless $M = X$

$Q(\{x \mid \|x\|<1\})$ = open unit ball in X/M

$\subseteq$: because $\|Q\| \le 1$

$\supseteq$: because $\|x+M\|<1$, $\exists$ $y \in x+M$ with $\|y\|<1$



False for closed unit balls. Can have

$\|x+M\| = 1$ but $\|y\| > 1$ for all $y \in x+M$

$\ell^1/M \cong \mathbb{C}$

as for all $x \in \ell^1$

$x = \alpha e_1 + y \in M$

$e_1 + M$ spans X/M

X sep $\to$ X/M sep.

$M, X/M$ sep $\to$ X sep

(M closed lin. subspace)

Linear Functionals ; Dual Spaces

Linear functional on X is a cont. lin. op. $F: X \to \mathbb{C}$.

Dual Space of X: $X^* = B(X, \mathbb{C})$

Ker F is a closed subspace (hyperplane)

Claim: $\ker F = \ker g \Rightarrow F = \alpha g$ for $\alpha \in \mathbb{C}^*$

PF: Suppose $\ker F \subsetneq X$. Take $x \in X \setminus \ker F$

Let $\alpha = F(x)/g(x)$. Then $F - \alpha g = 0$ on $\ker F \cup \{x\}$. But the span of $\ker F \cup \{x\}$ is X so $F - \alpha g = 0$

$F: X \to \mathbb{C}$ lin. then spanned by $\ker F$ & $x \notin \ker F$

PF: For $z \in X$, have $z = \alpha x + \underbrace{(z - \alpha x)}_{\in \ker F}$ if $\alpha = \frac{F(z)}{F(x)}$

Hence hyperplane. Throw in another variable and span everything

Claim: F cont. $\Leftrightarrow$ $\ker F$ closed

F disc. $\Leftrightarrow$ $\ker F$ is dense (not all of X)

F cont $\to$ $\ker F = F^{-1}(0)$ closed

F not cont $\to$ $\ker F \neq X$, $\exists$ seq unit vectors $\ni$ $F(x_n) \to \infty$ (F not bounded). For every $y \in X$

$y - F(y)/F(x_n)\, x_n \in \ker F \to y$

Thm: X^* is complete for every normed space X.

$\left(\|f\|_{X^*} = \sup_{\|x\| \le 1} |f(x)| \right)$

Pf: (sketch) $\{f_n\}$ Cauchy cts functions on $\overline{B}_1$.

Then for all $x \in \overline{B}_1$. Then $\{f_n(x)\}$ Cauchy so conv. $f_n(x) \to f(x)$ ← $\mathbb{C}$ is complete

+ everywhere in X

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\| \, \|x\|$$

Then like Weierstrass M-Test.

f lin. $\|f_n - f\| \to 0$ by same arg. as for $C[0,1]$ (Applied on $\overline{B}_1$ instead of $[0,1]$).

Table of Dual Spaces

	X	X^*
$1 \le p < \infty$	ℓ^p	ℓ^q ; $1/p + 1/q = 1$
	ℓ^∞	super-ugly (non-sep.)
	~~c_0~~	~~ℓ^∞~~
	ℓ^1	ℓ^∞
	L same as	for ℓ
	c_0	ℓ^1
	nothing	$L^1[0,1]$
	$C(K)$	{complex signed measures on K}

compact

isometric isomorphisms

Strategy: Find set $\{e_k\}$ in X such that $\overline{\text{span}\{e_k\}} = X$

Then every $f \in X^*$ is det. by $\{f(e_k)\}$.

Pf $c_0^* = \ell^1$

$\{e_1, e_2, \dots\}$ have dense span

Given $f \in c_0^*$. Let $y_n = f(e_n)$

$f(x) = \sum x_n y_n$ by lin. $\forall x \in c_{00} \subset c_0$

Claim $\sum_{n=1}^{\infty} |y_n| \le \|f\|$

For all N, let $x_n = \overline{y_n}/|y_n|$ if $y_n \ne 0$, 0 if $y_n = 0$ (and want $1 \le n \le N$, 0 otherwise)

These x's unimodular

Then $f(x) = \sum_1^N x_n y_n$

$= \sum_1^N \frac{y_n \overline{y_n}}{|y_n|}$

$= \sum_1^N |y_n| \le \|f\| \underbrace{\|x\|}_{=1}$

So have map $c_0^* \to \ell^1$ via $f \mapsto \{f(e_n)\}$

Onto: $\forall y \in \ell^1$. Let $f(x) = \sum x_n y_n$ converges as $|f(x)| \le \|x\|_{c_0} \|y\|_{\ell^1}$

This also shows norm preserving: $\|f\|_{c_0^*} = \|y\|_{\ell^1}$

Generally,

1) Map $X^* \xrightarrow{T} Y$

Show $\|T\| \le 1$

2) Show onto & isometry

ℓ^p; $1 < p < \infty$

$f \in (\ell^p)^*$. Let $y_n = f(e_n)$

Claim $y \in \ell^q$ & $\|y\|_q \le \|f\|$

We know $f(x) = \sum_1 x_n y_n$ for all $x \in c_{00}$

Given N, let $x_n = \begin{cases} \bar{y}_n |y_n|^{q-2} & x_n \neq 0 \text{ \& } n \le N \\ 0, & \text{otherwise} \end{cases}$

$$f(x) = \sum_1 \frac{\bar{y}_n y_n}{|y_n|^{2-q}}$$

$$= \sum_1^N |y_n|^q$$

Also $|f(x)| \le \|f\| \cdot \|x\|$

$$\|x\| = \left(\sum_1 |x_n|^p\right)^{1/p}$$

$$= \left(\sum_1 |y_n|^{(q-1)p}\right)^{1/p}$$

But $p = q/q-1$

$$= \left(\sum_1 |y_n|^q\right)^{1/p}$$

So $\sum_1 |y_n|^q \le \|f\| \left(\sum |y_n|^q\right)^{1/p}$

$$\longrightarrow \left(\sum_1 |y_n|^q\right)^{1/q} \le \|f\|$$

3.6: Hahn-Banach Theorem

Functions $X \to Y$ are subsets of $X \times Y$. Also $A \to Y$, where $A \subset X$

Subsets partially ordered by inclusion. Moreover,

$f \leq g$; f restriction of g.
or
g extension of f

Observe: if $\{f_\alpha\}$ is a totally ordered family of functions then $\bigcup_\alpha f_\alpha$ is also a function.

McShane-Whitney Extension Theorem

Suppose $A \subset \mathbb{R}$ & $f: A \to \mathbb{R}$.
Suppose $\exists L \ni |f(x) - f(y)| \leq L|x-y|$ $\forall x, y \in A$ (Lipschitz function). Then $\exists$ extension $F: \mathbb{R} \to \mathbb{R}$ which is also Lipschitz with constant L.

Pf: Choose $x \in \mathbb{R} \setminus A$ ($A = \mathbb{R}$ done).

min max

For all $a, b \in A$: $* \; f(a) - L|x-a| \leq f(b) + L|x-b|$

as $f(a) - f(b) \leq L|a-b| \leq L|x-a| + L|x-b|$

Let $\alpha = \sup_{a \in A} (f(a) - L|x-a|)$; $\beta = \inf_{b \in B} (f(b) + L|x-b|)$

Then $\alpha \leq \beta$

trying to extend to

Let $f(x)$ be a number in $[\alpha, \beta]$. Observe f is now Lipschitz (with constant L) on $A \cup \{x\}$

$\{L$ Lipschitz ext of $f\}$ has a max element by Zorn's Lemma. Since if miss a point in $\mathbb{R}$, can extend. means domain of max element is all of $\mathbb{R}$. ∎

Hahn-Banach Thm (VI): Let X be a normed space. M lin. manifold $f: M \to \mathbb{C}$ a bounded lin. functional, ie $|f(x)| \leq \|f\| \, \|x\|$

Then $\exists F: X \to \mathbb{C} \ni F|_M = f$ and $\|F\| = \|f\|$, F bounded lin funct.

Pf: $L := \|f\|$. Choose $x \notin M$ (otherwise done). Let $\alpha = \sup_{y \in M} f(y) - L\|x-y\|$ and $\beta = \inf (f(y) + L\|x-y\|)$

Again, $\alpha \leq \beta$

$[f(y) - L\|x-y\| \leq f(z) + L\|x-z\|]$

Let $f(x)$ be number in $[\alpha, \beta]$

Also

$f(tx+y) := tf(x) + f(y)$ $\forall t \in \mathbb{C}$ $\forall y \in M$

So f now defined on span$(M \cup \{x\})$

Need to prove

$|f(tx+y)| \leq L\|tx+y\|$

Equivalent to

$$|f(x) + f(y/t)| \le L\|x + y/t\|$$

This is true as...

$$f(x) \le f(-y/t) + L\|x + y/t\|$$

$[-y/t \in M]$

$$\ge f(-y/t) - L\|x + y/t\|$$

So f can be extended to larger lin. manifold with same norm.

$\{$all lin. ext. of f with norm $L\}$

satisfies assumptions of Zorn's lemma

so max element but as before domain now X. $\blacksquare$

Ex: $M = C[0,1]$; $X = L^\infty[0,1]$

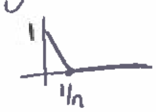
Define $f: M \to \mathbb{C}$ via $f(\phi) = \phi(0)$

f lin. function w/ norm 1. By H.B.

$\exists\ F \in (L^\infty[0,1])^*$ extending f.

This F is not of form $F(\phi) = \int_0^1 \phi\psi$

$\psi \in L^1[0,1]$: Suppose ψ exists.

Let $\phi_n(m) =$  . Clearly $\phi_n \to 0$

and bounded by 1. So $\phi_n\psi \to 0$ a.e.

and $|\phi_n\psi| \le |\psi|$. Lebesgue Dom. Conv. thm. $\int_0^1 \phi_n\psi \to 0$ contradicting

$F(\phi_n) = 1$.

Ex: (Norming Functionals)

Let X be normed. $x \in X$

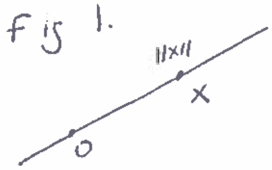
Then $\exists\ f \in X^* \ni \|f\| = 1$ and

$f(x) = \|x\|$

Pf: $M = \text{span}(x)$. Suppose $x \neq 0$. Define

$f(x) = \|x\|$ so $f(tx) = |t|\,\|x\|$

Norm of f is 1.



Extend by HB $\blacksquare$

Before used real scalars:

$$\sup_{y \in M} \left(f(y) - L\|x - y\|\right)$$

M manifold in X

$f: M \to \mathbb{C}$ lin. manifold

Need $F: X \to \mathbb{C}$; $\|F\| = \|f\|$

Pf: $f_1 = \text{Re}\, f$ ($\mathbb{R}$-lin.)

Extend $\hat{f_1}$ to $F_1: X \to \mathbb{R}$

Let $F(x) = F_1(x) - iF_1(ix)$

Claim: F is $\mathbb{C}$-lin

$$F((a+ib)x) = F_1((a+ib)x) - iF_1((ia-b)x)$$

$$= aF_1(x) + bF_1(ix) - iaF_1(x) + ibF_1(x)$$

$$= (a+ib)(F_1(x) - iF_1(ix))$$

$$= (a+ib)F(x)$$

Claim²: $\|F\| = \|F_1\| \overset{H.B.}{=} \|f_1\| = \|f\|$

$$\sup_{\|x\|\le 1} |Re\, f(x)| = \sup_{\substack{\|x\|\le 1 \\ 0\le\theta\le 2\pi}} |Re\, f(e^{i\theta}x)|$$

$\|Re\, f\|$

$$= \sup_{\substack{\|x\|\le 1 \\ 0\le\theta\le 2\pi}} |\underbrace{Re\, e^{i\theta} f(x)}_{|f(x)|}|$$

$$= \sup_{\|x\|\le 1} |f(x)|$$

$$= \|f\|$$

Claim: $F|_M = f$.

$Re(F|_M - f) = 0$ so by Claim² above, $F|_M - f = 0$

Thm: Suppose M lin. subspace of X and $x_0 \notin M$. Then $\exists\, f \in X^*$ such that $f(x_0) = 1$, $f|_M = 0$,

$$\|f\| = \frac{1}{dist(x_0, M)}$$

Pf: $x_0 + M \ne 0$ in quotient space as $\|x_0 + M\| = dist(x_0, M)$

Let g be a norming functional for $x_0 + M$

$\|g\| = 1$: $g(x_0 + M) = \|x_0 + M\|$

$(g \in (X/M)^*)$ Have quotient map

$$X \xrightarrow{\pi} X/M$$
$$x \mapsto x + M$$

Let $f := \dfrac{g\pi}{\|x_0 + M\|}$

$f(x_0) = 1$ ✓

$f|_M = 0$ ✓

$\|f\| = \frac{1}{dist}$ ✓ $(\|g\| = 1)$

☒

Ex: $\exists\, 0 \ne f \in (L^\infty[0,1])^*$ which vanishes on all cont. functions

3.7 Banach Limits

(Assign limit to every bounded seq.)

Def: $L \in (\ell^\infty)^*$ is a Bounded Banach limit if:

- $L(\{x\}) = \lim x_n$ if lim exists
- $\|L\| = 1$
- $Lx \ge 0$ if $x_n \ge 0$
- $L(\{x_2, x_3, \dots\}) = Lx$

Consider $M = \{(x_n - x_{n+1})_{n=1}^\infty : x \in \ell^\infty\}$ a lin. manifold

$= \{(y_n) \mid \text{partial sums bounded}\}$

We want $L|_M = 0$

Let $C = (1, 1, \cdots)$

Note: $dist(C, M) = 1$ (assume real scalars) $\uparrow$

≤ 1 as $\|C\| = 1$

but if $y \ni \underbrace{\|y - C\|}_{d} < 1$ then

$y_n \geq 1 - d > 0$. So partial sums are unbounded so $y \notin M$.

Apply previous theorem to $x_0 = C$ and $\overline{M}$. Get $\mathcal{L} \ni \mathcal{L}(C) = 1$

$\mathcal{L}|_M = 0$, $\|\mathcal{L}\| = 1$. If $x_n \geq 0$

then let $y_n = \|x\| - x_n$. Since $\|y\| \leq \|x\|$, we have $\mathcal{L}(y) \leq \|x\|$

But $\mathcal{L}$ lin so $\|x\| \mathcal{L}(C) - \mathcal{L}(x)$

Hence, $\mathcal{L}(x) \geq 0$.

As a consequence of this property, we get a 'Squeeze Thm'

If $a \leq x_n \leq b \rightarrow a \leq \mathcal{L}(x_n) \leq b$

as $\mathcal{L}(x_n - a) \geq 0$ & etc.

$\uparrow$ conj. seq.

$x_n \to \alpha$ then $\exists N \ni$

$\alpha - \epsilon \leq x_n \leq \alpha + \epsilon$ then

$\alpha - \epsilon \leq \mathcal{L}x \leq \alpha + \epsilon$ so

$\mathcal{L}x = \alpha$ ☒

Same idea could work for $\ell^\infty(\mathbb{Z})$. Get translation invariant $\mathcal{L}$. $\mathcal{L}S = \mathcal{L}$. Gives an invariant mean. Gives a "measure" on $\mathbb{Z}$ $\ni$

$\mu(\mathbb{Z}) = 1$

$\mu(A + K) = \mu(A)$

Namely,

$\mu(A) = \mathcal{L}(\chi_A) \in [0, 1]$

Finitely additive

(Banach) Adjoint Operator

Given $T : X \to Y$

$T^* : Y^* \to X^*$ via

$T(f) = f \circ T$

Differences from Hilbert Adjoint

- $(\alpha T)^* = \alpha T^*$ for 'B'-adj

but for H-adj. $(\alpha T)^* = \bar{\alpha} T^*$

- $dom\, T^* = Y^*$ B-adj.

$= Y$ for H-adj.

- $T^* T$ not def. for B-adj.

- $(T^*)^* \neq T$ for B-adj.

Dual of subspace M of normed space X.

Def: $M^\perp = \{ f \in X^* \mid f|_M = 0 \}$

$\hookrightarrow$ ann. of M

If $N \subset X^*$. ${}^\perp N = \{ x \in X \mid f(x) = 0 \ \forall f \in N \}$

$\hookrightarrow$ pre-ann. of N

Thm: $M^* \overset{\text{isometric}}{\cong} X^*/M^\perp$ & $(X/M)^* \overset{\text{isometric}}{\cong} M^\perp$

Obv iso. just check

ker $M^\perp$

$i: M \to X$ incl.

$i^*: X^* \to M^*$ rest.

ker $M^\perp$

$X^*/M^\perp \cong M^*$

Onto: Hahn Banach

Same Norm: $f \in X^*$

$\| f + M^\perp \| \geq \| f|_M \|$

$\hookrightarrow$ by def

$\leq$ by H.B.

Extend $f|_M$ to F w/ same norm

$F \in f + M^\perp$ as same values on M.

$\pi: X \to X/M$

$\pi^*: (X/M)^* \to X^*$

ran $\pi^* = \{ f \circ \pi : f \in (X/M)^* \}$

$\subseteq M^\perp$

For $\supseteq$: $g \in M^\perp$ let $f(x+M) = g(x)$ ~~[illegible]~~ (1)

then $f \in (X/M)^*$

π^* isometric: $\| \pi^* g \| = \| g \|$

X^{**}, bidual of X.

$e: X \to X^{**}$

$x \mapsto ev_x$

$\| ev_x \| = \sup_{\|f\| \leq 1} | f(x) |$

$= \| x \|$

Def: X is reflexive if e onto.

Ref.	Non-reflexive
ℓ^p, L^p; $1<p<\infty$	$\ell^1, \ell^\infty, L^1, L^\infty$
fin. dim.	$C[0,1]$
	c_0
	$B(H)$

If $1<p<\infty$ $\forall f \in (L^p)^*$

$\exists g \in L^q \ni f(\phi) = \int \phi g$

$(L^p)^* = L^q$. So $f \in (L^q)^*$

have $g \in L^p \ni f = ev_g$

$$\begin{cases} c_0^* = \ell^1, \quad \ell^{1*} = \ell^\infty \\ e: \underset{\text{sep}}{c_0} \to \underset{\text{not sep.}}{\ell^\infty} \text{ is inclusion} \end{cases}$$

Anything: max or sum in norm is then non-reflexive.

X^* reflexive $\to$ X^{**} reflexive $\to$ X reflexive ($e: X \to X^{**}$)

X ref. $\rightleftarrows$ X^* ref.

$e: X \to X^{**}$ isom iso.

$\Phi \in X^{***}$

WTS every $\Phi: X^{**} \to \mathbb{C}$ is evaluation @ $x \in X^*$

Φ can be identified with $f \in X^*$; $f = \Phi \circ e_X$

let

for $g \in X^{**}$ ~~$\Phi(g) = g(f)$ so $\Phi = ev_f$~~

have $g = e_X(x)$

$\Phi(g) = \Phi(e_X(x)) = f(x) = g(f)$

so $\Phi = ev_f$ for $f \in X^*$

3.12a: Open Mapping Theorem

Notation: $B_X(r) = \{x \in X \mid \|x\| < r\}$

Thm: A surj. $T \in B(X,Y)$ is an open map, where X,Y Banach spaces. More precisely, $\exists r > 0$ $\ni T(B_X(1)) \supseteq B_Y(r)$. Equivalently, every $y \in Y$ has preimage x of norm $\leq \|y\|/r$

Consequences:

*** 1) A cont. bij. lin. op. is an iso.

2) $\|T^* y^*\| \geq r \|y^*\| \quad \forall y^* \in Y^*$

So T^* has a lower bound so inj.:

$$\|T^* y^*\| = \sup_{\|x\| \leq 1} |T^* y^*(x)|$$
$$= \sup_{\|x\| \leq 1} |y^*(Tx)|$$
$$\geq \sup_{\|y\| < r} |y^*(y)|$$
$$= r\|y^*\|$$

Fails without completeness:

$Tx = (x_n/n)$ as $c_{00} \to c_{00}$; ℓ^2 norm

$T^{-1}x = (nx_n)$ is not bounded

Pf (Thm):

Part 1: $Y = TX = \bigcup_{n=1}^{\infty} \overline{TB_X(n)}$

Countable union of closed sets. So by Baire Cat. Thm, $\exists n \ni$

$\overline{TB_X(n)} \supseteq y_0 + B_Y(r)$

for some center $y_0 \in Y$, $r > 0$.

$$B_Y(r) \subseteq \overline{TB_X(n)} - y_0 \quad \leftarrow \text{translate}$$
$$= \overline{T(B_X(n) - x_0)} \quad \leftarrow \text{trans. b/f mapping}$$
$$\subseteq \overline{TB_X(n + \|x_0\|)} \quad \leftarrow \text{cont. in another bigger ball}$$

By scaling, $\exists c > 0 \ni$ (r changed app.)

$B_Y(cr) \subseteq \overline{TB_X(r)} \quad \forall r > 0$.

[Thus far have used completeness of Y]

$\hookrightarrow$ B.C.T.

Part 2: Claim $\overline{TB(\varepsilon)} \subseteq TB_X(2\varepsilon)$

Take $y_1 \in \overline{TB_X(\varepsilon)}$. $\exists x_1 \in B_X(\varepsilon)$

$\ni \|y_1 - Tx_1\| < c\varepsilon/2$

$y_2 := y_1 - Tx_1 \in B_Y(c\varepsilon/2) \subseteq \overline{TB_X(\varepsilon/2)}$

$\vdots$

Replace $x_2 \in B_X(\varepsilon/2) \ni \|y_2 - Tx_2\| < c\varepsilon/4$ and continue. We get $\|x_n\| < \varepsilon/2^{n-1}$

So $x = \sum_{n=1}^{\infty} x_n \in B_X(2\varepsilon)$

completeness

$Tx = \sum_{n=1}^{\infty} Tx_n = y_1$ as

$\underbrace{\underbrace{\underbrace{y_1 - Tx_1}_{y_2} - \cdots}_{y_3 \;\vdots} - Tx_n}_{y_{n+1}} \to 0$ as $\|y_n\| \leq c\varepsilon/2^{n-1}$

(cont. in app. balls)

∎

Rem: T surj. $\rightsquigarrow T^*$ inj.

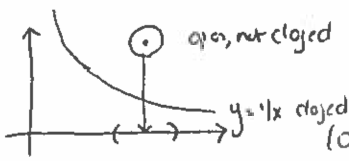
T inj. $\not\rightsquigarrow T^*$ surj.

$Tx = (x_n/n)$ in ℓ^2.

$T \in B(X,Y)$; T open $\rightsquigarrow$ T surjective

So then T open $\rightleftarrows$ T surjective

T closed map: For all closed $A \subset X$, TA closed in Y.



T closed $\rightleftarrows$ T lower bound OR $T = 0$

Ker T, ran T not trivial $\rightsquigarrow$ not closed by hyperbola ex.

Must have ran T closed.

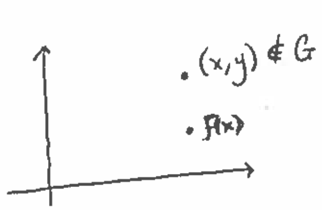
Ker $T = 0$ & ran T closed $\rightleftarrows$ T has lower bound

3.12 Closed Graph Theorem

$f: X \to Y$ Hausdorff top. spaces

has closed graph if $G = \{(x, f(x)) \mid x \in X\}$ closed in $X \times Y$.

Cont. $\rightsquigarrow$ Closed graph



$y \neq f(x)$. Then disjoint neigh. $f(x) \in V$, $y \in \mathcal{U}$ in Y. f cont., $\exists\, W \ni x$ $\ni f(W) \subset V$. Then $W \times \mathcal{U} \subset X \times Y$ is open neigh cont. (x,y) & not int. G.

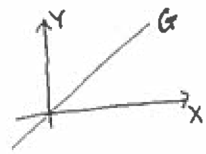
In general, closed graph $\not\Rightarrow$ cont.

$f(x) = 1/x$; $f(0) = 0$ $\quad \mathbb{R} \to \mathbb{R}$

G is closed.

Closed Graph Thm: If X, Y are Banach and $X \xrightarrow{T} Y$ is a lin op. w/ closed graph then T is bounded

Pf: $G \subset X \oplus Y$ (use norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$ on $X \oplus Y$)



So G lin. manifold so subspace. Hence G Banach.

$\pi_X : G \to X$; $\pi_X(x,y) = x$

bij. & $\|\pi_X\| \le 1$. By Open Mapping Thm, $(\exists\, r \ni \|\pi_X(x,y)\| \ge r \|(x,y)\|)$

So π_X^{-1} cont. So $\pi^{-1}(x) = (x, Tx)$ so T is cont. $\boxed{}$ $\hookrightarrow \Rightarrow$ bounded.

Goal $\left\{ \begin{array}{l} \text{To prove } T: X \to Y \text{ cont, suffices to } \\ \text{show if } x_n \to x \text{ in } X \\ \text{and } \overset{Tx_n}{y_n} \to y \text{ in } Y \Rightarrow y = Tx \end{array} \right.$

Compare to if $x_n \to x \Rightarrow Tx_n \to Tx$.

Suffices to show $Tx_n \to Tx$ in some weaker sense.

Ex: $p \in [1, \infty]$, $A = (a_{ij})$ is $\ni \forall x \in \ell^p$ $y_i = \sum a_{ij} x_j$ conv. & $y \in \ell^p$

Claim $x \to Ax$ bounded op. $\ell^p \to \ell^p$

Lem: $x \notin \ell^q$; $\frac{1}{p} + \frac{1}{q} = 1$. Then $\exists y \in \ell^p$ $\ni \sum_{k=1}^{\infty} x(k)y(k)$ diverges

Sketch: $x = (\underbrace{x_1, \dots, x_{b_1}}_{\text{1st block}}, \underbrace{x_{b_1+1}, \dots, x_{b_2}}_{\text{2nd block}}, \dots)$

ℓ^q norm > 2 $\qquad$ ℓ^q norm > 4 $\qquad \dots$

let $y = (\underbrace{y_1, \dots, y_{b_1}}_{}, \dots)$

ℓ^p norm 2^{-1}

$$\sum_{1}^{b_1} x_k y_k \geq 1$$

Now for claim: By lem, every row of A is in ℓ^q. Suppose $x_n \to x$ in ℓ^p & $Ax_n \to y$ in ℓ^p.

Must show $Ax = y$. But $Ax_n \to Ax$ coordinatewise. Since each coord of Ax is $\sum a_{ij} x(j)$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix}$$

So $y = Ax$.

Complemented Subspaces

In a Banach space, a subspace M has no 'canonical' complement.

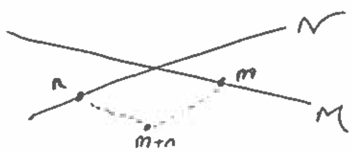
M, N comp. subspaces if $\forall x$ $\exists!\, m \in M \ni x = m + n$, $\exists!\, n \in N$

Thm: If M, N complementary then $X \cong M \oplus N$ $\quad (\|(m,n)\| = \|m\| + \|n\|)$

Pf: $T: M \oplus N \to X$

$T((m,n)) = m + n$

T bijective : $\|T\| \leq 1$ by Δ-ineq.



By OMT, T cont.

$C(\|m\| + \|n\|) \leq \|m+n\| \leq \|m\| + \|n\|$. ∎

Ex: ℓ^p ; $M = \{x \mid x(2n) = 0\}$

$N = \{x \mid x(2n-1) = 0\}$

$C[0,1]$ or $L^p[0,1]$: $M = \{f \mid f|_{[0,\alpha]} = 0\}$

$N = \{f \mid f|_{[\alpha,1]} = 0\}$

is not an example in $C[0,1]$ as Misses functions cont at α.

$X, \{0\}$ is allowed.

* { Given M subspace of X when can we find subspace $N \ni$ M, N complementary. } There is a lin. manifold $V \ni M + V = X$

and $M \cap V = \{0\}$: Hamel basis for M extend to hamel basis for X

Let $V = \text{span}\langle$added elements$\rangle$

This V generally not closed so not of much help.

Def: M is complementary if N such N exists in *.

So comp. iff can proj onto it.

Thm: M complementary $\Leftrightarrow$ $\exists$ idempotent $E \in \mathcal{B}(X) \ni \operatorname{ran} E = M$. May have to proj e.g.

Pf: $\Rightarrow$: $M \oplus N \cong X$. idempotent clear. Bounded clear. ran id $= M$ (Note $\|E\|$ may be > 1)

$\Leftarrow$: $N := \operatorname{Ker} E$.

$x = x + Ex - Ex = \underbrace{x - Ex}_{N} + \underbrace{Ex}_{M}$ unique

Rem: fin. dim Banach spaces are complemented

Thm: c_0 is not complemented in ℓ^∞.

Lem: $\{A_i \mid i \in I\}$, ~~uncountable~~ $\exists$ uncantable

$A_i \subset \mathbb{N}$ inf.

$A_i \cap A_j$ fin $\forall i \neq j$

Pf: $\mathbb{N} \xleftrightarrow{bij} \mathbb{Q} \longrightarrow \mathbb{R} \setminus \mathbb{Q}$

$I = \mathbb{R} \setminus \mathbb{Q}$ uncantable.

$\forall i \in I$, pick seq. of rationals

conv. to i.

$A_i = \{$ set of elements $\}$

Uniform Boundedness Principle

(For more of top. vector spaces, see Rudin's Funct. Analysis)

$M = \{f \in C[0,1] \mid f(0) = 0\}$

$f + M$ consists (contains) $f(0)$

$\downarrow$

$N := \{f \in C[0,1] \mid f \equiv \text{constant}\}$

Idea choose rep. such that set of such is subspace. So if M 2 fixed points then N set. lin. functions and so on.

$\times$

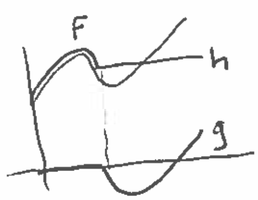
$M = \{f \in C[0,1] \mid f|_{[0,1/2]} \equiv 0\}$

$N = \{g \in C[0,1] \mid g|_{[1/2,1)} \text{ constant}\}$

$M \cap N =$ 0 on $[0,1/2]$ constant $[1/2,1]$ so 0.

$f = g + h$; $h = \begin{cases} f, & [0,1/2] \\ f(1/2), & [1/2,1] \end{cases}$

$g := f - h \in M$



Thm: Suppose X is Banach, Y normed. If $A \subset B(X,Y)$ pointwise bounded (meaning $\sup_{T \in A} \|Tx\| < \infty \ \forall x \in X$) then it is normed bounded (meaning $\sup_{T\in A} \|T\| < \infty$).

Conv. trivial

Rem 1: $\hat{Y}$ = completion of Y. Any map into Y is map to $\hat{Y}$ so just replace with $\hat{Y}$.

Rem 2: Fails if X is not complete

$X = c_{00}$; $Y = \mathbb{C}$

$T_n(x) = n x_n$: $\|T_n\| = n$

T_n ptwise bounded: $\{|T_n(x)|; n \in \mathbb{N}\}$ $\exists N \ni x_n = 0 \ \forall n > N$

$\leq N \sup_{n \le N} |x_n|$

Cor: If $\{T_n\} \subset B(X,Y)$ are such that $\lim_{n\to\infty} T_n(x)$ exists $\forall x \in X$ $\Rightarrow$ $Tx := \lim_{n\to\infty} T_n x$ is in $B(X,Y)$

⌈Ptwise conv. preserves cont.⌋

Pf: $\{T_n\}$ is pointwise bounded $\rightarrow \exists M \ni \|T_n\| \le M \ \forall n$ so $\|Tx\| \le M\|x\|$ since $\|T_n x\| \le M \|x\|$

Rem: In general, under these assumptions $\|T_n - T\| \not\to 0$.

Let $T_n = (S^*)^n$ on ℓ^2

$\hookrightarrow$ backwards shift

Obs $T_n x \to 0$ $\forall x$ but

$\|T_n\| = 1$ $\forall n$. So conv.

but not conv. in norm.

$T_n(e_k) = e_{k-n}$ $\forall k > n$

seq. in $\ell^2 \to 0$. So all cropping enough, left w/ 0.

Pf (Thm):

$$M(x) = \sup_{T \in A} \|Tx\|$$

Suppose to the contrary that A not norm bounded so can choose $x_n \in X$ $\ni$ $T_n \in A$

$x_n \to 0$ $\ni$ $\|T_n x_n\| \to \infty$ 'as fast as we want'? Idea let's $x = \sum x_n$ and show $\|T_n x\| \to \infty$

$$\|T_n x\| \geq \|T_n x_n\| - \overset{①}{\sum_{k<n} \|T_n x_k\|} - \overset{②}{\sum_{k>n} \|T_n x_k\|}$$

$\to \infty$ as fast as desire

Take suff. 'small' x_n so that x conv.

① $\leq \sum_{k<n} M(x_k)$: so choose T_n, x_n $\ni$

$$\|T_n x_n\| \geq n + \sum_{k<n} M(x_k)$$

To control ②, choose

$$\|x_n\| \leq \frac{2^{-n}}{\max_{k \leq n} \|T_k\|}$$

Then ② $\leq \sum_{k>n} \|T_n\| \|x_k\|$

$$\leq \sum \|T_n\| \frac{2^{-k}}{\|T_n\|}$$

$$\leq \sum 2^{-k} < 1$$

So $\|T_n x\| > n - 1$, $\Rightarrow\Leftarrow$ $\square$

Topological Vector Spaces

Def: X is a TVS if it is a vector space with a topology τ such that add. & scaler mult. cont.

$$X \times X \to X$$
$$a \quad b \mapsto a+b$$
$$\mathbb{F} \times X \to X$$
$$c \quad x \mapsto cx$$

Triv. Ex: $\tau = \{\phi, X\}$

Not even Hausdorff

$\tau = \{P(x)\}$

$\hookrightarrow$ cont. fails

- Every normed space is a TVS

$\forall a,b \in X \quad \forall \epsilon > 0 \quad \exists \delta \ni$

$N_\delta(x) + N_\delta(y) \subseteq N_\epsilon(x+y)$

by def $\delta = \epsilon/2$

$\forall \alpha \in \mathbb{C}, \ \forall x \in X \quad \forall \epsilon > 0 \ \exists \delta > 0$

$$\ni \quad \underset{\beta}{\underset{\shortparallel}{N_\delta(\alpha)}} \cdot \underset{y}{\underset{\shortparallel}{N_\delta(x)}} \subseteq N_\epsilon(\alpha x)$$

$$\begin{aligned} \|\beta y - \alpha x\| &\leq \|\beta y - \beta x\| + \|\beta x - \alpha x\| \\ &\leq |\beta| \delta + \delta \|x\| \\ &\leq (|\alpha| + \delta)\delta + \delta \|x\| \\ &< \epsilon \end{aligned}$$

if δ suff. small.

Why TVS? Many natural vector spaces don't have a natural norm:

- $C(\mathbb{R})$: cont. funct. on $\mathbb{R}$
- $C^\infty(\mathbb{R})$
- $C^\infty((a,b))$
- measurable functions on (Ω, μ)
- All seq. $\{x_n \in \mathbb{C}\}$

Ex: $\mathbb{C}^\infty = \{\text{seq. } x_n \in \mathbb{C}\}$
$= \mathbb{C} \times \mathbb{C} \times \cdots$

τ = prod top.

Open neig. system for $p \in \mathbb{C}^\infty$:

$$\mathcal{U}_{\epsilon, i_1, \dots, i_n}(p) := \{x \in \mathbb{C}^\infty \mid |x_{i_k} - p_{i_k}| < \epsilon \text{ for } k = 1, \dots, n\}$$

another way....

$$\mathcal{U}_{\epsilon,N}(p) = \{x \mid |x_n - p_n| < \epsilon \text{ for } n = 1, \dots, N\}$$

Check add. cty in $\mathbb{C}^\infty$.

Given $x, y \in \mathbb{C}^\infty$ and nbhd

$U_{\epsilon,N}(x+y)$,

$$\mathcal{U}_{\epsilon/2,N}(x) + U_{\epsilon/2,N}(y) \subset U_{\epsilon,N}(x+y)$$

Ways to define top. on vector space

a) directly

b) metric $d(x,y)$

c) seminorms $\{p_\alpha : \alpha \in A\}$

Ex: of Metric TVS

$L^p[0,1]$, $0<p<1$

$\quad \| $

$\{f \mid \int_0^1 |f|^p < \infty\}$

$d(f,g) = \int_0^1 |f-g|^p$

Key fact: $t \mapsto t^\alpha$ concave

$|f-h|^p \leq |f-g|^p + |g-h|^p \to \Delta$ ineq.

Seminorms

$C(\mathbb{R})$: seminorm $p_n(f) = \sup_{[-n,n]} |f|$

$\{p_n\}$ defining ONS:

$\mathcal{U}_{\epsilon, n_1 \dots n_k} = \{y \mid p_{n_i}(x-y) < \epsilon \quad i=1,\dots,k\}$

Def: Locally Convex Space (LCS)

a TVS where top. given by seminorms

$\{p_\alpha : \alpha \in A\} \ni \bigcap_{\alpha \in A} \{p_\alpha = 0\} = \{0\}$

* LCS $\to$ TVS ; seminorms imply cont. of $+, \cdot$

$\mathbb{C}^\infty(-)$ has norm $p_n(x) = |x_n|$

Locally Convex Space (LCS)

Recall: LCS has top. defined by seminorms $\{p_\alpha \mid \alpha \in A\}$

$\bigcap_\alpha \{p_\alpha = 0\} = \{0\}$ $\quad \forall \alpha \; p_\alpha(x) = 0 \to x = 0$

$p_\alpha(x) = 0 \; \forall \alpha \to x = 0$

Neigh of x are $U_{\epsilon, \alpha_1, \dots, \alpha_n}(x) = \{y \mid p_{\alpha_k}(x-y) < \epsilon, k = 1, \dots, n\}$

Claim: LCS $\to$ TVS

$(x,y) \mapsto x+y$ cont.

Given $U_{\epsilon, \alpha_1, \dots, \alpha_n}(x+y) =: W$

Consider $U_{\epsilon/2, \alpha_1, \dots, \alpha_n}(x) =: \mathcal{U}$, $U_{\epsilon/2, \alpha_1, \dots, \alpha_n}(y) =: V$

then $\mathcal{U} + V \subseteq W$

Sim. $(\alpha, x) \mapsto \alpha x$ just as with normed space.

Ex:

X = all functions on $\mathbb{R}$ to $\mathbb{C}$

$p_\alpha(f) = |f(\alpha)|$; $\alpha \in \mathbb{R}$

$f_n \to f$ in X?

$\Updownarrow$

$f_n \to f$ pointwise

So top. of pointwise conv.

$\Leftrightarrow$: $\forall \epsilon > 0 \; \forall \alpha \in \mathbb{R} \; \exists N \ni p_\alpha(f_n - f) < \epsilon$ for $n \geq N$. $|f_n(\alpha) - f(\alpha)| < \epsilon \rightleftarrows$ ptwise conv.

Weak & Weak* Top

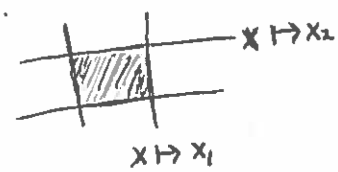
Let X be a normed space, X^* the dual.

Weak top. on X is induced by seminorm: $\{p_f(x) = |f(x)|;\ f \in X^*\}$

Weak* top. on X^* is induced by $\{p_x(f) = |f(x)|;\ x \in X\}$ (= top. ptwise conv.)

Adv. weaker top get more compacts (less open $\to$ more compact sets)

When $\dim X < \infty$, weak top on X is norm top.



When $\dim X = \infty$, every nonempty weakly open set is unbounded with respect to norm

Indeed, if $0 \in \mathcal{U}$, $\mathcal{U}$ open then $\exists \epsilon, f_1, \dots, f_n \ni$

$\mathcal{U} \supseteq \{x : |f_k(x)| < \epsilon, k = 1, \dots, n\}$

$\supseteq \bigcap_{k=1}^{n} \ker f_k = \infty$-dim'l subspace

Why "LCS" if it's about seminorms?

Org. locally convex means $\exists$ open neigh. system where all neigh. are convex.

seminorms produce convex nbhd (Δ-ineq.)

Conv., convex neigh come from seminorms (Minkowski gauge)

Note if $f : X \to \mathbb{C}$ cont. lin. then $f^{-1}(\{|z|<1\})$ convex & open.

Fact: $L^p[0,1]$; $0<p<1$

$d(f,g) = \int |f-g|^p$

has no convex open sets except ϕ, X. Therefore, no cont. lin. functionals except $f \equiv 0$

Lem: Let X be a LCS ; $\{p_\alpha\}$

1) $\forall \alpha$, p_α cont. on X

2) If p is cont. & $\underset{\text{seminorm}}{q} \le Cq$ then q cont.

3) if $\exists r, R>0 \ni \{p<r\} \subset \{q<R\}$ then $q \le Cp$

$|p_\alpha(x) - p_\alpha(y)| \le \underbrace{p_\alpha(x-y)}_{<\epsilon}$ by Δ

$|q(x) - q(y)| \le |q(x-y)| \le C\ \underbrace{|p(x-y)|}_{<\delta}$

By scaling

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4.2 Metrizable and Normable LCS

Ways to simplify the set of seminorms $\{p_\alpha \mid \alpha \in A\}$

Adding a cont. seminorm does not change topology:

$$\underset{\tau_1}{(X, \{p_\alpha \mid \alpha \in A\})} \underset{\text{homeo}}{\cong} \underset{\tau_2}{(X, \{p_\alpha\} \cup \{q\})}$$

↑ actual equality?

$\tau_1 \subseteq \tau_2$ trivially

If $U \in \tau_2$. For all $x \in U$ $\exists\ U_{\epsilon, p_{\alpha_1} \cdots p_{\alpha_n}, q}(x)$

$$\cap$$

$$V$$

$U_{\epsilon, p_{\alpha_1} \cdots p_{\alpha_n}}(x) \cap \{y \mid q(x-y) < \epsilon\}$

open τ_1 in τ_1 — q cont. so open

If $p_1 \leq C p_2$, then removing p_1 does not change the topology. Because $\{y \mid p_1(x-y) < \epsilon\}$

$$\cap\!\shortmid$$

$$\{y \mid p_2(x-y) < \epsilon / c\}$$

If $\{p_\alpha\}$ is countable, they can be replaced by ordered seminorm $q_1 \leq q_2 \leq \cdots$

So topology is generated by $U_{\epsilon, N}(x) = \{y \mid q_N(x-y) < \epsilon\}$

$q_1 = p_1$
$q_2 = \max(p_1, p_2)$
$q_3 = \max(p_1, p_2, p_3)$
$\vdots$

But then can remove p's since redundant: $p_n \leq q_n$.

Cont. so can add them.

Thm: LCS is metrizable iff its top. can be defined by a countable collection of seminorms.

Ex: On $C(\mathbb{R})$

$\{p_\alpha(f) = \sup_{[-\alpha, \alpha]} |f|, \alpha \in (0, \infty)\}$

equiv. to countable one $\sup_{[-n,n]} |f|$ so metrizable.

Pf: Given $\{p_n \mid n \in \mathbb{N}\}$

Define metric $d(x,y) = \sum_1 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}$

$f(t) = \frac{t}{1+t}$ bounded, subadditive, inc. $f(t+s) \leq f(t) + f(s)$

Conv. suppose $\exists$ metric d which induces LCS top. then

$B_{1/n}(0) = \{x \mid d(x, 0) < 1/n\}$

form a nbhd system local basis of top @ 0.

For each n, $\exists\ \epsilon, \alpha_1, \ldots, \alpha_{k_n}$ (dep. on n)

$\ni U_{\epsilon, p_{\alpha_1} \cdots p_{\alpha_{k_n}}}(0) \subseteq B_{1/n}(0)$

Then countable collection $\bigcup_{n=1}^{\infty} \{p_{\alpha_1}, \ldots p_{\alpha_{k_n}}\}$

induces the same top.

Ex: Let X be an inf. dim. normed space with weak top^w: $\{p_f(x) = |f(x)| \mid f \in X^*\}$

Then (X, w) is not metrizable.

Suppose d is an equiv. metric.
Hence $\forall r$, $\{x \mid d(x,0) < r\}$ is unbounded wrt norm. it contains a weak nbhd of 0. So $\exists \{x_n\}$ such that $d(x_n, 0) < 1/n$ and $\|x_n\| > n$

$d(x_n, 0) \to 0 \Rightarrow x_n \xrightarrow{w} 0$

Then $\forall f \in X^*$, $\{f(x_n)\}$ bounded

then UBP, $\{x_n\}$ is norm-bounded

Contradicts $\|x_n\| > n$.

Def: A ~~TVS~~ is called a Frechét Space if its top. is induced by metric d ~~wrt~~ $\ni$ space is complete. (or Polish space)

Eg $(C(\mathbb{R}), \{p_n(f) = \sup_{[-n,n]} |f|\})$ is Frechét

Suppose $\{f_k\}$ Cauchy wrt d.

$d(f_j, f_k) < \epsilon$ if $j, k > M$

Then $p_n(f_j - f_k) < \epsilon$ for large j, k

so Cauchy wrt to unf metric $[-n,n]$ so $f_j \to f$ unf. on $[-n, n]$. Still need to show conv. wrt metric d.

$d(f_j, f) \to 0$ as $p_n(f_j - f) \to 0$ $\forall n$.

Thm: LCS normable $\Leftrightarrow \exists$ open set $B \ni 0$ s.t. $\forall$ open set $U \ni 0$ $\exists r > 0$ $rB \subseteq U$

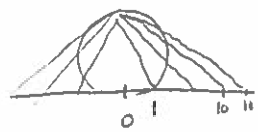
So fits in another set after rescaling.

$\|\cdot\|$ exists $B = \{x \mid \|x\| < 1\}$

Conv.: Fit a seminorm nbhd inside of B and this is a norm

Completeness of LCS

Ex: (on $\mathbb{R}$)



$d(x,y) = |\tan^{-1}x - \tan^{-1}y|$

$(\mathbb{R}, d)$: same top, not complete

$\{x_n = n\}$ Cauchy but lim x_n X

$x_n \to x$ depends on top. only

$\{x_n\}$ Cauchy depends on metric.

However, if d_1 & d_2 are trans. invariant metrics on TVS which define same top $\rightsquigarrow$ Completeness (Cauchyness) seq. are the same.

$d(x+z, y+z) = d(x,y)$

Pf: Cauchy. $\forall \epsilon > 0\ \exists N \ni$
$d(x_j, x_k) < \epsilon \quad \forall j,k \geq N$

$\forall \epsilon\ \exists N \ni d(x_j - x_k, 0) < \epsilon$
(trans. invariance) iff $\forall$ nbhd $\mathcal{U}$ of $0\ \exists N \ni x_j - x_k \in \mathcal{U}$
$\forall j,k \geq N$

4.3 Hahn Banach Thm for LCS (real)

Real Space $\to \{x \mid f(x) = c\}$

separates the space

Ex: Let $X = L^p[0,1]$; $0 < p < 1$

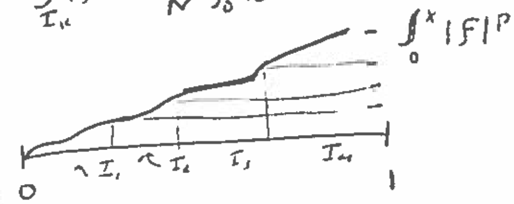
$d(f,g) = \int_0^1 |f-g|^p$

Claim: X has no convex open sets except ϕ & X.

Suppose $\exists\ \mathcal{U}$. By trans invariance can assume $0 \in \mathcal{U}$. So $\mathcal{U} \supset$
$\{f \mid \int_0^1 |f|^p < r\}$ for some r.

Given $f \in L^p[0,1]$, partition $[0,1]$ into $I_1, \cdots, I_n$ so that

$$\int_{I_k} |f|^p = \frac{1}{N}\int_0^1 |f|^p$$



$$f = \frac{(Nf\chi_{I_1}) + \cdots + (Nf\chi_{I_N})}{N}$$

$$d(Nf\chi_{I_k}, 0) = \int_0^1 |Nf\chi_{I_k}|^p =$$

$$= N^p \frac{1}{N} \int_0^1 |f|^p \to 0$$

as $N \to \infty$

Hence $f \in \mathcal{U}$. But then $\mathcal{U} = X$

So HB fails on $X = L^p[0,1]$; $0<p<1$

Let $M = \{\text{const funct}\} \subset X$

$f(c) = c$ is cont. on M.

$\nexists$ cont. ext. of f to X as only cont. lin. functional on X is $\equiv 0$.

$\hat{F}$

$\{x \mid |F(x)| < \epsilon\}$ is open, convex, containing 0 ($\neq \emptyset$) so it's $= X$. So $F \equiv 0$.

Things are better in LCS: HB holds then.

When is a lin. functional $f: X \to \mathbb{R}$ cts if X has $\{p_\alpha \mid \alpha \in A\}$

f cont. $\rightleftarrows$ $\exists$ ~~$p_{\alpha_1}, \ldots, p_{\alpha_n}$~~ $\alpha_1, \ldots, \alpha_n$ $\ni$ $|f(x)| \le M \max(p_{\alpha_k}(x))$

Indeed, $\{x \mid |f(x)| < 1\}$ is open $\to$ $\exists\, \alpha_1, \ldots, \alpha_n \ni U_{\alpha, \alpha_1, \ldots, \alpha_n}(0) \subset \{|f| < 1\}$

So if $\max |p_{\alpha_k}(x)| < \epsilon \rightsquigarrow |f(x)| < 1$

By scaling $|f(x)| \le 1/\epsilon \max |p_{\alpha_k}(x)| \; \forall x$

$\to$ ~~Equiv.~~ f cont. $\to$ $|f|$ cont. seminorm?

$\to$ equiv. to $\exists$ cont. seminorm $q \ni |f| \le q$

$c \cdot \max(p_{\alpha_k})$

Hahn-Banach Gen. Form

Suppose $q: X \to \mathbb{R}$ is sublin. meaning $q(x+y) \le q(x) + q(y)$ and $q(tx) = t\,q(x)$; $t > 0$

Suppose $M \subset X$ is a lin. manifold and $f: M \to \mathbb{R}$ lin. funct. such that $f \le q$ on M. Then $\exists$ lin. funct. $F: X \to \mathbb{R} \ni F \le q$ on X and $F|_M = f$

Cor: Cts lin. $f: M \to \mathbb{R}$ has cts ext. $F: X \to \mathbb{R}$. Indeed, $\exists\, q$ cts seminorm on $X \ni |f| \le q$ on M. / By HB $\exists\, F$ ext $F \le q$ on M

also $-F(x) = F(-x) \le q(-x) = q(x)$

So $|F| \le q$.

Pf (HB): By Zorn, only need to extend to $\operatorname{Span}(M \cup \{x_0\})$

Must choose $f(x_0) = c$?

Need $\forall y \in M$

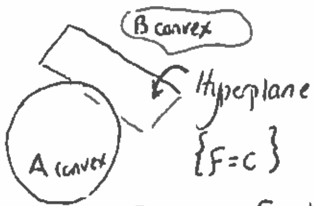
$f(y \pm t x_0)$ $\forall t \ge 0$ ($t=0$ triv. so $t>0$)

$= f(y) \pm tc \le q(y \pm t x_0)$

$z = y/t \to$ Cancel t, get $f(w) - q(w - x_0) \le c \le q(z + x_0) - f(z)$

$\forall w, z \in M$

Hahn-Banach & Separation Thms



$A \cap B = \emptyset$ $\quad F$ cts lin. functional

Three Kinds of Sep.

1) (Simply) separated: If $\exists\ F \in X^*$ and $\alpha \in \mathbb{R} \ni F \le \alpha$ on A & $F \ge \alpha$ on B

2) Strictly separated: If $\exists\ F \in X^*$ $\alpha \in \mathbb{R} \ni F < \alpha$ on A, $F > \alpha$ on B.

3) Strongly sep: If $\exists\ F$, $\exists\ \alpha < \beta$ $F < \alpha$ on A, $F > \beta$ on B.

Ex in ℓ^2

$A = c_{00} = \{x \mid$ fin. numb nonzero entries$\}$

$B = A + \underbrace{\left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)}_{x_0 \in \ell^2 \setminus c_{00}}$

A, B convex disjoint but cannot be sep.: (A dense)

$F \le \alpha$ on $A \to F$ constant ($F \le \alpha$ on ℓ^2 ↑). $(\equiv 0)$

HW: Even disjoint closed convex is not enough.

Thm: A point p can be separated from an open convex set C provided $p \notin C$. (on a LCS)

PF: $p = 0$ WLOG (subtract p).
Need lin. func. $F \in X^* \ni F > 0$ on C. Choose $x_0 \in C$
Let q be Minkowski gauge of $x_0 - C$. $\leftarrow q_\Omega(x) = \inf\{t \ge 0 \mid x/t \in \Omega\}$

So $x_0 - C = \{q < 1\}$

q_Ω is sublin: $q_\Omega(x+y) \le q_\Omega(x) + q_\Omega(y)$

$q(x_0) \ge 1$ as $x_0 \notin x_0 - C$.

Define $F(x_0) = q(x_0)$, extend lin.
$F(tx_0) = t\,q(x_0)$ for $t \in \mathbb{R}$
Note $F \le q$ on this line. By HB have $F \le q$ on X. So $\forall x \in C$

$F(x_0 - x) \le q(\underbrace{x_0 - x}_{\in x_0 - C}) < 1$

So $F(x) > F(x_0) - 1 = q(x_0) - 1 \ge 0$ ∎

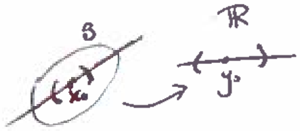
Thm 2 A, B disjoint convex. B open then they can be semistrictly sep.
$F \le \alpha$ on A $\quad F > \alpha$ on B

Moreover if A open, have strict sep.

PF: $C = \cancel{B \setminus A}$ $B - A$ open. $= \bigcup B - a$ and does not contain 0. Thm 1 gives F as above: $F > 0$ on C. So $F(b) - F(a) > 0 \to F(b) > F(a)$.

$\alpha := \sup_A f$ so $f \le \alpha$ on A
$f \ge \alpha$ on B

If B open, $f(B) \subset \mathbb{R}$
open ✓ nonconst lin. f.



So $f(B) \subset (\alpha, \infty)$ semistrict.

Then $f(A)$ open & $f(A) \subset (-\infty, \alpha)$ ☑

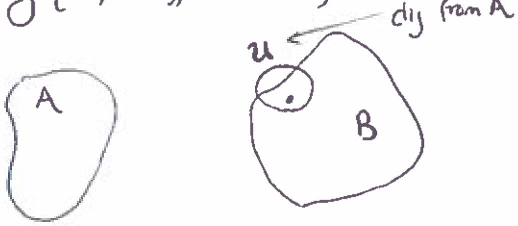
Thm 3:

A, B disjoint convex
A closed
B compact
Then strongly sep.

Pf: Step 1: $\exists$ convex nbhd $0, \mathcal{U}$
$\ni A + \mathcal{U}$ & $B + \mathcal{U}$ are disjoint

Step 2: By Thm 2 $A + \mathcal{U}$ &
$B + \mathcal{U}$ are strictly sep.

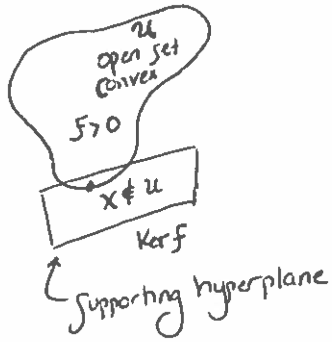
Step 3: Strict $\to$ strong by compactness:
$f > \alpha$ on $B \rightsquigarrow f \ge \alpha + \epsilon$ on B
by compactness, f attains inf.



$\exists b_1, \cdots, b_n \ni \mathcal{U}_{\epsilon/2}(b_i)$ cover B (compact)
$q =$ max all seminorms involved in $\mathcal{U}_{\epsilon/2}$
Observe $\forall a \in A, b \in B$, $q(a-b) > \epsilon/2$
$\mathcal{U} = \{x \mid q(x) < \epsilon/4\}$ works ☑

K02

Remarks on Convex sets



Suppose K is compact convex; $x_0 \in \partial K$

Does $\exists$ supporting hyperplane?

$f(x) \geq f(x_0) \quad \forall x \in K.$

f nonconstant $(\neq 0)$

Answer no:

$K = \{x \in \ell^2 \mid |x_n| \leq 1/n\}$

Compact: closed, bounded, flat

$0 \in \partial K \quad (\forall r>0, B_r(0) \not\subset K)$

$K = -K$. So if $f \geq 0$ on K then $f = 0$ on K. But $K \not\subset$ hyperplane

as orthogonal complement $= \{0\}$:

$x \in K^\perp \rightsquigarrow x \perp \frac{1}{n} e_n$ so $x(n) = 0$.

$y \in \ell^2$ if $y \neq 0$ $\exists x \in K \ni \langle x, y\rangle > 0$

$\langle -x, y \rangle < 0$.

If A closed convex with empty interior. Is it contained in ϵ-nbhd of some affine hyperplane?

If compact, yes.

Equiv: $\exists f$ with $|f| = 1 \ni$ diam $f(A) < \epsilon$

No: $A = \{x \in \ell^2 : x_n \geq 0 \ \forall n\}$

A convex closed

$x \in A$ & $r > 0$. $\exists N \ni |x_N| < r$ (term sq. sum so $\to 0$). Then $\exists$ $y \notin A$ with $\|y - x\| < r$

$y = x - t e_n$

$x_n < t < r$

So empty interior. But $A - A = \ell^2$ meaning $\forall x \in \ell^2$ $\exists a, b \in A \ni x = a - b$

$a_n = \max(x_n, 0)$

$b_n = \max(-x_n, 0)$

If $f \neq 0 \to \{f(a) - f(b)\} \underset{\mathbb{R}}{\overset{\shortparallel}{}}$ unbounded

Proj of A onto any line is unbounded.

3.13 #5

X compact top. space

$Y \subset X$ closed

Suppose $\exists$ bounded $T : C(Y) \to C(X) \ni$

$(Tg)|_Y = g$ "sim. ext. operator"

Prove $M = \{f \in C(X) : f|_Y = 0\}$ is complemented in $C(X)$

$N := \operatorname{ran} T$

WTS M, N complementary

$M \cap N = \{0\}$:

$Tg \in M \to Tg|_Y = 0$

then $g = 0 \to Tg = 0$

$M + N = C(X)$: $\forall f \in C(X)$

$g = T(f|_Y) \in N$

$h = f - g \in M$

3.12 #5: $A \in B(X,Y)$

TFAE:

1) $\exists c > 0$, $\|Ax\| \geq c\|x\|$

2) $\ker A = 0$ & $\operatorname{ran} A$ closed

$1 \to 2$: $\ker A = 0$ triv.

$\{y_n\}$ cauchy $y_n = Ax_n$

$\|x_n - x_m\| \leq c^{-1}\|y_n - y_m\|$

so x_n cauchy so conv. to x (Banach space) have $y_n \to Tx$

$2 \to 1$: OMT

$A: X \to \underbrace{\operatorname{ran} A}_{\text{Banach space on own}}$ onto so open

But $\ker A = 0$ so $A^{-1}: \operatorname{ran} A \to X$

bounded so $\|\underbrace{A^{-1}y}_{x}\| \leq M\|\underbrace{y}_{Ax}\|$

$\|Ax\| \geq \frac{1}{M}\|x\|$

A onto $\to A^*$ has lower bound

$A^*f = f \circ A$

Ch. 5: Weak Topologies

Duality of LCS

X is a LCS

$X^* = \{ f : X \to \mathbb{C} \text{ cont. lin.} \}$

often one writes x^* instead of f, ie $x^* \in X^*$.

$$\underbrace{\langle x, x^* \rangle = x^*(x)}_{\text{just notation. not inner product}}$$

Weak * top on X^* is induced by seminorms $p_x(x^*) = |\langle x, x^* \rangle|$; $x \in X$

Corresponding weak topology on X is induced by seminorms $p_{x^*}(x) = |\langle x, x^* \rangle|$ for $x^* \in X^*$.

$\sigma(X^*, X)$ $\qquad$ $\sigma(X, X^*)$

Weak Topology

on this space $\qquad$ From this space

How does this weak top. compare to org top, τ, on X?

Claim: $\sigma(X, X^*) \subseteq \tau$

A weak nbhd of x is $\bigcap_{k=1}^{n} \{ y : |\underbrace{\langle x-y, x_k^* \rangle}| < \epsilon \}$

cont. lin. func. in org. top. So open in τ as x_k^* cont.

If U open in $\sigma(X, X^*)$

then $\forall x$ it contains open set cont. $\tau \to U$ open in τ.

	Stronger	if/then arrow	Weaker
Open		$\longleftarrow$	
Closed		$\longleftarrow$	
Compact		$\longrightarrow$	
Dense		$\longrightarrow$	
Cont. Funct. on X		$\longleftarrow$	

incl. go opposite way

Thm: The dual of X^*, when given weak* top, is X equipped with weak* top.

Pf: $f : X^* \to \mathbb{C}$ cont. iff $\exists\, x_1, \dots, x_n$

Cont. $|f(x^*)| \le \max_k |\langle x_k, x^* \rangle|$

Then $\operatorname{Ker} f \supseteq \bigcap_{k=1}^{M} \operatorname{ker} f_k$, where

Choose $v_k \ni f_k(v_k) \neq 0$

$M \cup \{v_i\}_1^M$ span X. on M $f = \sum_1 c_k f_k$ is $0 = 0$ on. choose c_k to match values in $v_1, \dots, v_k$

$f_k(x^*) = \langle x_k, x^* \rangle$. Then

$f = \sum_1 c_k f_k$ for some $c_k \in \mathbb{C}$

So $f = \sum_1 c_k f_k \in X$.

Top. on dual of X^* is induced by maps $x \mapsto |\langle x, x^* \rangle|$ which is $\sigma(X, X^*)$

Rem: The dual of $(X, \sigma(X, X^*))$ is $(X^*, \sigma(X^*, X))$ as cont. lin funct. on X are also weakly cont.

Ex: $X = \mathbb{C}^\infty$ or $(\mathbb{C}^\omega)$

$=$ all seq. ; $p_n(x) = |x(n)|$

What is X^*?

$f : X \to \mathbb{C}$ cont.

$\Updownarrow$

$$f(x) \le C \max_{1 \le k \le N} |p_k(x)|$$

$$\le C \max_{1 \le k \le N} |x(k)|$$

So $\ker f \supseteq \{x_1 = \cdots = x_N = 0\}$

So $f(x) = \sum_{k=1}^{N} c_k x(k)$

So $X^* = C_{00}$

Exercise: $x_n^* \to x^*$ in $\sigma(X^*, X)$ iff $\bigcup_n \operatorname{supp}(x_n^*)$ is finite and $x_n^*(k) \to x^*(k)$ $\forall k$

Thm If $A \subset X$ is closed and convex then it's weakly closed

Pf: If $x \notin A \rightsquigarrow$ can be strongly sep from A so $\exists f$ and ϵ such that $\underbrace{\{y : |f(x-y)| < \epsilon\}}_{\text{weakly nbhd of } x}$ is disjoint

So A^c is weakly open.

5.3: Alaoglu Thm

Last time: convex + ~~norm~~-closed $\rightarrow$ weakly closed

X norm $\rightarrow$ X^* norm ⟩ compare

$\sigma(X, X^*)$ $\quad$ $\sigma(X^*, X)$

Ex: if X is ~~normal~~ normed, a norm-closed convex subset of X^* is not nec. w^* closed.

Let $X = c_0$, $X^* = \ell^1$

$M = \{x \in \ell^1 : \sum_n x(n) = 0\}$

M closed & convex but not w^* closed

Seq: $\{e_1 - e_n\}$

$$e_1 - e_n \xrightarrow{w^*} e_1 \notin M$$

$\forall f \in c_0$, $f(e_n) \to 0$. or $\forall x \in c_0$ $\langle x, e_n\rangle \to 0$.

This also shows $f(x) = \sum_{n=1}^{\infty} x(n)$ is not w^* cts on ℓ^1.

Note: $\sigma(X^*, X) \subset \sigma(X^*, X^{**}) \subset$ norm

Skip "polarity" in 5.1 & 5.2

Now 5.3....

Alaoglu $\rightarrow$ Thm: X normed space $\rightarrow$ closed unit ball $B \subset X^*$ is weak* - compact.

Hence any weak* closed bounded set is w* compact.

Def: A net is a map a directed set to X.

Ex: $(\mathbb{N}, \leq)$ so seq. a net.

Tail of a net is $\{x(i) : i \geq i_0\}$

$x_0 := \lim x(i)$ if every nbhd of x_0 contains some tail & x_0 is a cluster pt of $x(i)$ if every nbhd of x_0 intersects some tail.

x_0 is a cluster pt of $x(i)$ if every nbhd of x intersects any tail.

E compact $\Leftrightarrow$ every net in E has a cluster point.

E closed $\Leftrightarrow$ if $x(i) \in E$ & $x(i) \to x_0$ $\lim x_0 \in E$

Tychonoff Thm: Product compact $\rightarrow$ compact

Pf: (Alaoglu) Consider set of all functions from B_X to $\{z \in \mathbb{C} : |z| \leq 1\}$

$\hookrightarrow \{x \in X : \|x\| \leq 1\}$

This is $Z = \{|z| \leq 1\}^{B_X}$ = B_X prod of $\{-\}$

Top of ptwise conv. Z compact

Every $f \in B_{X^*}$ restricted to B_X is a map $B_X \to \{|z| \leq 1\}$

So element of Z.

Let W = set of these restrictions. Remains to check W closed. (then closed in compact)

Closed: $f(i) : i \in I$. Suppose $f(i) \to g \in Z$
$\in W$

Must show $g \in W$. Pick $x, y \in B_X$ and $\alpha, \beta \in \mathbb{C}$ $\ni \alpha x + \beta y \in B_X$

Know $f(i)(\alpha x + \beta y) = \alpha f(i)(x) + \beta f(i)(y)$

$$\Downarrow$$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

So $f \in W$. ☐

Sep. Case: If X sep. normed space then B_{X^*} is metrizable in w^* top. and is (seq.) compact.

Pf:

Metrizable: C countable dense subset of X τ_C = top $\sigma(X^*, C)$

seminorms $x^* \to |\langle c, x^* \rangle|$

τ_C is metrizable (countable family)

Claim: On B_{X^*}, τ_C = weak* top

$$\sigma(B_{X^*}, X) = \sigma(B_{X^*}, C)$$

given $U_{\epsilon, x_1, \ldots, x_n}(x^*)$

Choose $c_1, \ldots, c_n \ni \|x_k - c_k\| < \epsilon/2n$

Check $U_{\epsilon/2, c_1, \ldots, c_n}(x^*) \subset U_{\epsilon, x_1, \ldots, x_n}(x^*)$ in B_{X^*}.

In normed space: A, B convex

$\text{dist}(A,B) > 0 \iff A, B$ strongly sep

$\Leftarrow$: triv.

$\Rightarrow$: closure change nothing. Take $\varepsilon/3$ neigh. of each. open convex so $\exists$ strict sep.

A, B disjoint closed convex, A bounded in Banach space X

If X ref., strong sep. possible

Eg: $X = \ell^1$

A = closed unit ball in ℓ^1

$B = \{x \in \ell^1 \mid \underbrace{\sum \frac{n}{n+1} x_n}_{f(x)} = 1\}$

Note $\|f\| = 1$, $|f(x)| < 1$ in A

But $d(A,B) = 0$

$\frac{n+1}{n} e_n \in B$, $e_n \in A$

$\|\frac{n+1}{n} e_n - e_n\| = 1/n$

Idea of ex: $f(x) = \sum \frac{n}{n+1} x_n$ is a bounded lin. funct. on ℓ^1 that is not norm-attaining means $|f(x)| < \|f\|$ } Easy way of showing non-ref.

Meaning: If X is reflexive, every $f \in X^*$ is norm attaining.

PF: $X = (X^*)^* \rightarrow B_X$ is weak-compact.

X	X^*	X^{**}
$\sigma(X, X^*)$		$\sigma(X^{**}, X^*) \rightarrow w^*$

Can't be closed as closed in Hausdorff is compact.

w^*-cl B_X

sep. hyperplane in w^*-top - comes from ev. functional.

$\forall$ Banach space X, evaluation functionals $f \mapsto f(x)$ are norm-attaining on X^*

(why? They are weak* cont. & B_{X^*} is w^* compact)

Not new: Already knew $\forall x \neq 0$ $\exists f \ni \|f\| = 1$, $f(x) = \|x\|$ so ev_x attains norm on f

Thm (R.C. James). On every non-reflexive Banach space $\exists$ lin. funct. that is not norm attaining.

Cor: If X non-ref. $\rightarrow B_X = \{\|x\| \leq 1\}$ is not weakly compact.

Pf: (w/o James) Consider $B_X \subset X^{**}$

$B_X \subset B_{X^{**}}$

$\sigma(X, X^*) \mid \sigma(X^{**}, X^*)$

weak top. on B_X is the subspace top. from w^* top. on X^{**}.

weak = weak* restricted to X.

Claim: B_X is w^*-dense in $B_{X^{**}}$ (hence B_X is not w^* compact)

Suppose not: $\exists z \in B_{X^{**}}$ that is not in w^*-closure B_X. So $\exists x^* \in X^*$ $\ni \langle z, x^* \rangle > \alpha$ but $\langle x, x^* \rangle \leq \alpha$ $\forall x \in B_X$ $\rightarrow \|x^*\| \leq \alpha$

$\rightarrow \leq \|z\| \|x^*\|$

$\leq \|x^*\| \rightarrow$ contradiction

Conclusion: X ref. $\rightleftarrows$ B_X is weakly compact.

Application: X ref., $\Phi : X \to \mathbb{R}$ convex

$\lim_{\|x\| \to \infty} \Phi(x) = \infty$ "~~Coercive~~"

Φ cont. (in norm)

Then $\inf_X \Phi$ is obtained.

(Good for Cal. of Variations).

Pf: $\exists$ $B(R)$ closed $\ni$ $\Phi > \inf \Phi$ on $B(R)^c$. By Alaoglu, $B(R)$ is weakly compact.

$\forall t \in \mathbb{R}$, $\{x : \Phi(x) \le t\}$ is convex & closed & bounded, hence weakly ~~closed~~ closed. So Φ is lower semicont. in weak top. Lower semicont. obtains min on compact set. $\square$

Back to sep. case of Alaoglu....

Seq. compact: $\{f_n\} \subset B_{X^*}$

C countable dense subset of X

C_1: $|f_n(C_1)| \le \|C_1\|$

So $\exists$ conv. subseq. $f_{n_k}(C_1)$

C_2: .... conv. subsubseq. $f_{n_{k_j}}(C_2)$

Choose diagonal subsequence

Conv. at every point of C. but C dense & $\{f_n\}$ equicont. (bd norm) $\rightsquigarrow$ so conv. on X. $\hookrightarrow$ ptwise

5.7 Krein-Milman Theorem

Extreme points

Suppose K is convex. A point $x \in K$ is extreme if $\nexists$ $a, b \in K$, $a \neq b$, $\ni$ $x = \frac{a+b}{2}$

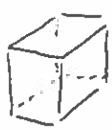
Equiv: $x \notin$ line seg. contained in K

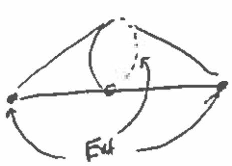
$\nexists$ $v \neq 0 \ni x+v, v-v \in K$

$K \setminus \{x\}$ is convex

Ext K = extreme points K

1)  a ball in $\mathbb{R}^n$
ext $B = \partial B$

2)  ex$t\, C = \{$vertices$\}$

3) 

So Ext not nec. closed

4) B = closed unit ball ℓ^∞

Ext B = all unimodular sequences ← ±1 seq.

PF: $v \neq 0$, $\exists$ $v_n \neq 0$

$\supset$: $|x_n+v_n|^2 + |x_n-v_n|^2 = 2|x_n|^2 + 2|v_n|^2 > 2$

So one ↓ of not in K

$\subset$: $x \in B \ni \exists n \ni |x_n| < 1$

$v = (1-|x_n|)e_n \to x \pm v \in B$

5) B = unit ball of $C = \{x \in \ell^\infty \mid \lim x_n \text{ exists}\}$

Same story ext $B = \{$unimod. seq. in $C\}$

6) B = unit ball in $C_0 = \{x \in \ell^\infty \mid \lim x_n = 0\}$

ext $B = \{$unimod. seq$\} = \phi$

Cor: C_0 and C are not isomet iso

If $T: C \to C_0$ isom isom.

$\to \bar{B}_C \xrightarrow{T} \bar{B}_{C_0}$ same for ext. pts

Inj. enough $\left\{ \begin{array}{l} \text{More gen, } T \text{ inv. lin. map} \rightsquigarrow T(\text{ext}_K) \\ = \text{ext } T(K) \end{array} \right.$

Krein-Milman Thm: K convex compact subset of a LCS, then K = closed convex hull of ext K.

Ext pt exists $\{$

Cor: Closed unit ball of X^* has ext. points

In particular, C_0 not dual space of any normed space. Not even isomorphic.

Same for L^1 (no ext. pts of unit ball) : $L^1 \neq X^*$

PF: exclude triv. case $K = \{x\}$ or $K = \phi$

Method: find a convex set $U \subset K \ni$

$K \setminus U$ = one pt (then pt is extreme)

1) Consider all open proper convex subsets of K. ordered by inclusion open means relative to K. ← subspace top of K

eg convex nbhd of any pt $x \in K$ intersected with K.

Check Zorn's Lemma applies

So $\exists$ max element : max. proper open convex subset $\mathcal{U} \subset K$

2) Fix $x_0 \in K$ and consider contraction

$Tx = x_0 + \lambda(x - x_0)$; $0 < \lambda < 1$

$T : K \to K$

$T^{-1}(\mathcal{U})$ open convex and is larger than $\mathcal{U}$.

(not imm. obvious)

Hence $\mathcal{U}$ contains all nontriv. convex comb. $\lambda a + (1-\lambda) b$; $0 < \lambda < 1$, $a \neq b$, $a, b \in K$

3) Suppose $a, b \in K \setminus \mathcal{U}$, $a \neq b$

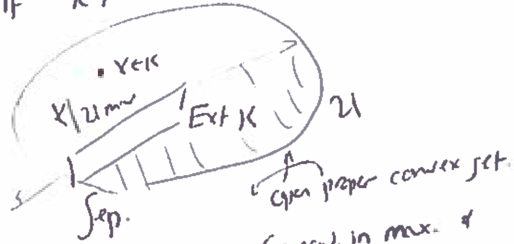
Let $V = \mathcal{U} \cup N$, N a nbhd of a, $b \notin N$

Since $\mathcal{U} \subseteq V \subseteq K$, V is convex also open & proper ($b \notin V$)

Contradiction: $\mathcal{U}$ max such set.

So $K \setminus \mathcal{U}$ one pt hence ext pt.

If $K \neq$ hull ext K



So cont. in max. & $K - \mathcal{U}_{max}$ ext.

$\Rightarrow\Leftarrow$

Final May 9th 5:15–7:15
4 out of 6.

$x \in ext(B_c) \rightleftarrows |x_n| = 1 \ \forall n, x \in c$

$\to$ if $\exists n\ |x_n| < 1 \to x \pm \epsilon e_n \in B_c$

So $x \notin ext\ B_c$

$\leftarrow x \notin ext,\ \exists x \ni x \pm v \in B_c \to$

$|x_n \pm v_n| \le 1 \ \forall n$

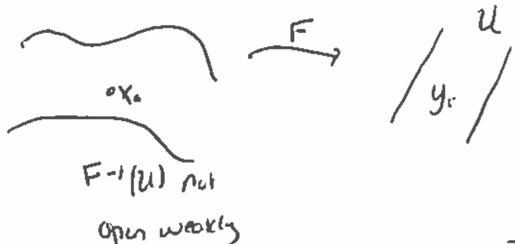
$|x_n|^2 + |v_n|^2 \le 1 \ \forall n$

So $\exists n \ni v_i \neq 0 \to |x_n| < 1$

5.9 Schauder's Fixed Point Thm

Nonlin. funct analysis:
nonlin. maps in various TVS

Weak top is less useful, b/c nonlin. maps are usually not cont. in weak top.



In fin. dim. Brower's Fixed pt. Thm $\to \mathbb{R}^n$ fin. dim

Also true if $B \subset \mathbb{R}^n$ is convex closed bounded set.

Ex: (HW) $\exists$ cont. map $f: B \to B$ $\to$ closed unit ball in ℓ^2 without fixed points.

So we need add. assumptions.

Def: F compact map: F cont. $\overline{F(V)}$ compact $\forall$ bounded set E

Sch. Fixed point Thm: Suppose X is a Banach space. $E \subset X$ convex closed bounded. $F: E \to E$ compact. Then F has a fixed point.

Special Case: E compact, then every cont. map $E \to E$ is compact.

Lem: For every compact set $K \subset X$ and every $\epsilon > 0$ $\exists$ cont. map $\Phi: K \to X$ such that...

- $\Phi(K) \subset$ fin. dim subspace
- $\|\Phi(x) - x\| < \epsilon \quad \forall x \in K.$

$\downarrow$ Id map = unif limit of finite rank-maps

Could Φ be chosen lin here? No in general.

If such Φ is lin $\to \forall$ compact op: ~~T~~ could construct Φ with $K = \overline{T(B)}$, get $\|\Phi T - T\| < \epsilon$
ΦT of fin. rank

1973, Enflo showed above approx. not always possible.

↳ ~~True~~ Not always true in Banach space (as in Hilbert).

$\forall K \; \exists M$ fin dim.

$K \subset N_\epsilon(M)$

proj onto M?

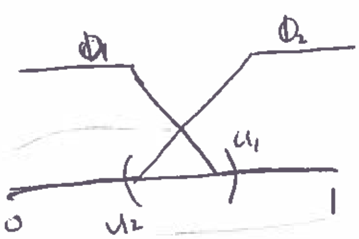
- nearest pt proj. not lin.
- If E idem. w/ $\overset{\text{ran}}{E} = M$

$\|E\|$ may be large. $Ex - x$ may be large

Pf (lem): Cover K by ~~by~~ balls of radius $< \epsilon$

Choose fin subcover. U_j.

$\exists$ cont. partition of unity Φ_j (cts fn. $0 \le \Phi_j \le 1$ & $\operatorname{supp} \Phi_j \subset U_j$ & $\sum \Phi_j = 1$ on K)



pick x_j - center of U_j.

$\Phi(x) = \sum_1 x_j \Phi_j(x)$

convex comb.? → (Cont $\in x_j$ only those lin)

$\|x - x_j\| < \epsilon$

Then $\|\Phi(x) - x\| < \epsilon$

Rem: We also have $\Phi(K) \subset co(K)$ convex hull

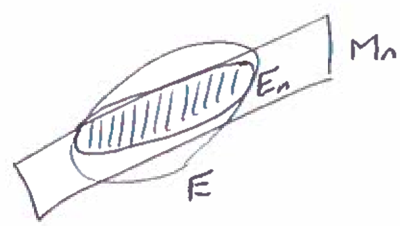
Pf: (Schauder's FPT)

$K = \overline{F(E)}$. Let $\epsilon = 1/n$ get Φ_n from lem.

$\|\Phi_n(x) - x\| < 1/n \quad \forall x \in K$

Let $M_n = \operatorname{span}(\Phi_n(K))$ fin dim. subspace

$E_n = E \cap M_n$



E_n closed convex in M_n

$f_n = \Phi_n \circ f$ map $E_n \to E_n$

$\Phi_n(K) \subset M_n$

$\Phi_n(K) \subset E$ since E convex $\supset co(K)$

Brouwer's says fixed point.

$\exists x_n \in E_n \ni f_n(x_n) = x_n$

$\{F(x_n)\}$ in compact set K so $\exists$ subseq $F(x_{n_j}) \to x_0$

$\|F(x_{n_j}) - x_{n_j}\| \le \|F(x_{n_j}) - f_{n_j}(x_{n_j})\| + \|f_{n_j}(x_{n_j}) - x_{n_j}\|$ ($< 1/n_j$, $\to 0$)

So $x_{n_j} \to x_0$

F cont. $F(x_0) = \lim F(x_{n_j}) = x_0$