MAT 705: Differentiable Manifolds

08/29/2016

Office Hour: MW: 11-12

Topological Manifold

Def: A manifold M of dim. n (or n-manifold) is a top. space such that

1) M Hausdorff
2) $\forall p \in M$, $\exists\ \mathcal{U}$ cont p such that $\mathcal{U} \cong$ open subset in $\mathbb{R}^n$
3) M has countable basis of open sets.

Ex:

1) $\mathbb{R}^n$; n-manifold

2) $\mathcal{U} \subset \mathbb{R}^n$ open set then $\mathcal{U}$ n-manifold

3) Circle, S^1: 
1-manifold
$* S^1 \not\cong \mathbb{R}^1$

4) 2-sphere, S^2 is a 2-manifold

$\cap$
$\mathbb{R}^3$ $\quad * S^2 \not\cong \mathbb{R}^2$

5) 2-Torus: T^2 2-manifold

$\cap$
$\mathbb{R}^3$ $* T^2 \not\cong \mathbb{R}^2$
$* T^2 \not\cong S^2$

Thm: A topological manifold is locally connected, locally compact, and a union of countable collection of compact subsets.

This follows easily from the fact that $\mathbb{R}^n$ is locally connected & compact & has such a union & the definition of a topological manifold.

Note: $B^n \subseteq \mathbb{R}^n$ & $B^m \subseteq \mathbb{R}^m$ are not homeo if $n \neq m$. So dim. of a top. manifold is invariant.

A manifold with boundary is a Hausdorff space M with countable basis of open sets such that $\forall p \in M$ $\exists$ open set $\mathcal{U}$ cont p and homeo. $\phi \ni$ either

1) $\mathcal{U} \overset{\phi}{\cong} \mathcal{U}' \subset H^n \setminus \partial H^n$

or

2) $\mathcal{U} \overset{\phi}{\cong} \mathcal{U}' \subset H^n$ and $\phi(p) \in \partial H^n$

where $H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$

*Note: p belongs to (1) or (2). Then $p \in M$ is called a boundary point, denoted $p \in \partial M$, if $p \in (2)$ and an interior point if $p \in (1)$.

Rem: A surface is a 2-manifold. A surface can be formed by gluing together manifolds with boundaries along their boundaries.

$S^2 \cong$ 

$T^2 \cong$ 

or

Thm: Every compact, connected, orientable surface is homeo. to a sphere with handles added.

Ex: Real Projective Space

The real proj. space of dim. n, denoted $\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$

where

$$(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$$

iff $\vec{x} = t\vec{y}$ for $0 \neq t \in \mathbb{R}$.

Denote equiv. class of $\vec{x}$ by $[x]$

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{R}P^n$$
$$x \mapsto [x]$$

Give topology to $\mathbb{R}P^n$ via quotient topology, ie

$\mathcal{U}$ open in $\mathbb{R}P^n \rightleftarrows$ $\pi^{-1}(\mathcal{U})$ is open. Could realize $\mathbb{R}P^n$ via $S^n / \sim$, where $\sim$ is identification of antipodal points. Easier to see it is a manifold this way.

Check: $\mathbb{R}P^n$ is a top. manifold in rigor. and that

$\mathbb{R}P^n \cong$ 

We will introduce local coordinates on $\mathbb{R}P^n$:

$$\mathcal{U}_i = \{ [x_1, \dots, x_{n+1}] \in \mathbb{R}^{n+1} \setminus \{0\} \mid x^i \neq 0 \}$$

Check: $\mathcal{U}_i$ open. The $\mathcal{U}_i$ cover $\mathbb{R}P^n$. ($n+1$ total such sets)

$$\phi_i([x]) = \underbrace{(x_1, \dots, x_{i-1}, \hat{x}_i, \dots, x_{n+1})}_{\hat{x}_i}$$

well defined for this equivalence class.

$$= \underbrace{(x_1, \dots, x_{i-1}, \dots, x_{n+1})}_{\sim x_i \text{ skipping } x_i}$$

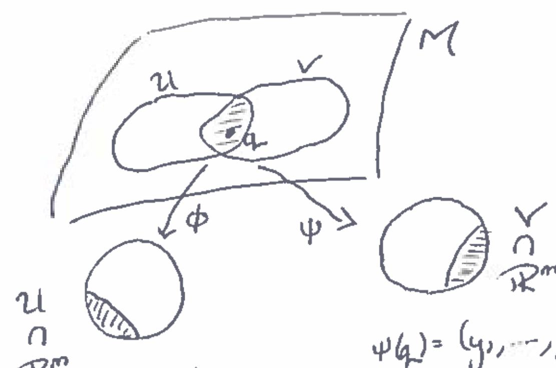
$$= \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right)$$

Then $\{ (\mathcal{U}_i, \phi_i) \}$ is a coordinate system.

08/31/2016

Let (U, ϕ) & (V, ψ) be coordinate neigh. with $\phi \cap \psi \neq \emptyset$.

$\psi \phi^{-1}$ is a homeomorphism where $\psi\phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$



$\phi(q) = \underbrace{(x_1, \dots, x_m)}_{\text{depends on } q}$ $\quad \psi(q) = \underbrace{(y_1, \dots, y_m)}_{\text{depends on } q}$

$$(x_1, \dots, x_m) \mapsto (y_1, \dots, y_m)$$

Makes sense because ϕ, ψ homeo.

So write $\psi\phi^{-1}$ in local coord.

$h_i(x_1, \dots, x_m) = y_i$ for $1 \leq i \leq m$

Similarly, inverse $\phi\psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.

Write $\phi\psi^{-1}$ in local coordinates

$g_j(y_1, \dots, y_m) = x_j$ for $1 \le j \le n$

Notice h_i, g_j are cont. and their composition gives the identity. So change of coordinates is a cont. function.

Def: $(\mathcal{U}, \phi), (V, \psi)$ are C^∞-compatible if $\mathcal{U} \cap V \neq \emptyset$ have change of coordinates as before with h_i, g_j are C^∞ maps. In other words, $\psi\phi^{-1}$ and $\phi\psi^{-1}$ are diffeomorphisms between $\phi(\mathcal{U} \cap V)$ and $\psi(\mathcal{U} \cap V)$.

Def: A C^∞-structure on a top. manifold M is a family of $\tilde{\mathcal{U}} = \{(\mathcal{U}_\alpha, \phi_\alpha)\}$ of coordinate neighborhoods such that

1) $M = \bigcup_\alpha \mathcal{U}_\alpha$

2) $(\mathcal{U}_\alpha, \phi_\alpha), (\mathcal{U}_\beta, \phi_\beta)$ are C^∞-compatible for all α, β

Max cond. $\to$ 3) For all (V, ψ) compatible with every $(\mathcal{U}, \phi) \in \tilde{\mathcal{U}}$ then $(V, \psi) \in \tilde{\mathcal{U}}$.

Def: A C^∞-manifold (smooth manifold) is a top. manifold together with a C^∞-structure.

However, very hard to check #3 condition. So we use this:

Thm: If M is a top. manifold and $\{(V_\alpha, \phi_\alpha)\}$ is a covering of M by C^∞-compatible coord. neigh, then $\exists!$ C^∞-structure on M containing this coord neigh.

PE: Define C^∞-structure to be collection $\tilde{\mathcal{U}}$ of all coord neigh. $(\mathcal{U}, \phi)$ that are C^∞-compatible with $\{(V_\alpha, \phi_\alpha)\}$. We check the conditions:

1) Obvious

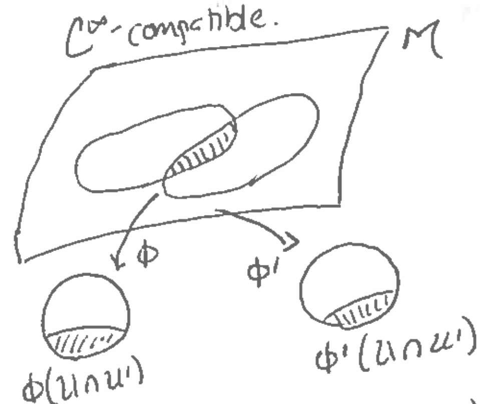
3) Follows from 2.

2) Suppose $(\mathcal{U}, \phi), (\mathcal{U}', \phi')$ are in $\tilde{\mathcal{U}}$. Suffices to show

$\phi'\phi^{-1} : \phi(\mathcal{U} \cap V) \to \phi'(\mathcal{U} \cap V)$

$\phi\phi'^{-1} : \phi'(\mathcal{U} \cap V) \to \phi(\mathcal{U} \cap V)$

C^∞-compatible.



For $p \in \mathcal{U} \cap \mathcal{U}'$, $\exists$ $V_\alpha \ni p \in V_\alpha$ and $V_\alpha \subset \mathcal{U} \cap \mathcal{U}'$. as $\bigcup_\alpha V_\alpha = A$

Let $W = \mathcal{U} \cap \mathcal{U}' \cap V_\alpha$

Now $p \in W$. Let $x = \phi(p) \in \phi(W)$

So we can write

$$\phi' \phi^{-1} = \underbrace{(\phi' \phi_\alpha^{-1})}_{C^\infty} \underbrace{(\phi_\alpha \phi^{-1})}_{C^\infty}$$

So $\phi' \phi^{-1}$ is C^∞ on W. Hence, it is C^∞ at $x = \phi(p)$. Since x arb, $\phi' \phi^{-1}$ and $\phi \phi'^{-1}$ are C^∞ on $\phi(\mathcal{U} \cap \mathcal{U}')$ and $\phi'(\mathcal{U} \cap \mathcal{U}')$

The uniqueness of $\tilde{\mathcal{U}}$ follows by def. ☒

Ex: $\mathbb{R}^n$ is a C^∞ manifold. Apply the thm by choosing $(V, \Phi) = (\mathbb{R}^n, id)$ to see this.

Ex: (Open Submanifold) An open submanifold $\mathcal{U}$ of a C^∞ manifold is always a C^∞ manifold. The C^∞ structure consists of $\{(V', \phi')\}$, where $V' = V \cap \mathcal{U}$ and $\phi' = \phi|_{\mathcal{U}}$ and $\{(V, \phi)\}$ is a C^∞-structure on M.

Thm: If M^m, N^n are C^∞ manifolds, then $M \times N$ is a C^∞ manifold of dim $= m+n$ with C^∞-structure determined by $\{(\mathcal{U} \times V, \Phi_u \otimes \Phi_v)\}$, where $\{(\mathcal{U}, \Phi_u)\}, \{(V, \Phi_v)\}$ are C^∞ structures of M, N respectively and $(\Phi_u \otimes \Phi_v)(p,q) \overset{def}{=} (\Phi_u(p), \Phi_v(q)) \in \mathbb{R}^{m+n}$

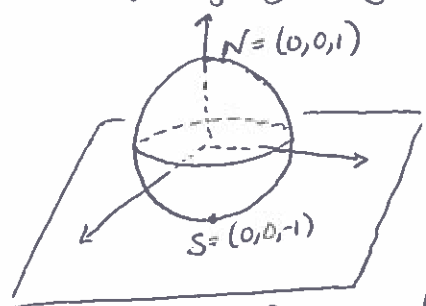
PF: Apply the thm.

Ex: $T^2 = S^1 \times S^1$ or

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$$

Ex: $S^n \subset \mathbb{R}^{n+1}$ is a n-manifold. We will show when $n=2$, suffices to find C^∞ compatible charts. covering of

We use sterrographic proj.



$\mathcal{U}_1 = S^2 \setminus \{N\}$ open $(y_1, y_2, -1)$

$\phi_1 : \mathcal{U}_1 \to \mathbb{R}^2$ via
$(x_1, x_2, x_3) \mapsto (y_1, y_2)$. Use sim Δs to get $((0,0,1), (x_1, x_2, x_3), (y_1, y_2, -1)$ lies along a line)

$$\frac{x_3 - 1}{-1-1} = \frac{x_2}{y_2} = \frac{x_1}{y_1} \quad \text{Solve}$$

$$(y_1, y_2) = \left(\frac{2x_1}{1-x_3}, \frac{2x_2}{1-x_3} \right)$$

Cont. as $x_3 \neq 1$. So C^∞.

Do sim. for south pole.
$\mathcal{U}_2 = S^2 \setminus \{S\}$

Inv. maps here are C^∞ via the Inverse Function Thm. So $\phi_2 \phi_1^{-1}$ and $\phi_1 \phi_2^{-1}$ are C^∞.

So $(\mathcal{U}_1, \phi_1), (\mathcal{U}_2, \phi_2)$ are C^∞ compatible. ☒

09/07/2016

Thm: If X top. space, Hausdorff, with countable basis. If $\{\mathcal{U}_i\}$ covering of X by C^∞-compatible countable neigh then X is a C^∞- manifold.

Today, we see further examples.

X top space

$\sim$ denote an equiv. relation

$[x] = \{y \in X \mid x \sim y\}$, the equiv. class of x. Suppose $A \subset X$, define $[A] = \{$equiv. class $a \in A\}$

$X/\sim$ set of all equivalence classes in X.

$\pi : X \to X/\sim$ the natural map

$x \mapsto [x]$

The quotient topology: $\mathcal{U} \subset X/\sim$ is open iff $\pi^{-1}(\mathcal{U})$ is open in X.

It is simple to show then that π is cont.

Eg: $\mathbb{Z}$ set int. with equiv. relation $a \sim b \Leftrightarrow a-b \in \mathbb{Z}$. Then $\mathbb{R}/\sim = S^1$

Def: An equiv. relation on X is called open if when $A \subset X$, then $\pi(A) = [A]$ is open in the quotient top. Namely, $\sim$ is open if π is an open map.

Lem: If $\sim$ is open and X has a countable basis of topology, then $X/\sim$ also has a countable basis of topology

Pf: By def., if $W \subset X/\sim$ is open, then $\pi^{-1}(W)$ is open in X. Assume $\{\mathcal{U}_i\}_{i=1}^\infty$ is a countable basis for X.

Now $\sim$ is open $\Leftrightarrow$ π open map. So $\pi(\{\mathcal{U}_i\}_{i=1}^\infty)$ is a countable family and is an open covering of $X/\sim$.

Since $\mathcal{U}_i$ is a basis for the top. X, then $\exists\, \mathcal{U}_i$ such that $\mathcal{U}_i \subset \pi^{-1}(W)$. So $\pi(\mathcal{U}_i) \subset \underbrace{\pi(\pi^{-1}(W))}_{=W}$

then $\pi(\{\mathcal{U}_i\}_{i=1}^\infty)$ is a basis of the top. on W. ∎

Lem: Let $\sim$ be open on X. Then $R = \{(x,y) \mid x \sim y\} \subset X \times X$ is closed if and only if X is Hausdorff

Pf:

$\Leftarrow$: We show complement open. If $x \not\sim y$, ie $(x,y) \notin R$, by Hausdorfness $\exists$ open sets $\mathcal{U}_x, \mathcal{U}_y \in X/\sim$ $\ni \mathcal{U}_x \cap \mathcal{U}_y = \emptyset$.

Define $\mathcal{U}'_x = \pi^{-1}(\mathcal{U}_x)$ & $\mathcal{U}'_y = \pi^{-1}(\mathcal{U}_y)$

These are open by def. quotient top.

Note $x \in \mathcal{U}'_x$ and $y \in \mathcal{U}'_y$. Then $(x,y) \in \mathcal{U}'_x \times \mathcal{U}'_y$. Suppose $\exists\, (x',y') \in \mathcal{U}'_x \times \mathcal{U}'_y$ $\ni x' \sim y'$. So $\pi(x') = \pi(y')$.

Then $\pi(\mathcal{U}_x') \cap \pi(\mathcal{U}_y') \neq \phi$.

But this is $\pi(\pi^{-1}(\mathcal{U}_x)) \cap \pi(\pi^{-1}(\mathcal{U}_y))$
$= \mathcal{U}_x \cap \mathcal{U}_y = \phi$. So $\mathcal{U}_x' \times \mathcal{U}_y' \subset$
$R^c = X \times X \setminus R$. So R^c is open.

$\Rightarrow$: Given any $\pi(x), \pi(y) \in X/\sim$ with $\pi(x) \neq \pi(y)$. So $x \nsim y$ so that $(x,y) \in R^c$. But this is open so $\exists$ $\mathcal{U}_x \times \mathcal{U}_y$ cont. (x,y) so that $\mathcal{U}_x \times \mathcal{U}_y \subset R^c$. Then $\pi(\mathcal{U}_x) \cap \pi(\mathcal{U}_y) = \phi$. ▨

Ex: $\mathbb{R}P^n$ - the real projective space.

Take $X = \mathbb{R}^{n+1} \setminus \{0\}$. $\vec{x} = (x_1, \cdots, x_{n+1})$
$\vec{y} = (y_1, \cdots, y_{n+1})$, then $\vec{x} \sim \vec{y}$ iff
$\vec{y} = t\vec{x}$ for some $t \neq 0$.

Show $\mathbb{R}P^n = X/\sim$ is a C^∞ manifold of $\dim = n$:

1) Now $\pi: X \to \mathbb{R}P^n$ is open so $\sim$ is open.

Note $\Phi_t: X \to X$ for $t \neq 0$
$x \mapsto t \cdot x$

is a homeomorphism. (artinian, onto, so image Φ_t is artinian.)

For all $\mathcal{U} \subset X$ open, then

$\pi^{-1}(\pi(\mathcal{U})) = \bigcup_{t \neq 0} \Phi_t(\mathcal{U})$ is open

But then $\pi(\mathcal{U})$ is open by quotient top. But then π is open.

2) Now X has a countable basis so that $X/\sim$ has a countable basis. (Lem 1).

3) Look at $X \times X$. Define $F(x,y) =$
$\sum_{i \neq j} x_i x_j [x_i y_j] = F(x,y)$. ~~Now~~ Now $F(x,y)$ is cont so that $F^{-1}(0)$ is closed. Then $F^{-1}(0) =$
$\{(x,y) \in X \times X \mid x \neq ty\}$ for $t \neq 0$
$= R$ by definition. So R closed by second lemma. $X/\sim = \mathbb{R}P^n$ is Hausdorff

4) Check $\mathbb{R}P^n$ is a C^∞-manifold by finding a open covering of C^∞ compatible coordinate neigh. For $i = 1, \cdots, n+1$. Define $\tilde{\mathcal{U}}_i = \{ (x_1, \cdots, x_{n+1}) \mid x_i \neq 0 \}$. and $\mathcal{U}_i = \pi(\tilde{\mathcal{U}}_i)$. $\tilde{\mathcal{U}}_i$ open so $\mathcal{U}_i$ is open. Moreover, $\mathcal{U}_i$ from $i = 1, \cdots, n+1$ covers $\mathbb{R}P^n$.

Define $\Phi_i: \mathcal{U}_i \to \mathbb{R}^n$
$(x_1, \cdots, x_{n+1}) \mapsto \underbrace{\frac{(x_1, \cdots, \hat{x}_i, \cdots x_{n+1})}{x_i}}_{\text{omit \& 'normalize'}}$

Check: Φ_i well defined & diffeomorphism. Well-defined is obvious.

'Smoothness' is obvious.

Injective: $\phi_i(x) = \phi_i(y)$

$$\frac{(x_1, \dots, \hat{x}_i, \dots, x_{n+1})}{x_i} = \frac{(y_1, \dots, \hat{y}_i, \dots, y_{n+1})}{y_i}$$

$$\frac{(x_1, \dots, x_{n+1})}{x_i} = \frac{(y_1, \dots, y_{n+1})}{y_i}$$

i^{th} spot both $= 1$ so fine.

Then $\vec{x} = \underbrace{\left(\frac{x^i}{y^i}\right)}_{\text{scalar}} \vec{y}$ so $x \sim y$. $\neq 0$

Surj: Obvious

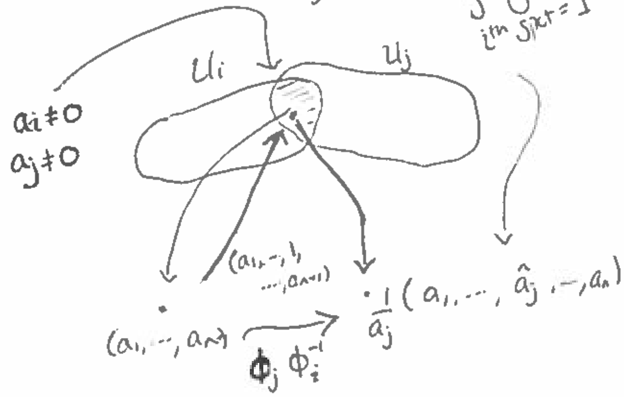
$\hat{a} = (a_1, a_2, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_{n+1}) \in \mathcal{U}_i$

and $\phi_i(\hat{a}) = (a_1, \dots, \hat{1}, \dots, a_{n+1})$

Now just need C^∞-compatible condition.

$\phi_j \phi_i^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \to \phi_j(\mathcal{U}_i \cap \mathcal{U}_j)$

$(a_1, \dots, a_n) \mapsto \frac{1}{x_j}(a_1, \dots, a_n)$ j^{th} gone & i^{th} spot $= 1$



But $a_j \neq 0$ so this is C^∞.

Example done 💡 Onto another more special example. One should check all details at least once for the following.

Ex: (Grassmanian, $G(k,n)$)

$G(k,n)$ is the set of all k-planes containing 0 in $\mathbb{R}^n$.

e.g. $k=1$ then $G(1,n) = \mathbb{R}P^{n-1}$

Let $F(n,k)$ be set of all k-frames in $\mathbb{R}^n$, where k-frame in $\mathbb{R}^n$ is a lin. indep. of k-row vectors in $\mathbb{R}^n$.

$x_I = (x_1^I, \dots, x_n^I)$ & $x^k = (x_1^k, \dots, x_n^k)$

A k-frame is identified with a $k \times n$ matrix $X = \left.\begin{pmatrix} x^I \\ \vdots \\ x^k \end{pmatrix}\right\}$ $k \times n$ matrix

Clearly, X is nonsingular. Then $F(n,k) \underset{\text{open}}{\subset} \mathbb{R}^{n \cdot k}$: determined by $\binom{n}{k}$ equations that are determinants of all $k \times k$ submatrices. Then $F(n,k)$ is a C^∞-manifold.

<u>09/12/2016</u>

Ex: $\overset{\text{real}}{\wedge}$ Grassmanian, $G(k,n)$. All k-planes in $\mathbb{R}^n$. $F(k,n)$ set of all k-frames in $\mathbb{R}^n$. A k-frame is k-lin. indep. row vectors in $\mathbb{R}^n$.

$x_1 = (x_1^1, \dots, x_n^1)$

$x_k = (x_1^k, \dots, x_n^k)$

Write $X = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} x_1^1 & \cdots & x_n^1 \\ \vdots & \ddots & \vdots \\ x_1^k & \cdots & x_n^k \end{pmatrix}$

At least one nonzero $k \times k$ determinant (of submatrix) $\longrightarrow$ nonsingular $k \times n$ matrix

Recall $M(k,n)$ is the set of all $k\times n$ matrices $\cong \mathbb{R}^{kn}$

We want to see $F(k,n) \underset{\text{open}}{\subset} M(k,n)$

$$F(k,n) = X = \bigcup_{1\le j_1\le \cdots \le j_k \le n} \{ M \in M(k,n) \mid \underset{\det}{\hat{}}(M)_{j_1,\dots,j_k}^{1,2,\dots} \neq 0 \}$$

submatrix of all rows and $j_1,\dots,j_k$ colums

This is an open submanifold in $\mathbb{R}^{kn}$.
det given by poly.

Then this a C^∞-manifold.
$F(k,n)$

The equivalence relation $\sim$ is defined by
$X\sim Y \rightleftharpoons \exists\, \mathcal{U}\in GL(k,\mathbb{R}) \ni X = \mathcal{U}Y$

So X, Y span same plane - just a change of basis. Then
$G(k,n) = F(k,n)/\sim$

Check: $G(k,n)$ has countable base for topology, Hausdorff, $\sim$ open, $\mathcal{R}$ closed, $F(k,n)$ has countable basis
immediate

Let $J = (j_1,\dots,j_k) \subset \{1,\dots,n\}$
j_k strictly increasing. Then
$J' = (i_1,\dots,i_{n-k}) = \{1,\dots,n\}\setminus J$
ordered.

Let X = as before - the k-frame. Then $X_J =$

$$\begin{pmatrix} x^1_{j_1} & \cdots & x^1_{j_k} \\ \vdots & \cdots & \\ x^k_{j_1} & \cdots & x^k_{j_k} \end{pmatrix}_{k\times k}$$

&

$X_{J'} =$

$$\begin{pmatrix} x^1_{i_1} & \cdots & x^1_{n-k} \\ \vdots & \ddots & \vdots \\ x^k_{i_1} & \cdots & x^k_{i_{n-k}} \end{pmatrix}_{k\times(n-k)}$$

Let $\hat{\mathcal{U}}_J = \{ X\in F(k,n) \mid \det X_J \neq 0\}$

be the open set in $F(k,n)$.

Let $\mathcal{U}_J = \pi(\tilde{\mathcal{U}}_J) \underset{\text{open}}{\subset} G(k,n)$

For any $Y\in \tilde{\mathcal{U}}_J$, $\exists!\, X\in \tilde{\mathcal{U}}_J$ such that $X_J = 1_k$
$k\times k$ id. matrix.

and $X\sim Y$. Let $J = \{1,\dots,k\}$

$$X = \left(\begin{array}{ccc|ccc} 1 & & 0 & x^1_{k+1} & \cdots & x^1_n \\ & \ddots & & \vdots & & \\ 0 & & 1 & x^{k+1}_k & \cdots & x^k_n \end{array}\right)$$

Define $\varphi_J : \mathcal{U}_J \to \mathbb{R}^{k(n-k)} \cong M(k,n-k)$

$Y \mapsto Y_J^{-1} Y_{J'}$

Check φ_J well defined. Namely,
$X\sim Y$, $Y\sim Y' \rightsquigarrow \varphi_J(Y) = \varphi_J(Y')$

$Y\sim Y' \rightleftharpoons \mathcal{U}$ inv. w/ $Y = \mathcal{U}Y'$

2) $V = \bigcup_i W_i$; $F: V \to \mathbb{R}$ is a function and F is C^∞ on W_i for all i, then F is C^∞ on V.

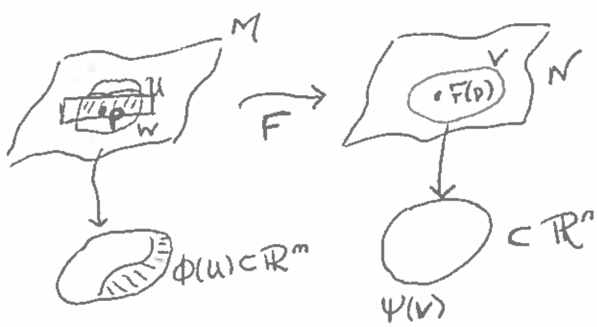
09/14/2016

dim

M^m, N^n are C^∞ manifolds

$W \subset M$ open

$F: W \to N$ a map (cont.)



$\exists$ (U, Φ), (V, Ψ) coord. neigh. on M, N, respectively, $p \in U$, $F(p) \in V$

possibly making U smaller, we have F represented by local coord.

$\hat{F} \overset{def}{=} \Psi \circ F \circ \Phi^{-1}: \Phi(U \cap W) \to \Psi(V)$

$(x_1, \cdots, x_m) \mapsto (y_1, \cdots, y_n)$

'near' p.

Def: $F: W \to N$ is a C^∞ map if $\forall p \in W$, $\exists (U, \Phi), (V, \Psi)$ as above such that $\hat{F}$ is C^∞ on $\Phi(U \cap W)$

Rem: The smoothness of F does not depend on the choices of (U, Φ) & (V, Ψ): using (U', Φ'), (V', Ψ') then

$\hat{F}' \overset{def}{=} \Psi' \circ F \circ (\Phi')^{-1}$

$= \underbrace{(\Psi' \Psi^{-1})}_{C^\infty} \underbrace{(\Psi \circ F \circ \Phi^{-1})}_{\hat{F}} \underbrace{(\Phi \Phi'^{-1})}_{C^\infty}$

So $\hat{F}$ C^∞ iff $\hat{F}'$ C^∞. So this is well defined.

Write $\hat{F} = (F_1, \cdots, F_m)$

From $\Phi(U \cap W) \to \Psi(V)$

$(x_1, \cdots, x_m) \mapsto (y_1, \cdots, y_n)$

Namely,

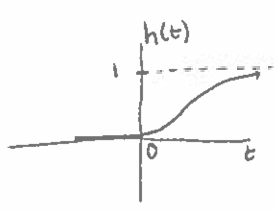
$F_j(x_1, \cdots, x_m) = y^j \quad \forall\, 1 \le j \le n$

Thm: $F \subset M$ closed. $K \subset M$ compact. ($\to C^\infty$, $\to$ cc) Then if $F \cap K = \emptyset$, $\exists$ smooth function f on M with values in $[0,1]$ such that $f|_K \equiv 1$ & $f|_F \equiv 0$.

Note: Thm holds if $M = \mathbb{R}^m$.

Pf:

(1) $\forall B_\epsilon(a) \subset \mathbb{R}^m$ (open ball with center a & radius ϵ). Then $\exists$ C^∞ function $g(x)$ on $\mathbb{R}^m$ with values in $[0,1]$ such that $g|_{B_{\epsilon/2}} \equiv 1$ and $g|_{\mathbb{R}^m \setminus B_\epsilon(a)} \equiv 0$. Why? $h(t) = \begin{cases} 0, & t \le 0 \\ e^{-1/t}, & t > 0 \end{cases}$



Check $h(t)$ is C^∞ on $\mathbb{R}$.

$Y_J = \mathcal{U} Y_J'$, $Y_{J'} = \mathcal{U} Y_{J'}'$. Then

$\Psi_J(Y) = Y_J^{-1} Y_{J'} = (\mathcal{U} Y_J')^{-1}(\mathcal{U} Y_{J'}')$

$\Psi_J(Y') \xleftarrow{\text{is}} = Y_J'^{\,-1} Y_{J'}'$

So Ψ_J is well-defined. Claim

Ψ_J is a 1-1, onto, smooth map.

Onto obviously by form of X given

$X = (\,1 \,\vdots\, -\,)$

Smooth is also obvious given image.

1-1: $X = (\,1 \,\vdots\, -\,)$ change any x's and change lin. span of rows. Changes output. This is all just basic lin alg. which is omitted.

So $\{(\mathcal{U}_J, \Psi_J)\}$ is a covering that is C^∞-compatible so long as we have C^∞-compatible. Namely,

$\Psi_K \circ \Psi_J^{-1}$ is smooth. This is a map $\Psi_J(\mathcal{U}_J \cap \mathcal{U}_K) \to \Psi_K(\mathcal{U}_J \cap \mathcal{U}_K)$

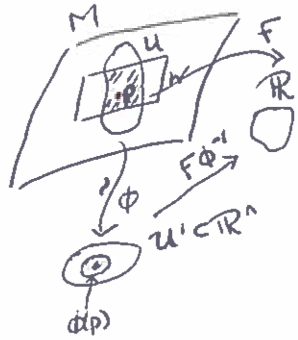
this is not trivial, it takes some work. But obvious because always mult. by a C^∞ matrix. //

Differentiable functions & Maps

M: C^∞-manifold

$W \underset{\text{open}}{\subset} M$

$f: W \to \mathbb{R}$ function.



For all $p \in W$, $\exists (\mathcal{U}, \phi)$ ∗ coord. neigh $p \in \mathcal{U}$

$\phi(p) = (x_1(p), \dots, x_n(p))$

local coordinate. Represent F by

$F \circ \phi^{-1} : \phi(\mathcal{U} \cap W) \to \mathbb{R}$

Without confusion, denote $F \overset{\text{def}}{=} F\phi^{-1}$ as the map $\phi(\mathcal{U} \cap W) \to \mathbb{R}$.

Def: $f: W \to \mathbb{R}$ is C^∞ if for all $p \in W$, $\exists (\mathcal{U}, \phi)$ coord. neigh. of p such that $\underbrace{f\phi^{-1}: \phi(W \cap \mathcal{U}) \to \mathbb{R}}_{\text{local representative}}$ is C^∞

Note: What if you change coord. neigh? Already have change neigh. that is C^∞.

Well-def that is C^∞ compatible. { Well defined follows from this.

That is, C^∞ on one coord neigh then is on all coord. neigh.

Each coord. is C^∞. Proj map C^∞ composed with $\phi \leftarrow$ -inv?

Facts:

1) $F: W \to \mathbb{R}$ is C^∞, $V \underset{\text{open}}{\subset} W \to$

$F|_V$ is C^∞ on V.

Let $\bar{g}(x) = \dfrac{h(\epsilon^2 - |x|^2)}{h(\epsilon^2 - |x|^2) + h(|x|^2 - 1/4\,\epsilon^2)}$

where $|x|^2 = \sum_{i=1}^{m} x_i^2$ Never both 0 as 'x's' in diff neigh.

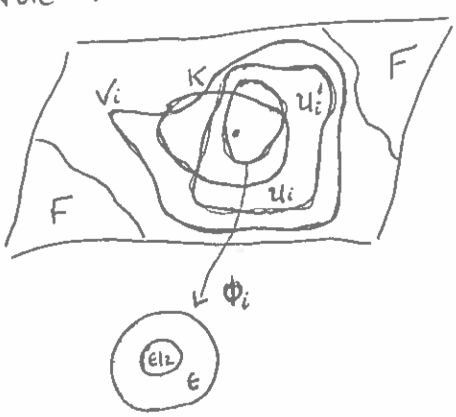
Check $0 \le \bar{g} \le 1$. $\bar{g} \equiv 0$ on $\mathbb{R}^m \setminus B_\epsilon(0)$ and $\bar{g} \equiv 1$ on $\overline{B_{\epsilon/2}(0)}$. Then $g(x)$ is $\bar{g}(x-a)$.

(2) F is closed in M. $\exists\, (V_i, \phi_i)$ coord. neigh, $\mathcal{U}_i' \subset \mathcal{U}_i \subset V_i$ such that

$\mathcal{U}_i \underset{\text{diffeo.}}{\cong} B_\epsilon(a) \subset \mathbb{R}^m$ and

$\mathcal{U}_i' \underset{\text{diff}}{\cong} B_{\epsilon/2}(a) \subset B_\epsilon(a)$

Note $V_i \subset M \setminus F$.



$\phi_i|_{\mathcal{U}_i} : \mathcal{U}_i \to B_\epsilon(a_i)$ homeo.

$\phi_i : \mathcal{U}_i' \to B_{\epsilon/2}(a_i)$ homeo.

Can do this by possibly shrinking targets using preimages & cont.

All possible such $\mathcal{U}_i$ cover K by finding $\mathcal{U}_{i,a}$ for each $a \in K$.

But K is compact!

* Used F closed in choosing V_i (open) so can find 'small enough' neigh to avoid F.

$$F^c \supset \underbrace{\bigcup_{i=1}^{k} \mathcal{U}_i'}_{} \supset K$$

using compactness.

But then.....

Define: $\sigma_i = \begin{cases} g_i \phi_i^* & \text{on } \mathcal{U}_i \\ 0, & \text{on } \mathcal{U}_i^c \leftarrow \text{comp. in } M \end{cases}$

Check $\left\{ \begin{array}{l} \text{Claim } \sigma_i \text{ is } C^\infty \text{ on } M,\ \operatorname{Im}\sigma_i \in [0,1], \\ \text{and } \equiv 1 \text{ on } \mathcal{U}_i'.\ \sigma_i \text{ is } 0 \text{ already on outside} \\ \mathcal{U}_i \text{ so define } 0 \text{ on } V_i \text{ 'and beyond'.} \end{array} \right.$

Define:

$$f = 1 - \prod (1 - \sigma_i) \in C^\infty(M)$$

Then $F \in [0,1]$ & $F \equiv 0$ on F,

$F \equiv 1$ on K. ☒

Cor: $\mathcal{U} \underset{\text{open}}{\subset} M$, $p \in \mathcal{U}$, $F \in C^\infty(\mathcal{U})$ then $\exists\, V$, $V \underset{\text{open}}{\subset} \mathcal{U}$, $p \in V$, $F^* \in C^\infty(\mathcal{U})$ and $F^* = F$ on V and is 0 on $M \setminus \mathcal{U}$.

Pf: Let $F = M \setminus \mathcal{U}$ closed. $V \subset \bar{V} \subset \mathcal{U}$, $\bar{V}$ compact. By Thm, $\exists\, g \in C^\infty(M)$, $0 \le g \le 1 \ni g|_{M \setminus \mathcal{U}} \equiv 0$ & $g|_{\bar{V}} \equiv 1$

Then $F^* = Fg$. ☒

Def: A C^∞ map $F: M \to N$ is a diffeomorphism if F is a homeomorphism & F^{-1} is C^∞. In this case, M & N are called diffeomorphic.

Rem: Diffeomorphism is an equivalence relation.

Ex: $F: \mathbb{R} \to \mathbb{R}$ is C^∞ homeomorphism $t \mapsto t^3$. But $F^{-1} = t^{1/3}$. This inv. is not even C^1 at $t=0$. So F is not a diffeomorphism.

Define $\mathbb{R}_1$ by one coord neigh (U_1, ϕ_1) ; $U_1 = \mathbb{R}$, $\phi_1(t) = t$.

Define $\tilde{\mathbb{R}}$ by one coord. neigh (U_2, ϕ_2) ; $U_2 = \mathbb{R}$ & $\phi_2(t) = t^3$

Note: $\phi_1 \phi_2^{-1}(t) = t^{1/3}$ not C^∞
So (U_1, ϕ_1), (U_2, ϕ_2) are not C^∞ compatible.

So $\mathbb{R}, \tilde{\mathbb{R}}$ have different C^∞ structure.

$F: \mathbb{R} \to \tilde{\mathbb{R}}$ is a diffeomorphism
$t \mapsto t^{1/3}$. ?

In local coord.,

$\phi_2 F \phi_1^{-1}(t) = t$

$\phi_1 F \phi_2^{-1}(t) = t$.

And both C^∞. By definition diffeomorphisms.

09/19/2016

Rank of a map, immersion

Recall

$A \in M(m,n)$

rank A = max lin. indep row vectors (# of)
= # max lin. indep column vectors
= max # nonvanish. minor det.

* rank $A \le \min\{m,n\}$

$F: \underbrace{\mathcal{U}}_{\text{open}} \to \mathbb{R}^m$ - C^r map
$\mathcal{U} \subset \mathbb{R}^n$

If $x \in \mathcal{U}$, $DF(x) = \dfrac{\partial(F_1, \dots, F_m)}{\partial(x_1, \dots, x_n)}$

$$= \begin{pmatrix} \frac{\partial f^1}{\partial x_1}, & \dots, & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x_1} & \dots & \frac{\partial f^m}{\partial x_n} \end{pmatrix}_{m \times n}$$

the Jacobian of F.

Inverse Function Theorem

Assume $F = (F_1, \dots, F_n)$

$F: \mathcal{U} \to \mathbb{R}^n$ is a C^r-map
$\mathcal{U} \subset \mathbb{R}^n$

Assume DF has maximal rank n at $x \in \mathcal{U}$. Then $\exists\, W \subset \mathcal{U}$ open, $x \in \mathcal{U}, W \ni F$ is a diffeomorphism from W to $F(W)$. Moreover, if $y = F(x)$ then $DF^{-1}(y) = (DF(x))^{-1}$

Rem: $F: \mathcal{U} \to \mathbb{R}^m$ is a C^r map, $\mathcal{U} \subset \mathbb{R}^n$

If $DF(x)$ is of rank k, then $\exists$ open neigh V of $x \ni$ rank of DF is at least k in V. Moreover, if $k = \min\{m, n\}$, then rank of DF is constantly k in V.

Rank Thm

$\mathcal{U}_0 \subset \mathbb{R}^n$ open

$V_0 \subset \mathbb{R}^m$ open

$F: \mathcal{U}_0 \to V_0$ C^r map

and $DF|_{V_0}$ constant rank k.

Then $\exists\, \mathcal{U} \subset \mathcal{U}_0 \ni V \subset V_0$ and C^r diffeo.

$x_0 \in \mathcal{U}_1$, $y_0 \in V_1$

$G: \mathcal{U}_1 \to \mathcal{U}$

$H: V_1 \to V$

such that

$H \cdot F \cdot G^{-1}: \mathcal{U} \to V$

$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$

Pf: WLOG, $x_0 = 0, y_0 = 0$.

Replace F by $F(x + x_0) - F(x_0)$

By permuting coord. in

$x = (x_1, \dots, x_n)$; $y = (y_1, \dots, y_m)$

we expose $k \times k$ minors of nonzero determinate of $DF(x_0)$ $y \dots$

$$\frac{\partial \vec{f}}{\partial \vec{x}} = \left(\;\Sigma\; \right)$$

Define C^r-map $G: \mathcal{U}_0 \to \mathbb{R}^n$

$(x_1, \dots, x_n) \mapsto (F^1(x), \dots, F^k(x), x^{k+1}, \dots, n)$

Then DG

$$\underbrace{\begin{pmatrix} \frac{\partial \vec{f}}{\partial \vec{x}} & * \\ 0 & 1_{n-k} \end{pmatrix}}_{\text{nonzero det. at } x=0}$$

So DG nonsing. at $x = 0$. Apply inverse Function thm: $\mathcal{U}_1' \subset \mathcal{U}_0 \ni G$ diffeo from $\mathcal{U}_1'$ onto $G(\mathcal{U}_1') = \mathcal{U}'$.

$FG^{-1}(0) = 0$ so $FG^{-1}(\mathcal{U}')$

$= F(\mathcal{U}_1') \subset F(\mathcal{U}_0) \subset V_0$.

$F(g^{-1}(x)) = (x_1, \dots, x_k, \bar{F}^{k+1}(x), \dots \bar{F}^n(x))$

where $\bar{F}^j(x) = F^j G^{-1}(x)$

Compute

$$D(FG^{-1}) = \begin{pmatrix} I_k & 0 \\ \frac{\partial F_i}{\partial x_1} \cdots & \frac{\partial F_i}{\partial x_m} \\ \vdots & \vdots \end{pmatrix}_{n\times n}$$

$D(FG^{-1}) = DF \cdot DG^{-1}$

(still of rank K on $\mathcal{U}'$), Then

$$\begin{pmatrix} \frac{\partial F_{K+1}}{} & \cdots & \frac{\partial F}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial F^{K+1}} & \cdots & \frac{\partial F^n}{\partial x^n} \end{pmatrix}$$

ie, $(F_{K+1}, \cdots, F_m)$ does not depend on $(x_1, \cdots, x_n)$. Define

$T: V'$ so $\subset \mathbb{R}^m \to V_0$ to $\mathbb{R}$;

not small enough.

$(y_1, \cdots, y_m) \mapsto (y_1, \cdots, y_k, y_{k+1} + \bar{F}^{k+1}(y_1, \cdots, y_n), \cdots, y_m + \bar{F}_m(y_1, \cdots y_k))$

Check $T(0) = 0$ &

$$DT = \begin{pmatrix} I_k & 0 \\ * & I_{m-k} \end{pmatrix} \text{ in } V'$$

Apply IVT yet again ?

T is a diffeo. from $V \subset V'$ onto $V_1 = T(V) \subset V_0$, $0 \in V$

Choose $\mathcal{U} \subset \mathcal{U}' \ni FG^{-1}(\mathcal{U}) \subset V_1$, $0 \in \mathcal{U}$

Let $H = T^{-1}$. Then

$$\mathcal{U} \xrightarrow[\text{diffeo.}]{G^{-1}} G^{-1}(\mathcal{U}) = \mathcal{U}_1 \xrightarrow{F} V_1 \xrightarrow[\text{diffeo.}]{H = T^{-1}} V$$

Therefore, $HFG^{-1}(x) = (x_1, \cdots, x_k, 0, \cdots, 0)$ ◻

If $F: N^n \to M^m$ be a C^∞ map

Let $p \in N$. Assume $(\mathcal{U}, \phi)$ (V, ψ) coord neigh of p & $F(p)$.

In local coord.

$\hat{F} = \psi \circ F \circ \phi^{-1}$

$\phi(\mathcal{U}) \to \phi(V)$

$(x_1, \cdots, x_n) \mapsto (F_1(x), \cdots, F_m(x))$

Def: Rank of F at p is rank of $\hat{F}$ at $\phi(p)$. Namely, the rank of the Jacobian at $\phi(p)$.

Note: The value of rank of F does not depend on choices of $(\mathcal{U}, \phi)$, (V, ψ). Namely $(\mathcal{U}', \phi')$, (V', ψ')

$\hat{F}' \stackrel{\text{def}}{=} \psi' F \phi'^{-1}$

$= \psi' \psi^{-1} \underbrace{\psi F \phi^{-1}}_{\hat{F}} \phi \phi'^{-1}$

$D\hat{F}' = \underbrace{D(\psi' \psi^{-1})}_{\text{Full rank}} D\hat{F} \underbrace{(D \phi \phi'^{-1})}_{\text{Full rank}}$

So rank $D\hat{F}' = $ rank $D\hat{F}$

Thm (Rank Thm): $F: N^n \to M^m$ a C^∞-map. If rank $F = K$ at every point in N and $\exists$ neigh $(\mathcal{U}, \phi)$, (V, ψ) of p & $F(p)$. $\ni$

$\phi(p) = 0$, $\psi(F(p)) = 0$ and

$\exists$ (after possible shrinking) $\hat{F} = \psi F \phi^{-1}: \phi(\mathcal{U}) \to \phi(V)$

$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_k, 0, \cdots, 0)$

Def: $F: N^n \to M^m$ a C^∞-map

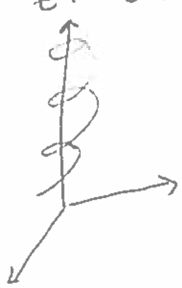
is an immersion or submersion if rank $F = n$ or rank $F = m$, respectively. If F is an injective immersion, let

$\tilde{N} = F(N) \subset M$

$\tilde{N}$ is called a submanifold of M if top. & C^∞ structure on $\tilde{N}$ induced by M coincides with top. & C^∞ structure induced by N using F. Thus, $F: N \to \tilde{N}$ is a diffeomorphism.

Ex: $F: \mathbb{R} \to \mathbb{R}^3$ Helix

$t \mapsto (t\cos 2\pi t, t\sin 2\pi t, t)$



submanifold $\mathbb{R}^3$

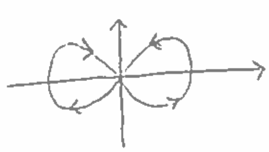
Ex: $F: \mathbb{R} \to \mathbb{R}^2$

$t \mapsto (\cos 2\pi t, \sin 2\pi t)$

Immersion but not injective.

Ex: $F: \mathbb{R} \to \mathbb{R}^2$

$t \mapsto \left(2\cos\left(t - \frac{\pi}{2}\right), \sin\left(2\left(t - \frac{\pi}{2}\right)\right)\right)$



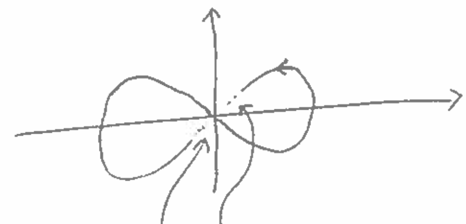
immersion but not injective

Ex: $F: \mathbb{R} \to \mathbb{R}^2$

Define $g(t) = \pi + 2\tan^{-1} t$

Inc. with $g(0) = \pi$ & $g(\infty) = 2\pi$

$t \mapsto \left(2\cos\left(g(t) - \frac{\pi}{2}\right), \sin\left(g(t) - \frac{\pi}{2}\right)\right)$



Never gets there

Immersion & Injective

Not submanifold.

09/21/2016

Rank Thm:

$F: N^n \to M^m$ C^∞-map

Assume rank $F = k$ at every point of N. Then $\exists (U, \phi), (V, \psi)$ on N and N such that

$$\psi F \phi^{-1}: \phi(U) \to \psi(V)$$

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$$

Immersion: rank $F = n \le m$

Submersion: rank $F = m \le n$

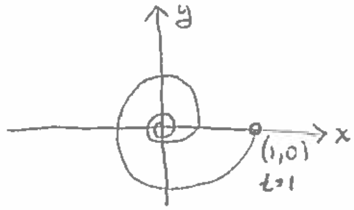
$F: N \to M$ is an injective immersion and $F(N)$ is a submanifold of M if $F(N)$ has an induced topology and C^∞ structure from M.

Ex: Helix in $\mathbb{R}^3$.

Ex: $F: (1,\infty) \to \mathbb{R}^2$

$$t \mapsto \left(\tfrac{1}{t}\cos 2\pi t, \tfrac{1}{t}\sin 2\pi t\right)$$

$$\lim_{t\to\infty} F(t) = (0,0)$$

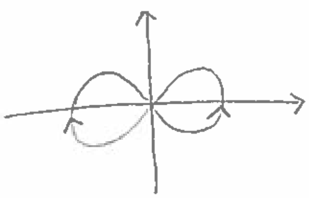


Rank 1 and injective. Check this is a submanifold.

Ex:

$F: \mathbb{R} \to \mathbb{R}^2$

$$t \mapsto \left(2\cos(t-\tfrac{\pi}{2}), \sin(2(t-\tfrac{\pi}{2}))\right)$$

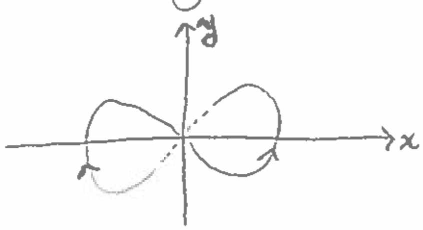


Not injective.

Ex: $F: \mathbb{R} \to \mathbb{R}^2$

$$t \mapsto \left(2\cos(g(t)-\tfrac{\pi}{2}), \sin(2(g(t)-\tfrac{\pi}{2}))\right)$$

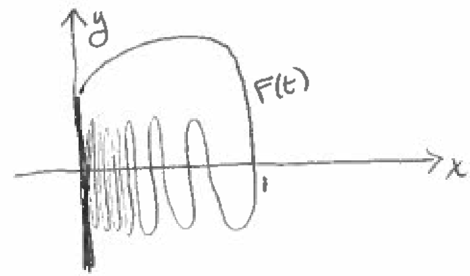
Choose $g(t) = \pi + 2\tan^{-1} t$



Injective but not submanifold.

Ex: $F: \mathbb{R} \to \mathbb{R}^2$

$$F(t) = \begin{cases} (1/t, \sin(\pi t)), & t \in [1,\infty) \\ \text{smoothly} & t \in [-1,1] \\ (0, t+2) & t \in (-\infty,-1] \end{cases}$$



Injective, immersion, not a submanifold

Def: An embedding (Nowadays) (imbedding) is an injective immersion $F: N \to M$ and F is a homeomorphism N to image $F(N) \subset M$ with the induced topology (M, N C^∞).

+Check: $F(N)$ is a submanifold.

Thm: $F: N^n \to M^m$ ~~g~~ immersion with $m \geq n$, then $\forall p \in N$, $\exists\, \mathcal{U} \subset_{\text{open}} N$ $p \in \mathcal{U}$ such that $F|_{\mathcal{U}}$ is an embedding.

Pf: By Rank Theorem, can find (U,Φ), (V,Ψ) on N & M, $p \in U$ and $F(p) \in V$, such that

$$\hat{F} \stackrel{\text{def}}{=} \Psi F \Phi^{-1} : \Phi(U) \to \Psi(V)$$

$$(x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0,\dots,0)$$

$\hat{F}$ is a homeo. So $\hat{F}$ embedding iff F is an embedding on Ψ on $\Psi(U)$. ∎

So the only barrier preventing immersion $\Rightarrow$ embedding is a global one.

Submanifolds

Def: A subset N of a C^∞-manifold M has n-dim. submanifold property if $\forall p \in N$, $\exists$ coord. neigh. $(\mathcal{U}, \phi)$ on M with local coord. $(x_1, \dots, x_m)$ such that

Can make this happen via trans & restriction

1) $\phi(p) = 0$
2) $\phi(\mathcal{U}) = [-\epsilon, \epsilon] \times \cdots \times [-\epsilon, \epsilon]$
3) $\phi(\mathcal{U} \cap N) = \{ x \in [-\epsilon, \epsilon]^m \mid x^{n+1} = \cdots = x^m = 0 \}$

content

for some $\epsilon > 0$.

If N has this property, coord. neigh of this type is called the preferred coord.

Lem: $N \subset M$ has n-dim. submanifold prop. then with the induced topology, N is a n-dim top. manifold.

Moreover, each preffered coord. system $(\mathcal{U}, \phi)$ of M defines a local coord. neighborhood $(V, \tilde{\Phi})$ on N by $V = \mathcal{U} \cap N$, $\tilde{\Phi} = \pi \phi|_V$, where

$\pi : \mathbb{R}^m \to \mathbb{R}^n$

$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$

is the projection map.

The local coord. on N defines a C^∞ structure on N such that the inclusion $i : N \to M$ is an embedding.

PE: $(V, \tilde{\Phi})$ is a topological coord. neigh. Namely, $\{V\}$ is an open covering of N. Moreover, $\tilde{\Phi} = \pi \phi|_V$: $V \to [-\epsilon, \epsilon]^n$ is a homeo. by submanifold property (ϕ homeo.).

Note they are C^∞ compatible $\because$ suppose $(V, \tilde{\Phi})$, $(V, \tilde{\Phi}')$ are two such coord. neigh. Then

$\tilde{\Phi}' \tilde{\Phi}^{-1} : (x_1, \dots, x_n) \mapsto$

$\pi \phi' \phi^{-1} (x_1, \dots, x_n, 0, \dots, 0)$

is C^∞. So N is a C^∞ manifold.

Now need to check i is an embedding. In preffered coordinates, $(\mathcal{U}, \phi)$, $(V, \tilde{\Phi})$ of N and M. $\phi i \tilde{\Phi}^{-1} (x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$. i is injective, immersion, & homeo. thus this is an embedding. $\boxtimes$

Def: A regular submanifold of a C^∞ submanifold M is a subset N with the submanifold property with the C^∞ structure determined by the preffered coord. system.

Ex: (S^2: 2-dim sphere)

$$S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$$

For all $x \in S^2$ in spherical coord. (ρ, θ, ϕ) of $\mathbb{R}^3 \setminus O$,

$$x = (r\cos\theta\cos\phi, r\cos\theta\sin\phi, r\sin\theta)$$

Let $\tilde{r} = r - 1$; $\hat{\theta} = \theta - \theta_0$; $\tilde{\Phi} = \Phi - \phi_0$

S^2 is defined by $\{\tilde{r} = 0\}$

Thm: $F: N' \to M$ (prime, not a 1) is an embedding of a C^∞-manifold of dim n in a C^∞ manifold M with dim m. Then $N = F(N')$ has n-dim submanifold property and thus a regular submanifold. Moreover, $F: N' \to N$ is a diffeo.

Pf: For all $p \in N'$, $F(p) = q \in N \subset M$. Choose coord. neigh (U, ϕ) of N' at p such that $\phi(p) = 0 \in \mathbb{R}^n$. $\phi(U) = [-\epsilon, \epsilon]^n \subset \mathbb{R}^n$ and (V, ψ) of M at q such that $\psi(q) = 0 \in \mathbb{R}^m$, $\psi(V) = [-\epsilon, \epsilon]^m \subset \mathbb{R}^m$. Moreover, $\hat{F} \overset{def}{=} \psi F \phi^{-1} : \phi(U) \to \psi(V)$

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$$

by the rank theorem. So N has n-dim. submanifold property. Moreover in preferred coord, $\tilde{\Psi} F \phi^{-1}$ but $\tilde{\Psi} = \pi \psi$.

So $\tilde{\Psi} F \phi^{-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$ is 1.

So of course this is a diffeomorphism. ∎

09/26/2016

Thm: $F: N \to M$ injective immersion where N is compact. Then F is an embedding & $\tilde{N} = F(N)$ is a regular submanifold.

Pf: $S \subset N$ closed. Then by basic top., S is compact. Then $F(S)$ is compact so $F(S)$ is closed. Then F maps open sets to open sets as F is 1-1 & onto from N to $\tilde{N}$. Then F is a homeo. from N to $\tilde{N}$. ∎

Thm: N^n, M^m C^∞-manifolds. $F: N \to M$ a C^∞-map with constant rank, say k, on N. Then for all $q \in F(N)$, $F^{-1}(q)$ is a closed regular submanifold of dim $n-k$.

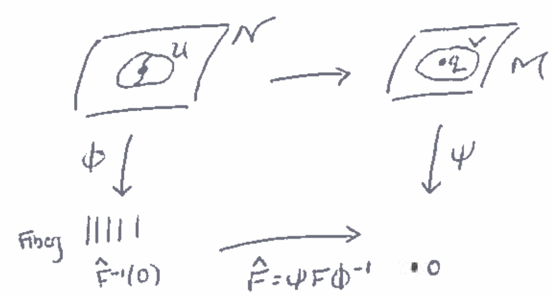
Pf: $A \overset{def}{=} F^{-1}(q)$ a closed subset in N. For all $p \in A$, F has rank k near p. By Rank Thm, $\exists$ (U, ϕ) at p and (V, ψ) at $q = F(p)$ with $\phi(p) = 0$, $\psi(q) = 0$, $\phi(V) = [-\epsilon, \epsilon]^n \subset \mathbb{R}^n$ and

$$\psi F \phi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0)$$

Then $F^{-1}(q) \cap U = \phi^{-1} \phi F^{-1} \psi^{-1}(0)$

$$= \phi^{-1}(\{x \in [-\epsilon, \epsilon]^n \mid x_1 = x_2 = \dots = x_k = 0\})$$

∎



Cor: $F: N^n \to M^m$ a C^∞-map with $m \le n$. Suppose rank $F = m$ (a submersion) at every point of $A = F^{-1}(q)$, where $q \in M$. Then A is a closed regular submanifold of dim $n-m$.

Pf: rank $F = m$ on an open neigh of A. Apply previous thm by replacing N by $\mathcal{U}$. ∎

Ex: $F: \mathbb{R}^n \to \mathbb{R}$

$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} x_i^2$$

F has rank 1 on $\mathbb{R}^n \setminus \{0\}$. By previous thm, for all $s \in \mathbb{R} \setminus \{0\}$, $F^{-1}(s)$ is a regular submanifold. In particular, $F^{-1}(1)$ is the unit sphere in $\mathbb{R}^n$ of dim $n-1$.

Ex: $F: \mathbb{R}^3 \to \mathbb{R}$

$$(x_1, x_2, x_3) \mapsto \left(a - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2$$

If $a > b > 0$, check rank $F = 1$ on $F^{-1}(b^2)$. Then $F^{-1}(b^2)$ is torus in $\mathbb{R}^3$.

Discrete Group Action on Manifolds & Covering Manifolds

Def: A discrete group Γ is a group with countably many elements and the topology is defined with each point is an open set.

The group action $\Gamma \times \tilde{M} \to \tilde{M}$ of Γ on $\tilde{M}$ is a map such that

1) for 1^{st}_{Γ} $e \in \Gamma$, $\Theta_e(x) = x$ for all $x \in \tilde{M}$.

2) For all $h_1, h_2 \in \Gamma$, $\Theta_{h_1}(\Theta_{h_2}(x)) = \Theta_{h_1 h_2}(x)$ for any $x \in \tilde{M}$

where $\Theta_h: \tilde{M} \to \tilde{M}$ is a diffeo. for all $h \in \Gamma$.

For notational ease, we denote h for Θ_h and $h \cdot x$ for $\Theta_h(x)$.

Consider the set of orbits $M = \tilde{M} / \Gamma$ with the quotient topology, ie $\mathcal{U} \subset M$ is open $\rightleftarrows$ $\pi^{-1}(\mathcal{U})$ is open in $\tilde{M}$, where $\pi: \tilde{M} \to M$, $x \mapsto \Gamma x$

Check: π is continuous and an open map: obv.

$$\pi^{-1}(\pi(\mathcal{U})) \overset{?}{=} \bigcup_{h \in \Gamma} (h \cdot \mathcal{U}) \text{ is open}$$

so $\pi(\mathcal{U})$ is open so π open.

Def: We say a discrete group Γ acts properly dis continuously if

1) For all $x \in \tilde{M}$, $\exists\, \mathcal{U} \subset \tilde{M}$ open with $x \in \mathcal{U}$ such that $\{h \in \Gamma \mid \mathcal{U} \cap h \cdot \mathcal{U} \neq \emptyset\}$ is finite.

2) given $x, y \in \tilde{M}$ not in same orbit, then $\exists\, \mathcal{U}$ cont. x, V cont. y $\ni$ $\mathcal{U} \cap \Gamma \cdot V = \emptyset$.

Rem: 2) in definition implies $M = \tilde{M}/\Gamma$ is Hausdorff.

Rem: 1) in definition may be replaced by

1') The isotrophy group Γ_x (for all $x \in \tilde{M}$) is finite and for each $x \in \tilde{M}$ admits an open neigh. $\mathcal{U} \ni$

$$\mathcal{U} \cap h \cdot \mathcal{U} = \phi$$

for all $h \notin \Gamma_x$ & $h \cdot \mathcal{U} = \mathcal{U}$ if $h \in \Gamma_x$.

Def: Isotrophy group: Γ_x is $\{h \in \Gamma \mid h \cdot x = x\}$.

Notice - as stated - $(1) + (2) \rightleftarrows (1') + (2)$ in definition.

Proof: $(2) \rightleftarrows (2)$ obvious. So

$(1)+(2) \to (1')$: Note Γ_x is contained in $\underbrace{\{h \in \Gamma \mid \mathcal{U} \cap h \cdot \mathcal{U} \neq \phi\}}_{\text{Finite}}$

By (1), all but finite elements in Γ, $\tilde{\mathcal{U}} \cap h \cdot \tilde{\mathcal{U}} = \phi$. Let $S = \{h \in \Gamma \mid \tilde{\mathcal{U}} \cap h \cdot \tilde{\mathcal{U}} \neq \phi\} \setminus \Gamma_x$. This is finite.

Given $s \in S$, $sx \neq x$, using $\tilde{M}$ Hausdorff, $\exists\ \tilde{\mathcal{U}}' \subset \tilde{\mathcal{U}}$ with $x \in \tilde{\mathcal{U}}'$ and $s \cdot x \in V$ open with $\tilde{\mathcal{U}} \cap V = \phi$.

Define $\tilde{\tilde{\mathcal{U}}} = s^{-1}V \cap \tilde{\mathcal{U}}'$, open neighborhood of x.

We have avoided s. Continue this process for the finite # elements in S. Then $\tilde{\tilde{\mathcal{U}}} \cap h \cdot \tilde{\tilde{\mathcal{U}}} = \phi$ for $h \notin \Gamma_x$ by induction.

Moreover, define $\mathcal{U} = \bigcap_{\sigma \in \Gamma_x} \sigma \tilde{\tilde{\mathcal{U}}}$

for $\sigma \in \Gamma_x$ is open

Then for $h \in \Gamma_x$

$$h \cdot \mathcal{U} = h \cdot \bigcap_{\sigma \in \Gamma_x} \sigma \tilde{\tilde{\mathcal{U}}}$$

$$= \bigcap_{\sigma \in \Gamma_x} (h \cdot \sigma) \tilde{\tilde{\mathcal{U}}}$$

$$= \bigcap_{\sigma \in \Gamma_x} \sigma \tilde{\tilde{\mathcal{U}}} = \mathcal{U}$$

The 'reverse' direction is simpler.

$(1') \to (1)$: $\exists\ \mathcal{U}$ cont x such that $\mathcal{U} \cap h \cdot \mathcal{U} = \phi$ for $h \notin \Gamma_x$. Then $\{h \in \Gamma \mid h\mathcal{U} \cap \mathcal{U} \neq \phi\}$ contained in Γ_x is finite. ☒

Def: A group G acts freely on a set X if $gx = x$ implies $g = 1_G$ for all $x \in X$, ie $\underset{\text{set}}{1_G} = \bigcap_{x \in X} \Gamma_x$

09/28/2016

Γ discrete group

$\Gamma \times \tilde{M} \to \tilde{M}$ a map such that for all $h \in \Gamma$, $h: \tilde{M} \to \tilde{M}$ is a diffeomorphism such that

1) $e(x) = x$ for $x \in \tilde{M}$

e identity in Γ

2) $h_1 h_2 (x) = h_1(h_2(x))$

Def: Γ acts on $\tilde{M}$ freely if $g(x) = x$ for all $x \in \tilde{M}$ then $g = e \overset{\text{def}}{=} 1_\Gamma$

Def: Γ acts on $\tilde{M}$ properly discontinuously if

1) I_x is finite for all $x \in \tilde{M}$. $\exists\, \mathcal{U}$ cont x such that $\mathcal{U} \cap g\mathcal{U} = \emptyset$ for all $g \notin \Gamma_x$ and $\mathcal{U} = g\mathcal{U}$ for all $g \in \Gamma_x$.

2) For all $x, y \in \tilde{M}$ not in the same orbit, $\exists\, \mathcal{U}, V$ containing x, y, respectively, such that $\mathcal{U} \cap \Gamma V = \emptyset$

Thm: If Γ is a discrete group acting freely and properly discont. on $\tilde{M}$, then $\exists!$ C^∞ structure on $\tilde{M}/\Gamma \overset{\text{def}}{=} M$ such that $\forall p \in M$ $\exists$ a connected open set $\mathcal{U}$ cont p with

i) $\pi^{-1}(\mathcal{U}) = \bigcup_\alpha \tilde{\mathcal{U}}_\alpha$ with $\mathcal{U}_\alpha$ connected open components in $\tilde{M}$

ii) $\pi: \tilde{\mathcal{U}}_\alpha \to \mathcal{U}$ a diffeomorphism for all α.

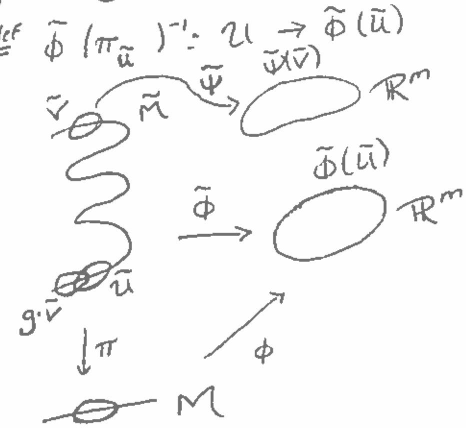
PF: (2) Gives that M is Hausdorff. Now $\pi: \tilde{M} \to M$ given by $x \mapsto \Gamma x$ is an open map. $\tilde{M}$ has a countable basis of topology so M has a countable basis of topology

Using the free action and (1) (from prop. discont. def.) for all $x \in \tilde{M}$, $\exists$ open set $\tilde{\mathcal{U}}$ cont. x such that $h\tilde{\mathcal{U}} \cap \tilde{\mathcal{U}} = \emptyset$ for $h \neq e$.

$\pi\tilde{\mathcal{U}} \overset{\text{def}}{=} \mathcal{U}$

Then $\pi|_{\tilde{\mathcal{U}}} : \tilde{\mathcal{U}} \to \mathcal{U}$ is a homeomorphism. Assume $(\tilde{\mathcal{U}}, \tilde{\Phi})$ is a coord. neigh. on $\tilde{M}$ by possibly shrinking $\tilde{\mathcal{U}}$. Let

$\phi \overset{\text{def}}{=} \tilde{\Phi}(\pi_{\tilde{\mathcal{U}}})^{-1}: \mathcal{U} \to \tilde{\Phi}(\tilde{\mathcal{U}})$



Then M is locally Euclidean and M is a top. manifold of dim the same as $\dim \tilde{M} = m$.

Check: $(\mathcal{U}, \phi)$ are C^∞ compatible and $\pi: \tilde{\mathcal{U}} \to \mathcal{U}$ is a diffeo.

Let $\mathcal{U} = \pi(\tilde{\mathcal{U}})$
$V = \pi(\tilde{V})$
$\mathcal{U} \cap V \neq \emptyset$
$\phi = \tilde{\Phi}(\pi_{\tilde{\mathcal{U}}})^{-1}$
$\psi = \tilde{\Psi}(\pi_{\tilde{V}})^{-1}$

There is a $g \in \Gamma$ such that $\tilde{\mathcal{U}} \cap g\tilde{V} \neq \emptyset$. Now....

$$\phi\psi^{-1} = \tilde{\Phi}(\pi_{\tilde{\mathcal{U}}}^{-1})\pi_{\tilde{V}}\psi^{-1}$$
$$= \tilde{\Phi}\underbrace{\pi_{\tilde{\mathcal{U}}}^{-1}\pi_{g\cdot\tilde{V}}}_{=1} g\tilde{\Psi}^{-1}$$

same map on intersection

$$= \tilde{\Phi} g \tilde{\Psi}^{-1}$$

on $\psi(\tilde{\mathcal{U}} \cap g\tilde{V})$ is C^∞ (change of variable on $\tilde{\mathcal{U}} \cap g\tilde{V}$). So M is a C^∞ manifold, π diffeo.

Ex: $\mathbb{Z}_2 \overset{\text{def}}{=} \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}_2$ acts freely & prop. discont. on S^n for $n \geq 2$ or 1, via treating the action as the antipodal map, ie

$1(x) = -x$ & $0(x) = x$

$\mathbb{Z}_2$ additively.

Then $S^n/\mathbb{Z}_2 \equiv \mathbb{R}P^n$

Def: $\tilde{M}$ is a covering manifold of M with C^∞ covering map $\pi : \tilde{M} \to M$ if π is onto $\tilde{M}$ is connected and $p \in M$ $\exists$ connected open neigh $\mathcal{U}$ cont p such that $\pi^{-1}(\mathcal{U}) = \bigcup \tilde{\mathcal{U}}_\alpha$, where $\{\tilde{\mathcal{U}}_\alpha\}$ open connected components such that $\pi|_{\tilde{\mathcal{U}}_\alpha} : \tilde{\mathcal{U}}_\alpha \to \mathcal{U}$ is a diffeomorphism for all α. $\mathcal{U}$ is called an admissible neigh & π is a covering map or proj.

Ex: $\pi : \mathbb{R} \to S^1$

1) $t \mapsto e^{2\pi i t}$

2) $\pi : \mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} \to T^n$

$(t_1, \dots, t_n) \mapsto (e^{i\theta t_1}, \dots, e^{i\theta t_n})$

$(\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, $\theta_i \neq 0$

3) $\pi : S^n \to \mathbb{P}^n$

$-x, x \mapsto [x]$

In general, if Γ acts on $\tilde{M}$ freely and properly discontinuously then $\tilde{M}$ carries $\tilde{M}/\Gamma$.

$\pi : \tilde{M} \to \tilde{M}/\Gamma$

$x \mapsto \Gamma x$

Def: $\pi : \tilde{M} \to M$ a covering. Then $h : \tilde{M} \to \tilde{M}$ diffeo. is a deck transformation if $\pi h = \pi$.

Thm: Let $\tilde{\Gamma}$ be the set of deck trans. then $\tilde{\Gamma}$ acts freely and properly discont. on $\tilde{M}$.

Pf: See text. $\boxtimes$

Tangent Space at a Point

M^m : C^∞ manifold

Def: (Tangent Space) $\overrightarrow{T_pM}$ to M at a point $p \in M$ is the set of all maps $X_p : C_p^\infty \to \mathbb{R}$ such that

Linear $\rightarrow$ 1) $X_p(\alpha F + g) = \alpha X_p(F) + X_p(g)$ for $\alpha \in \mathbb{R}$, $F, g \in C_p^\infty$.

Leibniz Rule $\rightarrow$ 2) $X_p(Fg) = X_p(F)g_{(p)} + F_{(p)} X_p(g)$

and a vector space operation on T_pM with

1) $(X_p + Y_p)(F) = X_p(F) + Y_p(F)$

2) $(\alpha X_p)(F) = \alpha X_p(F)$

A tangent vector to M at p is such $X_p \in T_pM$

* Check: T_pM is a vector space.

Thm: $F: M^m \to N^n$ C^∞ map

Then $F^*: C^\infty_{F(p)} \to C^\infty_p$
$f \mapsto fF$

This induces a homo. between these two algebras. $F_*: T_pM \to T_{F(p)}N$
$X_p \mapsto F_*(X_p)$

where $F_*(X_p)(f) = X_p(f \circ F)$

this is a vector space homo. When $F: M \to M$ is the identity, then both F^* and F_* are the identity homo.
If $H = GF$, where $G: M_2 \to M_3$,
then $H_* = G_* F_*$
$H^* = F^* G^*$

10/02/2016

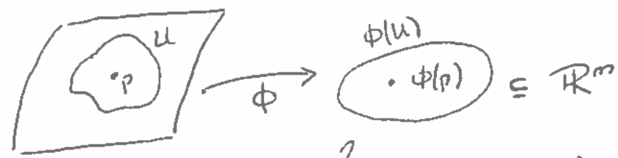
Remark: $f_* = dF$ the push forward of tangent vectors.

Cor: If $F: M \to N$ is a diffeo, then for all $p \in M$,
$$F_*: T_pM \to T_{F(p)}N$$
is an isomorphism.

Then we have the following:

Now if $p \in M$, choose a coord. neigh at $p \in M^m$
$$\phi(p) = (x_1(p), \dots, x_m(p))$$

then...
$$\phi_*: T_pM \to T_{\phi(p)}\mathbb{R}^m$$
is an isomorphism.



$\left\{\frac{\partial}{\partial x^i}\right\}_{i=1,\dots,m}$ is the standard basis on $T_{\phi(p)}\mathbb{R}^m \cong \mathbb{R}^m$

$\frac{\partial}{\partial x^i} \cong (\dots, 0, 1, 0, \dots)$ (i^{th})

Let $E_{ip} = (\phi_*)^{-1} \frac{\partial}{\partial x^i} \in T_pM$

Then $\{E_{1p}, \dots, E_{mp}\}$ is called the coordinate frame.

Cor: Fix a coord. neigh. (U, ϕ) at p in M. Then there exists a natural basis $\{E_{1p}, \dots, E_{mp}\}$ of T_pM. If $f \in C^\infty_p$, we write $\hat{f} = f\phi^{-1} \in C^\infty_{\phi(p)}$.

Notice $E_{ip}(f) = (\phi^{-1})_* \left(\frac{\partial}{\partial x^i}\right) f$
$$= \frac{\partial}{\partial x^i}(\hat{f})(\phi(p))$$

In fact for $X_p \in T_pM$
$$X_p = \sum_{i=1}^{m} X_p(x_i) E_{ip}$$

Namely,

$$E_{ip}(F) = \frac{\partial}{\partial x^i}\left(F\circ\phi^{-1}\right)\Big|_{\phi(p)}$$

Write $X_p = \sum_{i=1}^{m} \alpha_i' E_{ip}$

Use
$$X_p(x_j) = \sum_{i=1}^{m} \alpha_i E_{ip}(x_j) = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial x^i}(x_j) = \sum_{i=1}^{m} \alpha_i \delta_{ij} = \alpha_j$$

Substitute into the above.

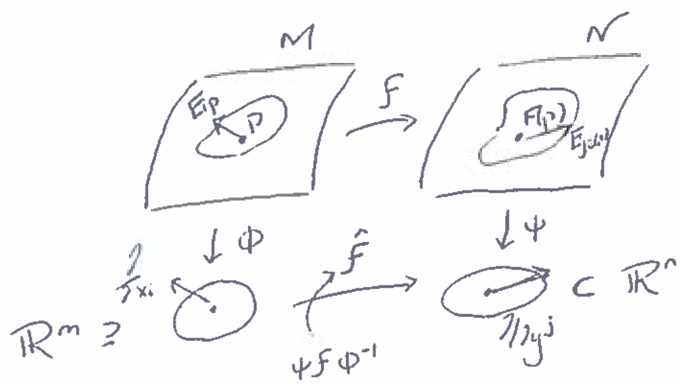
How do these α_j coefficients change under change of coordinates?

Thm: $F: M^m \to N^n$ C^∞-map. Let $p \in M$. $(U,\phi), (V,\psi)$ coord. neigh p, $\phi(p)$, respectively. F is locally given by
$$y^j = F^j(x_1, \dots, x_m)\,;\ 1 \le j \le n$$
where $f = (f_1, \dots, f_m)$. Let
$$E_{ip} = (\phi_*^{-1})\left(\frac{\partial}{\partial x^i}\right)$$
$$\tilde{E}_{jF(p)} = (\psi_*)^{-1}\left(\frac{\partial}{\partial y^j}\right)$$
basis T_pM & $T_{F(p)}N$, resp. Then
$$F_*(E_{ip}) = \sum_{j=1}^{n} \frac{\partial f^j}{\partial x^i} \tilde{E}_{jF(p)}$$
for $1 \le i \le m$

So if...
$$X_p = \sum_{i=1}^{m} \alpha_i E_{ip}$$
$$F_*(X_p) = \sum_{j=1}^{n} \beta_j \tilde{E}_{jF(p)}$$

$$\begin{pmatrix}\beta_1\\ \vdots \\ \beta_n\end{pmatrix} = \underbrace{\frac{\partial(f^1,\dots,f^n)}{\partial(x_1,\dots,x_m)}}_{\text{Jacobian}} \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{pmatrix}$$

evaluated at $\phi(p)$



Pf: For $g \in C^\infty_{F(p)}$,

$$F_*(E_{ip})(g) \overset{\text{def}}{=} E_{ip}(g\circ f)$$
$$\overset{\text{def}}{=} \frac{\partial}{\partial x^i}\left(g\circ f\circ\phi^{-1}\right)$$
$$= \frac{\partial}{\partial x^i}\left(g\,\psi^{-1}\hat{f}\right)$$
$$= \sum_{j=1}^{n} \frac{\partial(g\psi^{-1})}{\partial y^j}\frac{\partial f^j}{\partial x^i}$$
$$= \sum_{j=1}^{n} \frac{\partial f^j}{\partial x^i}\left(\psi_*^{-1}\right)\left(\frac{\partial}{\partial y^j}\right)(g)$$
$$= \sum_{j=1}^{n} \frac{\partial f^j}{\partial x^i}\tilde{E}_{jF(p)}(g)$$

But then $F_*(E_{ip}) = \sum_{j=1}^{n} \frac{\partial f^j}{\partial x^i} \tilde{E}_{j\,F(p)}$ $\boxed{}$

Write: $F_*\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$

Einstein notation & good way to memorize it. We often even drop summation.

Cor: $(U, \phi), (\tilde{U}, \tilde{\phi})$ C^∞ compatible coord. neigh of M at $p \in U \cap \tilde{U}$. with local coord. ~~neigh.~~ $(x_1, \dots x_m)$ & $(\tilde{x}_1, \dots, \tilde{x}_m)$. Then

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}$$

Jacobians

&

$$\frac{\partial}{\partial \tilde{x}^i} = \sum_j \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial}{\partial x^j}$$

Cor: rank F at p is dim. of the image of $F_*(T_pM)$.

F_* is an isomorphism from T_pM to $F_*(T_pM)$ if and only if rank $F = m$ & F_* is onto iff rank $f = n$.

Rem: $M \subset N$ (sub. man.); $F: M \hookrightarrow N$

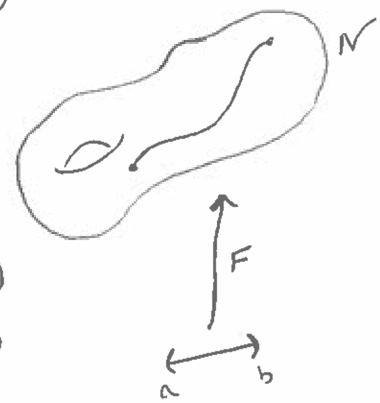
rank $F = m$ so $F_*(T_pM) \to T_{F(p)}N$ is injective & $T_pM \subset T_{F(p)}N$ is a subspace.

really p

Ex: $M = (a,b) \subset \mathbb{R}$

$F: M \to N$ C^∞-map

Defines a C^∞ map in N



$t_0 \in (a,b)$

$f \in C^\infty_{F(t_0)}$

$F_*\left(\frac{d}{dt}\right)(f)$

$= \frac{d}{dt}(fF)(t_0)$ then $F_*\left(\frac{d}{dt}\right)(f)$ is called the velocity vector to curve F at t_0.

(U, Φ) coord. neigh. at $F(t_0)$ & F is given by $\hat{F} = \phi F(t) = (x_1 t, \dots, x_n t)$

then $F_*\left(\frac{d}{dt}\right) = \sum_{i=1}^{n} \dot{x}_i(t_0) \frac{\partial}{\partial x^i}$

where $\dot{x}_i(t_0) = \frac{dx^i}{dt}(t_0)$

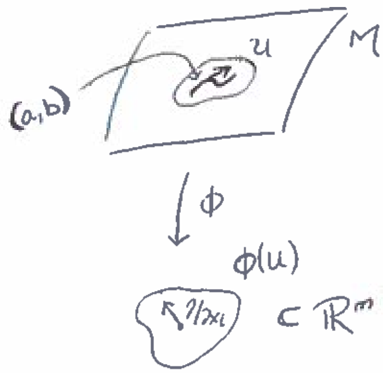
This coincides with case when $N = \mathbb{R}^n$.

10/10/2016

$X_p \in T_pM^m$

(U, ϕ) neigh. at p. Local coord $(x_1, \dots, x_n)$ Basis of T_pM

$\{\partial/\partial x_1, \dots, \partial/\partial x_m\} = (\phi^{-1})_* \frac{\partial}{\partial x_i}$



Ex: $F: (a,b) \to M$ C^∞ curve

$$F_*\left(\frac{d}{dt}\right) = \sum_{i=1}^{m} \frac{dx_i}{dt} \frac{\partial}{\partial x_i} \in T_F M$$

where $(x_1(t), \dots, x_m(t)) = \phi F(t)$

Ex: $M \subset \mathbb{R}^3$ 2-dim submanifold in $\mathbb{R}^3$ (surface) $W \subset M$ open.

$\Theta: W \to \mathbb{R}^3$ parametrization

$(u,v) \mapsto (x,y,z)$

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$$

Θ immersion $\Leftrightarrow$ $\dfrac{\partial(f,g,h)}{\partial(x,y)}$ is rank 2 everywhere in W. Define

$$\vec{X}_u = \Theta_*(\partial/\partial u)$$
$$\vec{X}_v = \Theta_*(\partial/\partial v)$$

Then $\vec{X}_u = \frac{\partial f}{\partial u}\frac{\partial}{\partial x} + \frac{\partial g}{\partial u}\frac{\partial}{\partial y} + \frac{\partial h}{\partial u}\frac{\partial}{\partial z}$

$\vec{X}_v = \frac{\partial f}{\partial v}\frac{\partial}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial}{\partial z}$

& $\{\vec{X}_u, \vec{X}_v\}$ are lin indep. and hence span 2D plane in $\mathbb{R}^3$, ie tangent plane $T_{\Theta(u,v)}M$

Consider a curve on M through $p \in M$.

$F: I \to M$ C^∞

I open $\subset \mathbb{R}$

$F(t_0) = p$

F is given by $(u(t), v(t))$, a parametrization.

& $F\Theta = (f(u,v), g(u,v), h(u,v))$

each depend on t.

Then $(\Theta \circ F)_*\left(\frac{d}{dt}\right) =$

$$\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t}\right)\frac{\partial}{\partial x} +$$
$$\left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial t}\right)\frac{\partial}{\partial y} +$$
$$\left(\frac{\partial h}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial t}\right)\frac{\partial}{\partial z}$$

$$= \frac{du}{dt}\vec{X}_u + \frac{dv}{dt}\vec{X}_v$$

Vector Fields

[Ours will be a C^∞ smooth v.F.]

Def: A vector field X on M is a function assigning to each point $p \in M$ a vector $X_p \in T_pM$ such that if (U, ϕ) coord neigh with local coord $(x_1, \dots, x_m)$ & write $X_p = \sum \alpha^i(p) \frac{\partial}{\partial x_i}\Big|_p$ then α^i is $C^\infty(U)$ for each i.

Rem:

1) The definition does not depend on the choices of coordinate neighs

2) Every v.f. $X: C^\infty(M) \to C^\infty(M)$ is a linear map with (1), (2) of tangent vector.

Ex:

1) $\mathbb{R}^3 \setminus \{0\}$

$$X = \sum_1^3 \frac{x_i}{r} \frac{\partial}{\partial x_i} \; ; \; r = \sqrt{\sum_1^3 (x^i)^2}$$

2) For a coord neigh $(\mathcal{U}, \phi)$ with local coord. ~~neighs~~ $(x_1, \cdots, x_m)$ on $M \rightsquigarrow \partial/\partial x_i$ is a v.f. on $\mathcal{U}$ for all i.

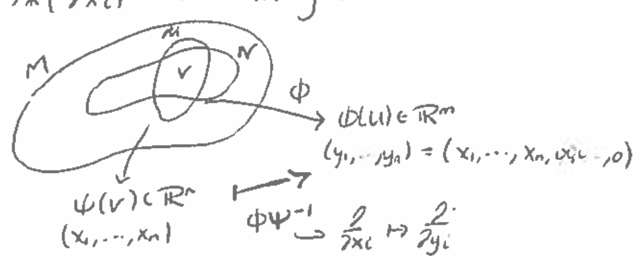
Lemma: $N \subset M$ submanifold
X: v.f. on M such that
$p \in N \subset M$, $X_p \in T_p N$. Then
$X|_N$ is a v.f. on N.

Pf: By choosing admissible coord neigh. (V, ψ) of N and $(\mathcal{U}, \Phi)$ of M at p, the inclusion map is given by

$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 0, \cdots, 0)$$

Then for $p \in N$, $1 \le i \le n$

$$(\psi^{-1})_* \left(\frac{\partial}{\partial x_i}\right) = (\Phi^{-1})_* \left(\frac{\partial}{\partial y_i}\right)$$



$$X_p = \sum_i \alpha^i (\Phi^{-1})_* \left(\frac{\partial}{\partial y_i}\right)$$

$$= \sum_i \alpha^i (\Phi \psi^{-1})_* \left(\frac{\partial}{\partial x_i}\right)$$

Then $X_p|_N = \sum_i \alpha^i (x_1, \cdots, x_n, 0, \cdots, 0)$

$(\psi^{-1})_* (\partial/\partial x_i)$

$\blacksquare$

(Local) 1-parameter Group Action on a Manifold

Θ an action of $\mathbb{R}$ on M is

$\Theta: \mathbb{R} \times M \to M$ C^∞

Satisfying

1) $\Theta_0(p) = p$ for $p \in M$

2) $\Theta_s(\Theta_t(p)) = \Theta_{s+t}(p)$

for $p \in M$; $s, t \in \mathbb{R}$

Ex: $\mathbb{R}^3$ and fix nonzero vector $\vec{a}$. Define $\Theta_t(x) = \vec{x} + \vec{a}t$. This is a C^∞ action of $\mathbb{R}$ on $\mathbb{R}^3$.

Let $\Theta: \mathbb{R} \times M \to M$ be a C^∞-action

Define $X_p: C_p^\infty \to \mathbb{R}$ by, $f \in C_p^\infty$,

$$X_p(f) = \lim_{\Delta t \to 0} \frac{f(\Theta_{\Delta t}(p)) - F(p)}{\Delta t}$$

Check: 1) $X_p \in T_p M$

2) X v.f. on M, called infinitesimal gen. of Θ

Let $(\mathcal{U}, \phi)$ be the coord. neigh at p. $I_\delta = (-\delta, \delta)$, $V \underset{open}{\subset} \mathcal{U}$ such that $\Theta(I_\delta, V) \subset \mathcal{U}$. Choose $p \in V$ also.

Then in local coord., Θ is given by

$(h_1(t, x_1, \dots, x_m), \dots, h_m(t, x_1, \dots, x_m))$

Then $X_p(f) = \lim_{\Delta t \to 0} \dfrac{f(h_1(\Delta t, x_1, \dots, x_m), \dots, h_m(\cdots)) - f(h_1(0,x), \dots, h_m(0,x))}{\Delta t}$

This limit clearly exists. (a comp of smooth functions).

$$= \sum_{i=1}^{m} \frac{\partial h_i}{\partial t}(0,x) \cdot \frac{\partial f}{\partial x_i}(x)$$

$$= \left(\sum_{i=1}^{m} \frac{\partial h_i}{\partial t}(0,x) \cdot \frac{\partial}{\partial x_i} \right)(f)$$

Then $X = \sum_{i=1}^{m} \frac{\partial h_i}{\partial t}(0) \frac{\partial}{\partial x_i}$ is $C^\infty(\mathcal{U})$

Def: $\Theta: G \times M \to M$ is a group action on M. Then v.f. X on M is called G-invariant if $(\Theta_g)_* X = X$ for all $g \in G$.

Thm: $\Theta: \mathbb{R} \times M \to M$ C^∞ action of $\mathbb{R}$ on M then the infitesimal generator X is $\mathbb{R}$-invariant

Pf: Merely check the definition. For $f \in C^\infty_{\Theta_t(p)}$

$$(\Theta_t)_* X_p(f) = X_{p}(f\Theta_t)$$

$$= \lim_{\Delta t \to 0} \frac{f\Theta_t(\Theta_{\Delta t}(p)) - f(\Theta_t(p))}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(\Theta_{\Delta t}(\Theta_t(p))) - f(\Theta_t(p))}{\Delta t}$$

$$= X_{\Theta_t(p)}(f)$$

So $(\Theta_t)_* X_p = X_{\Theta_t(p)}$ ☐

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Cor: If $X_p = 0$ then for all q in the orbit of p, $X_q = 0$. Equivalently, X vanishes identically or X is never 0 (for points in the orbit).

Thm: The orbit of p is either a single point or an immersion of $\mathbb{R}$ in M by $t \mapsto \Theta_t(p)$, depending on whether $X_p = 0$

Pf: The orbit of p = image of $\Theta: t \mapsto \Theta_t(p)$. Then Θ is an immersion $\rightleftharpoons \frac{d\Theta}{dt} \neq 0$ for all t.

$$\Theta_*\left(\frac{d}{dt}\right)(f) = \frac{d}{dt}(f\Theta)$$

$$= \lim_{\Delta t \to 0} \frac{f(\Theta_{\Delta t}(p)) - f(p)}{\Delta t}$$

$$= X(f)$$

Then $X = \Theta_*\left(\frac{d}{dt}\right)$

By the thm, $\frac{d\theta}{dt}(t) = (\theta_t)_* \left(\frac{d\theta}{dt}(0) \right)$

diffeo.

So that $\frac{d\theta}{dt} \neq 0$ if $\frac{d\theta}{dt}(0) \neq 0$. But then $X(p) \neq 0$. Then if $X(p) \neq 0$, θ is an immersion. OTOH, if $X_p = 0$, $\frac{d\theta}{dt}(t) = 0$ for $t \in \mathbb{R}$. In local coord., $\theta = (h_1, \dots, h_m) \rightsquigarrow \frac{dh_i}{dt} = 0$ for all i.

But then $h_i \equiv 0$ for all i so that $\theta_t(p) = p$. ☒

Def: Given a v.f. X on M, the curve $\gamma : J \subset \mathbb{R} \to M$ is called an integral curve of X if $\frac{d\gamma}{dt} = X_{\gamma(t)}$ on J.

Rem: Orbit of action $\theta : M \to N$ is an integral curve of infinitesimal gen.

Ex: $\theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$$(t, (x,y)) \mapsto (x+t, y)$$

$$X = \frac{\partial}{\partial x}$$

Let $M = \mathbb{R}^2 \setminus \{0,0\}$, then $(x_0, 0)$ cannot be repelled by $t = -x_0$.

Let $W \in \mathbb{R} \times M$ given by

$$W = \left(\bigcup_{y \neq 0} \mathbb{R} \times \{(x,y)\} \right) \cup \{ (t, x_0) \mid x(x+t) > 0 \}$$

$(x, 0)$

$\theta : W \to M$ onto. In general, M: C^∞ manifold with for all $p \in M$, $\exists\ \alpha(p) < \beta(p) \ni$ $W \cap \mathbb{R} \times \{p\} = \{(t, p)\}$ S.T.A, A

$$t \in (\alpha(p), \beta(p))$$

Def: A local 1-parameter group action or flow on M is a C^∞ map, $\theta : W \to M$ satisfying

1) $\theta_0(p) = p \quad \forall\ (0,p) \in W$

2) $\alpha(\theta_s(p)) = 1$
$\beta(\theta_s \cdot p) = \beta(p) - s$

and for $t \in (\alpha p - s, \beta p - s)$

$\theta_t \theta_s(p) = \theta_{t+s}(p)$ well defined.

Thm: Let $V_t = \{ p \in M \mid (t, p) \in W \}$. Then V_t is open for $t \in \mathbb{R}$ so $\theta_t : V_t \to V_{-t}$ is diffeo. ☒

$(\theta_{-t}^{-1}) \cdot \theta_t \frac{1}{t}$

Rem: $W = \bigcup_t \{t\} \times V_t = \bigcup_{p \in M} (\alpha(p), \beta(p)) \times \{p\}$

Pf: For $p_0 \in V_t$, ie $(t_0, p_0) \in W$,

since W open, $\exists\, \mathcal{U} \underset{\text{opn}}{\subset} M$, $p_0 \in \mathcal{U}$,

$\exists\, \delta > 0 \ni (t_0 - \delta, t_0 + \delta) \times \mathcal{U}_0 \subset W$

Then $\mathcal{U}_0 \subset V_{t_0} \rightarrow V_{t_0}$ is open. For

$p \in V_t$, $\exists\, \alpha(p) < 0 < \beta(p) \ni$

$t \in (\alpha(p), \beta(p)) \rightarrow \underset{\alpha(\theta_t(p))}{\underset{\shortparallel}{\alpha(p) - t}} > 0$ &

$\underset{\beta(\theta_t(p))}{\underset{\shortparallel}{\beta(p) - t}} < 0.$

Since $\underbrace{\alpha(p) - t}_{\alpha(\theta_p(t))} < -t < \underbrace{\beta(p) - t}_{\beta(\theta_t(p))}$

$\theta_{t(p)} \in V_{-t}$. Therefore, $\theta_{-t} \cdot \theta_t(p) =$

$\theta_0(p) = p$ & $\theta_t \theta_{-t}(p) = p$ $\square$

So in prev. example:

$t = 1$, $V_1 = \mathbb{R}^2 \setminus ([-1, 0] \times \{0\})$

$t = -1$, $V_{-1} = \mathbb{R}^2 \setminus ([0, 1] \times \{0\})$

Thm: $\theta: W \to M$ local flow

X: inf. generator, $p \in M$, $X_p \neq 0$

then $\exists\, (V, \psi)$ at p, $v > 0$, $p \in V' \subset V$

such that $\theta: (-v, v) \times V' \to V$

$(t, y_1, \dots, y_m) \mapsto (y_1 + t, y_2, \dots, y_m)$

and $X = (\psi^{-1})_* \left(\frac{\partial}{\partial y_1}\right)$ in V_1

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Pf: Choose local coordinate $(\mathcal{U}, \phi)$ at p with $\phi(p) = (0, \dots, 0)$. $\theta(t, x)$ is written $(h_1(t,x), \dots, h_m(t,x)) = h(t,x)$ in local coordinates and

$$X_p = (\phi^{-1})_* \left(\frac{\partial}{\partial x_1}\right) \text{ at } p$$

By calculation,

$$X_p = \sum_{i=1}^{m} \frac{\partial h_i}{\partial t}(0,0) \cdot \frac{\partial}{\partial x_i}$$

Then $\frac{\partial h_1}{\partial t}(0,0) = 1$ and

$\frac{\partial h_i}{\partial t}(0,0) = 0$ if $i > 1$. Choose δ small such that $V'' = \phi^{-1}((-\delta, \delta)^m) \subset \mathcal{U}$ and $\theta((-\delta, \delta) \times V'') \subset \mathcal{U}$. Define

$F: \underbrace{(-\delta, \delta)^m}_{\text{cube}} \to \phi(\mathcal{U})$ via

$$(y_1, \dots, y_m) \mapsto (h_1(y_1, 0, \dots, y_m), \dots\dots, h_m(y_1, 0, \dots, y_m))$$

Calculate jacobian of F at $\underbrace{(0, \dots, 0)}_{\text{call } 0}$

$$DF(0, \dots, 0) = \begin{pmatrix} \frac{\partial h_1}{\partial t} 0 & \cdots & \frac{\partial h_1}{\partial y_m} 0 \\ \vdots & & \vdots \\ \frac{\partial h_m}{\partial t} 0 & \cdots & \frac{\partial h_m}{\partial y_m} 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & I \\ 0 & \end{pmatrix}$$

Since $h_i(0,0,y_2,\dots,y_m)=y_i$
then $\frac{\partial h_i}{\partial y_j}=\delta_{ij}$ for $j>2$.

Therefore, F is a local diffeomorphism at 0. By the inverse function thm, $\exists\ V$ open, $p\in V \ni F$ is a diffeomorphism from $\phi(V)\subset(-\delta,\delta)^m$ to $F(\phi(V))$. Let $\psi=F^{-1}\phi$. and

$\psi(p)=F(\phi(p))=(0,\dots,0)$ and

$h_i(t+y_1,0,y_2,\dots,y_m)=$
$h_i(t,h(y_1,0,y_2,\dots,m))$,

[Def of 1-para group action gives $\theta_{t+s}(x)=\theta_t(\theta_s(x))$]

So $\theta:(-\delta,\delta)\times V\mapsto\mathcal{U}$
$(t,y_1,\dots,y_m)\mapsto(y_1,\dots,y_m)$
by composition (with F^{-1}). ▨

Recall $p\in M$, $I_p=(\alpha(p),\beta(p))$

$\theta_t(p): t\in I_p\to\theta_t(p)\in M$

is a C^∞ curve is an orbit of p.

If $X_p\neq0$. X_p along orbit of $p\Leftrightarrow\theta_t(p)$ orbit of p. Then it is an integral curve.

Question: Fix $p\in M$. Given a C^∞ vector field X on M, can we get an integral curve $\gamma(t)$ passing through $p\in M$?

Namely, $\dot\gamma(t)=X_{\gamma(t)}$

Thm (Existence Thm for ODE):

Let $\mathcal{U}\subset\mathbb{R}^n$ open, $\epsilon>0$
$f_i(t,x_1,\dots,x_n)\in C_r((-\epsilon,\epsilon)\times\mathcal{U})$ ← C^r smooth
for $i=1,\dots,n$. Then for all $x\in\mathcal{U}$ $\exists\,\delta>0$, $V\subset\mathcal{U}$, $x\in V$ such that

1) $\forall a\in(a_1,\dots,a_m)\in V$
$\exists\, x(t)=(x_1(t,\dots),\dots,x_n(t)):(-\delta,\delta)\to\mathcal{U}$
with $\begin{cases}\frac{dx_i}{dt}=f_i(t,x_1,\dots,x_n)\\ x_i(0)=a_i\end{cases}$

2) $X(t)$ is unique and
$X(t,a_1,\dots,a_n)\in C_r((-\delta,\delta)\times V)$

Remark: By uniqueness, if $f_i\equiv0$ for any i, the only solution is $x_i(t)=a_i$

translate to manifolds

Thm: X: C^∞ vector field on manifold M. If $p\in M$, then $\exists\ V$ open in M, $p\in V$, $\delta>0$

$\theta^V:(-\delta,\delta)\times V\to M$ C^∞ map

such that
1) $\dot\theta^V(t,q)=X(\theta^V(t,q))$
2) $\theta^V(0,q)=q$
for $t\in(-\delta,\delta)$, $q\in V$

If $F(t)$ is an integral curve of X with $F(0)=q$, then $F(t)=\theta^V(t,q)$ for $|t|<\delta$
In fact, θ^V is unique.

Thm: X: C^∞ vector field on M

Then $\exists$ a unique open interval $I(p)$, $0 \in I(p)$, such that

1) $\exists$ C^∞ integral curve $F(t)$ on $I(p) \ni F(0) = p$

2) For all int. curves $G(t)$ with $G(0) = p$, then the interval of definition of $G(t) \subset I(p)$ and $F(t) \equiv G(t)$.

Frobenius Thm

Ex: Consider the system of P.D.E.

$$\begin{cases} \frac{\partial z}{\partial x} = g(x,y,z) \\ \frac{\partial z}{\partial y} = h(x,y,z) \end{cases} \quad \text{on } W \underset{\text{open}}{\subset} \mathbb{R}^3$$

for all $(a,b,c) \in W$, suppose $z = F(x,y)$ is a solution, ie $\begin{cases} F_x(x,y) = g(x,y,F(x,y)) \\ F_y(x,y) = h(x,y,F(x,y)) \\ c = F(a,b) \end{cases}$

Geometrically, consider the set defined by $F(x,y) - z = 0$. The tangent vectors have a basis

$$\begin{cases} X = \frac{\partial}{\partial x} + g(x,y,z)\frac{\partial}{\partial z} \\ Y = \frac{\partial}{\partial y} + h(x,y,z)\frac{\partial}{\partial z} \end{cases}$$

Check: $X(F(x,y) - z) = f_x + -g(x,y,z) = 0$

$Y(F(x,y) - z) = f_y + -h(x,y,z) = 0$

for $h \in C^\infty$, calculate

$X \circ Y(h) - Y \circ X(h)$

$= 0$

In general, M: C^∞ manifold

If $p \in M$, $\exists$ (U, ϕ) with local coord. $(x_1, \dots, x_m)$. Then

$T_pM = \text{span}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right\}$

For $i,j \in \{1, \dots, n\}$, $h \in C^\infty(U)$

$$\underbrace{\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}h - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}h}_{h \in C^\infty,\ \text{order unimportant}} = 0$$

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$\mathfrak{X}(M)$, set of C^∞ v.f. over M

$\mathfrak{X}(M)$ (inf dim) v.s. over $\mathbb{R}$

module over $C^\infty(M)$

Def: (Lie Bracket) The Lie bracket over $\mathfrak{X}(M)$ is $[X,Y] \overset{\text{def}}{=} XY - YX$ for any $X, Y \in \mathfrak{X}(M)$

Check:

(1) $[X,Y] \in \mathfrak{X}(M)$

(2) $[X,Y] = -[Y,X]$

(3) $[\alpha_1 X_1 + \alpha_2 X_2, Y] = \alpha_1 [X_1, Y] + \alpha_2 [X_2, Y]$

$\alpha_1, \alpha_2 \in \mathbb{R}$. sim. for Y "coordinate"

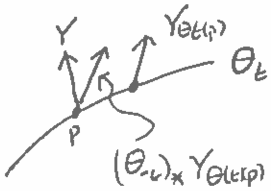
(4) Jacobi Identity:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

Def: (Lie Derivative) The lie deriv. of Y wrt X is a vector field given by:

$$(\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{1}{t}\left((\theta_{-t})_* Y_{\theta_t(p)} - Y(p)\right)$$

where $\dot{\theta}(t) = X_{\theta(t)}$ (integral curve so unique from our work before).



Like a M.V.T. ↓

Basically Fund. Thm of Calculus (proof)

Lem: $X \in \mathfrak{X}(M)$, $\theta: W \to M$ C^∞ map is 1-parameter grp generated by X. Fix $p \in M$, $f \in C^\infty(\mathcal{U})$, $p \in \mathcal{U}$, $\delta > 0$ $V \subset_{open} \mathcal{U}$, and $\theta((-\delta,\delta)\times V) \subset \mathcal{U}$, then $\exists$ C^∞ function $g(t,q)$ on $(-\delta,\delta)\times V$ such that $f(\theta_t(q)) = f(q) + t g(t,q)$ for all $t \in (-\delta,\delta)$ and $X_q(f) = g(0,q)$

$q \in V$

Pf: Define $r(t,q) = f(\theta_t(q)) - f(q)$ $\in C^\infty((-\delta,\delta)\times V)$ and $r(0,q) = 0$.

Define $g(t,q) = \int_0^1 \dot{r}(ts,q)\,ds$

By F.T.C., $t g(t,q) = \int_0^1 \dot{r}(ts,q)\,t\,ds$

$$= \int_0^t \dot{r}(u,q)\,du \underset{F.T.C.}{=} r(t,q) - r(0,q)$$

$$= f(\theta_t(q)) - f(q)$$

Proves the first part. For the second,

$$X_q(f) \overset{def}{=} \lim_{t\to 0} \frac{f(\theta_t(q)) - f(q)}{t}$$

$$= \lim_{t\to 0} \frac{t g(t,q)}{t} = g(0,q). \quad \square$$

Gives Lie derv. v.f. $\left\{ \begin{array}{l} \text{Thm: } X, Y \in \mathfrak{X}(M) \text{ then} \\ \mathcal{L}_X Y = [X,Y] \end{array}\right.$

Pf: Fix arb. $p \in M$ & $f \in C_p^\infty$.

Let $\dot{\theta} = X$.

$$\frac{1}{t}\left((\theta_{-t})_* Y_{\theta_t(p)} - Y(p)\right) \overset{?}{\to} [X,Y](p)$$

$t \mapsto -t$ ⟶ $\|$

necessarily so same

$$\lim_{t\to 0} \frac{Y(p) - (\theta_t)_* Y_{\theta_{-t}(p)}}{t}$$

By the lemma, $f(\theta_t) = f + t g_t$

then

$$\lim_{t\to 0} \frac{(\theta_t)_* Y_{\theta_t(p)}(f) - Y_p(f)}{t}$$

$$= \lim_{t\to 0} \frac{Y_p(f) - Y_{\theta_{-t}(p)}(f\theta_t)}{t}$$

$$= \lim_{t\to 0} \frac{Y_p(f) - Y_{\theta_{-t}(p)}(f + t g_t)}{t}$$

$$= \lim_{t\to 0} \frac{Y_p(f) - Y_{\theta_{-t}(p)}(f)}{t} - \lim_{t\to 0} Y_{\theta_{-t}(p)}\, g_t$$

$t \mapsto -t$

$$= \lim_{t\to 0} \frac{Y_{\theta_t(p)}(f) - Y_p(f)}{t} - \lim_{t\to 0} Y_{\theta_t(p)}\, g_{-t}$$

$$= \lim_{t\to 0} \frac{Y(f)(\theta_t(p)) - Y(f)(p)}{t} - \lim_{t\to 0} Y_{\theta_t(p)} \underbrace{g_{-t}}$$

$\dot{\theta}_t(f)(p)$ $\qquad g_0 = g(0,q) = X_q(f)$

$$= XY(f)(p) - YX(f)(p)$$

$$= [X,Y](f)(p)$$

Then $\mathcal{L}_X Y(p) = [X,Y](p)$ so

$\mathcal{L}_X Y = [X,Y]$ ☒

Rem: This shows $[X,Y]$ is a v.f. $([\,,\,])$

Thm: Suppose $F: N \to M$ C^∞-map

$X_1, X_2 \in \mathfrak{X}(N) \rightsquigarrow [F_* X_1, F_* X_2]$

$= F_*[X_1, X_2]$

Pf: For $p \in N$, $g \in C^\infty_{F(p)}$, $X_p \in T_pN$ $\quad \underbrace{X \in \mathfrak{X}(N)}$

$(F_*X)(F(p)) = F_*(X_p)$. In general, F_* is not a v.f. Then $(F_*X)(g)(F)$

$= X(gF)$. Therefore,

$F_*X_1\, F_*X_2\,(g)(F) - F_*X_2 F_*1$

$= F_*X_1\,(X_2(gF)) - F_*X_2[gF]$

$= X_1X_2(gF) - X_2X_1(gF)$

$= [X_1, X_2](gF)$

$= F_*([X_1,X_2])(g)(F)$

$\to [F_*X_1, F_*2] = F_*[X_1, X_2]$ ☒

Thm: $X, Y \in \mathfrak{X}(M)$, θ gen by X, σ: gen by Y.

Then $[X,Y] \leftrightarrow$

$(\sigma_s \theta_t)(p) = \theta_t \sigma_s(p)$

for all p.

Pf: $\Leftarrow$ Obvious

$d/ds(\theta_t(p))_{t=?}\ \theta_t(p) =$

Then $[X,Y] = \mathcal{L}_X Y =$

$$\lim_{t\to 0} \frac{(\theta_{-t})_* Y_{\theta_t(p)} - Y(p)}{t}$$

$$\lim_{s\to 0} \frac{\sigma_s(\theta_t(p)) - \theta_t(p)}{s} = \lim_{s\to 0} \frac{\theta_t(\sigma_s(p)) - \theta_t(p)}{s}$$

$\quad \| \qquad\qquad\qquad \|$

$Y(\theta_t(p)) \qquad (\theta_t)_* Y_p$

$\Rightarrow$: Let $Z_q = (\theta_{-t})_* Y_{\theta_t(q)}$

Show: $\dot{Z}_q(t) = 0$

Let $q' = \theta_t(q)$.

$$\dot{Z}_q = \lim_{\Delta t\to 0} \frac{(\theta_{-t-\Delta t})_* Y_{\theta_{t+\Delta t}(q)} - (\theta_{-t})_* Y_{\theta_t(q)}}{\Delta t}$$

$$= (\theta_{-t})_* \lim_{\Delta t \to 0} \frac{(\theta_{-\Delta t})_* Y_{\theta_{\Delta t}(q')} - Y_{q'}}{\Delta t}$$

$$= (\theta_{-t})_* ((\mathcal{L}_X Y)(q'))$$

$$= (\theta_{-t})_* [X,Y](q') \equiv 0$$

So $Z_{q(t)}$ does not depend on t

So $(\Theta_{-t})_* Y_{\Theta_t(q)} = Y(q)$ } Constant vector

Then $(\Theta_t)_* \dot{\sigma}_q = \dot{\sigma}(\Theta_t(q))$

$$\frac{d}{ds}\Theta_t(\sigma_s(q)) \qquad \frac{d}{ds}\sigma_s(\Theta_t(q))$$

Integrate & use fact have same start value.

$\Theta_t(\sigma_0(q)) = \sigma_0(\Theta_t(q)) = \Theta_t(q)$

By uniqueness of integral curve, $\Theta_t \sigma_s = \sigma_s \Theta_t$ ☐

10/24/2016

Tensors & Tensor Fields

Our v.s. will be finite dimensional.

Recall V v.s. over $\mathbb{R}$. Define V^* dual space. Call $\sigma \in V^*$ a covector.

(1) $F_* : V \to W$ a linear map then $F^* : W^* \to V^*$ defined by $\sigma \in W^*$, $F^*(\sigma)(v) = \sigma(F_*(v))$

&

$V \xrightarrow{F_*} W \xrightarrow{G_*} U$; F, G linear, V, W, U v.s.

$(G_* \circ F_*)^* = F^* \circ G^*$

} Bad notation?

(2) Let $\{e_1, \dots, e_n\}$ be a basis of V, $\exists!$ basis of V^* : $\{\sigma_1, \dots \sigma_n\}$ $\ni$ $\sigma_i(e_j) = \delta_{ij}$

(3) $V \cong (V^*)^*$

Def: A C^∞ covector field σ on manifold M is a smooth linear functional on space of vector fields over M in following sense:

(U, ϕ) coord neigh $(x_1, \dots, x_n)$ then $\sigma(\partial/\partial x_i) \in C^\infty(U)$

Rem:

1) Equiv, for all $X \in \mathfrak{X}(M)$, $\sigma(X) \in C^\infty(M) \rightleftarrows \sigma$ is a C^∞-covector field on M.

2) For all σ: C^∞ covector field $\sigma \in \mathfrak{X}(M) \to C^\infty(M)$ is a $C^\infty(M)$-lin. functional, ie

$\sigma(f_1 X_1 + f_2 X_2) = f_1 \sigma X_1 + f_2 \sigma X_2$ for $f_1, f_2 \in C^\infty(M)$; $X_1, X_2 \in \mathfrak{X}(M)$

Example: $f \in C^\infty(M)$. Define df by $df(X) = X(f)$ for $X \in \mathfrak{X}(M)$

Then df is C^∞ vector field on M called differential of f.

Let $(\mathcal{U}, \Phi)$ be coord. neigh with $(x_1, \dots, x_m)$. For $X = \sum_i a^i \partial/\partial x_i$ where $a_i \in C^\infty(\mathcal{U})$. Then

$$df(X) = df\left(\sum_i a_i \partial/\partial x_i\right) = \sum_i a_i \frac{\partial f}{\partial x_i}$$

<u>Note</u>: $dx_i\left(\partial/\partial x_j\right) = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$

Then $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$

<u>Fact</u>: $\{dx_1, \dots, dx_m\}$ is a basis of C^∞ covector field on $\mathcal{U}$.

$F: M \to N$ C^∞-map. Then

$F_p^*: T^*_{F(p)} N \to T_p^* M$

where $T_p^* M = (T_p M)^*$ is defined by

$$(F^*\sigma)(v) = \sigma(F_*(v))$$

where σ: C^∞ covector at $F(p)$, for $v \in T_p M$.

<u>Thm</u>: For σ C^∞ covector field on N, then $F^*\sigma$ is a C^∞ covector field on M.

Pf: Let $(\mathcal{U}, \Phi), (V, \Psi)$ be coord. neigh at p & $F(p)$ with coordinates $(x_1, \dots, x_m)$, $(y_1, \dots, y_n)$ and $\mathcal{U} \subset F^{-1}(V)$. For σ: C^∞ covector field at $F(p)$. Write $\sigma = \sum_j \sigma_j dy_j$ where $\sigma_j \in C^\infty(V)$

Then $F^*\sigma = F^*\left(\sum_j \sigma_j dy_j\right)$

$$= \sum_j \underbrace{\sigma_j F}_{\text{from "pulling out"}} \cdot F^*(dy_j)$$

By def,

$F^*(dy_j)\left(\partial/\partial x_i\right)$

$$= dy_j\left(F_*\left(\partial/\partial x_i\right)\right) = dy_j\left(\sum_k \frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k}\right)$$

$$\sum_k \frac{\partial y_k}{\partial x_i} \delta_{jk} \in C^\infty(V)$$

Then $F^*(dy_j) = \sum_i \frac{\partial y_j}{\partial x_i} dx_i$

Then $F^*\sigma$ is a C^∞ covector field on M. ∎

<u>Fact</u>: $F^*(dy_j) = \sum_i \frac{\partial y_j}{\partial x_i} dx_i$

&

$$F^*\left(\sum_j \sigma_j dy_j\right) = \sum_{j=1}^{n} \sum_{i=1}^{m} \sigma_j(F) \frac{\partial y_j}{\partial x_i} dx_i$$

* Be very familiar.

<u>Partitions of Unity</u>

Def: A covering $\{A_\alpha\}$ of M is called locally finite if for all $p \in M$, $\exists$ open set $\mathcal{U}_p$, $p \in \mathcal{U}$, such that

$$\#\{\alpha \mid A_\alpha \cap \mathcal{U}_p \neq \emptyset\} < \infty$$

A covering $\{B_\beta\}$ is a refinement of A_α if each $B_\beta \subset A_\alpha$ for some α. f: function on M, support of f is set $\operatorname{supp} f = \underbrace{\overline{\{x \in M \mid f(x) \neq 0\}}}_{\text{closed set.}}$

Def: A C^∞ partition of unity on M is a collection of smooth functions $\{f_\gamma\}$ on M such that....

1) $f_\gamma \geq 0 \quad \forall \gamma$
2) $\{\operatorname{supp}(f_\gamma)\}_\gamma$ is locally finite covering of M
3) $\sum_\gamma f_\gamma \equiv 1$ on M

Rem: $\sum_\gamma f_\gamma$ is essentially a finite sum near each point $p \in M$.

A partition of unity is subordinate to an open covering $\{A_\alpha\}$ if for all γ, $\exists A_\alpha$ such that $\operatorname{supp} f_\alpha \subset A_\alpha$. Namely, $\{\operatorname{supp} f_\gamma\}$ is a refinement of $\{A_\alpha\}$.

Prop: For all $\{A_\alpha\}$ open covering of M. $\exists$ a countable locally finite refinement $\{(U_i, \phi_i)\}$ of $\{A_\alpha\}$, where each (U_i, ϕ_i) is a coord neigh with $\phi_i(U_i) = \underbrace{B_3(0)}_{\text{open ball}}$ such that $\phi_i^{-1}(B_1(0)) = V_i \subset U_i$ also covers M.

Pf: Prop trivial if M is compact. If not, let $\{P_i\}$ be a countable basis of open sets $\ni \bar{P}_i$ is compact. Define a sequence of compact sets $K_1, K_2, \dots$ inductively: $K_1 = \bar{P}_1$. Assume up to K_n def. & r first integers $\ni r > i$ & $K_i \subset \underbrace{\bigcup_{j} P_j}_{\text{union}}$

then define $K_{i+1} = \overline{P_1 \cup \cdots \cup P_r}$
$= \bar{P}_1 \cup \bar{P}_2 \cup \cdots \cup \bar{P}_r$

Let $K_{-1} = K_0 = \emptyset$.

Know $\operatorname{int} K_{i+1} \supset K_i$ then $(\operatorname{int} K_{i+2} \setminus K_{i-1}) \cap A_\alpha$ is open. For p in above $\exists$ coord neigh $(U_{p,\alpha}, \phi_{p,\alpha})$ such that $U_{p,\alpha} \subset (\operatorname{int} K_{i+2} \setminus K_{i-1}) \cap A_\alpha$ and $\phi_{p,\alpha}(U_{p,\alpha}) = B_3(0)$. Let $V_{p,\alpha} = \phi_{p,\alpha}^{-1}(B_1(0))$ then $p \in V_{p,\alpha} \subset (\operatorname{int} K_{i+2} \setminus K_{i-1}) \cap A_\alpha$

Moreover, finitely many $V_{p,\alpha}$ covers compact set $K_{i+1} \setminus \operatorname{int} K_i \subset \operatorname{int} K_{i+2} \setminus K_{i-1}$ because of compactness. Write $V_{p,\alpha}$ as $\{V_{i_1}, \dots, V_{i_k}\}$ for k finite. Then $U_{p,\alpha}$ write as $\{U_{i_1}, \dots, U_{i_k}\}$. Then

$\bigcup_{i,j} V_{ij} = M$
$\hookrightarrow \bigcup K_{i+1} \setminus \mathring{K}_i$

Know countable & refinement. Only need locally finite: obvious as given open set at point use A_α to cover point but containing $\operatorname{int} K_{i+2} \setminus K_{i-1}$ & these use fin. many (check). ∎

10/26/2016

Countable Cover $\bigcup_{\substack{i=1,2,\dots \\ j=1,2,\dots,i_k}} \bigcup U_{i,j}$ is locally finite.

For all p, $\exists P_r \ni p \in P_r$

$P_r \subset K_{i+1}$

Then $\bigcup_{j=} \bigcup U_{ij} \supset K_{i+1} \supset P_r$

Then $S \supset K_{i+1} \supset P_r$. If $\alpha > i+2$,

then $\bigcup_{\alpha,\alpha_k} \cap K_{i+1} = \phi \rightsquigarrow$

$U_{\alpha,\alpha_k} \cap P_r = \phi$.

Therefore, $\{\mathcal{U}_i\}$ is locally finite.

Thm: Given each countable locally fin. coord covering $\{\mathcal{U}_i, \phi_i\}$ with $\phi_i(\mathcal{U}_i) = B_3(0)$ & let $V_i = \phi_i^{-1}(B_1(0))$ then $\exists$ a ~~[illegible]~~ partition of unity $\{f_i\} \ni$

1) $f_i > 0$ on V_i
2) $\operatorname{Supp} f_i \subset \phi_i^{-1}(B_2(0))$

In particular, every open covering $\{A_\alpha\}$ admits a partition of unity that is subordinate to it.

Pf: $\exists$ nonnegative smooth functions $\tilde{g}(x)$ on $\mathbb{R}^m$ such that

$$\underbrace{\tilde{g} = \begin{cases} 1 & \text{on } B_1(0) \\ 0 & \text{on } \neg B_1(0) = \mathbb{R}^m \setminus B_2(0) \end{cases}}_{\text{saw this earlier in the course.}}$$

Define $g_i = \tilde{g} \circ \phi_i \in C^\infty(M)$

$g_i \geq 0$ on M. Define $g_i|_{V_i^c} \equiv 0$.

and $\operatorname{supp} g_i \subset \phi_i^{-1}(B_2(0))$, $g_i \equiv 1$ on V_i since $\mathcal{U}_i$ is locally finite

$f_i = \dfrac{g_i}{\sum_j g_j}$ is well defined

and satisfies the partition of unity. $\blacksquare$

Thm: Any compact manifold M admits a C^∞ embedding as a submanifold in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Pf: Let $\{\mathcal{U}_i, V_i, \phi_i\}$ be finite covering of M as in the thm. Namely,

$\phi_i(\mathcal{U}_i) = B_3(0)$, $\phi_i(V_i) = B_1(0)$

Let $\{\cancel{g_i}\ f_i\}$ be partition of unity subordinate to $\mathcal{U}_i$. In fact, $f_i \equiv 1$ on V_i, $f_i \equiv 0$ on $(\phi_i^{-1}(B_2(0)))^c$

Then $f_i \phi_i : M \to B_3(0) \subset \mathbb{R}^m$ C^∞-map

By defining $(f_i\Phi_i)|_{U_i^c} \equiv 0$

Moreover, $f_i\Phi_i : V_i \to B_1(0)$ is a diffeomorphism. Define

$$F: M \to \underbrace{\underbrace{\mathbb{R}^m \times \cdots}_{K} \times \underbrace{\cdots \times \mathbb{R}^m}_{K}}_{2K}$$

$$p \mapsto (f_1\Phi_1, \dots, f_K\Phi_K, f_1, \dots, f_K)$$

F is C^∞. Only need check diffeo.

$DF \in M(m, mK+K) \to DF$ has at most rank m. OTOH, $p \in M$ $\exists V_i$ with $p \in V_i$ and $(\text{rank } DF)_p \geq \text{rank}(D f_i\Phi_i)_p = m$

diffeo.

Then rank $DF \equiv m$. So F is an immersion. For $p_1, p_2 \in M$

$F(p_1) = F(p_2)$ so then

$f_i\Phi_i(p_1) = f_i\Phi_i(p_2)$ for all $i = 1, \dots, K$.

$f_i(p_1) \neq 0$ since $p_1 \in V_i$.

So $\exists i \ni f_i\Phi_i(p_1) = f_i\Phi_i(p_2) \to \Phi_i(p_1) = \Phi_i(p_2) \to p_1 = p_2$.

So F inj. $\to F$ embedding as inj. immersion from compact manifold. ∎

Thm (Whitney): Any C^∞ manifold M^m can be C^∞ embedded as a closed submanifold of $\mathbb{R}^{2m}$.

Tensor Fields:

Def: A tensor Φ on a v.s. V is a multilinear map

$$\Phi: \underbrace{V \times V \times \cdots \times V}_{r} \times \underbrace{V^* \times \cdots \times V^*}_{s} \to \mathbb{R}$$

r: covariant order

s: contravariant order

Fix (r,s), let $T^r_s \to T^r_s(V)$ be the set of all tensors on V.

Check: $T^r_s(V)$ is a vector space.

Thm: $\dim_{\mathbb{R}} V = n \to \dim_{\mathbb{R}} T^r_s(V) = n^{r+s}$

Pf: A basis of $T^r_s(V)$ is $\Omega^{j_1,\dots,j_r}_{i_1,\dots,i_s}$ for $1 \leq j_1, \dots, j_r \leq n$, $1 \leq i_1, \dots, i_s \leq n$

given by

$$\Omega^{j_1\cdots j_r}_{i_1\cdots i_s}(e_{k_1}, \dots, e_{k_r}, f_{e_1}, \dots, f_{e_s}) = \delta_{j_1 k_1} \cdots \delta_{j_r k_r} \delta_{i_1 e_1} \cdots \delta_{i_s e_s}$$

where

$\{e_1, \dots, e_n\}$ basis of V

$\{f_1, \dots, f_n\}$ dual basis.

Check: This is a basis.

Fact:

$$\phi = \sum_{\substack{j_1 \cdots j_r \\ i_1 \cdots i_s}} \phi(e_{j_1}, \dots, e_{j_r}, \xi_{i_1}, \dots, \xi_{i_s}) \Omega^{j_1, \dots, j_r}_{i_1, \dots i_s}$$

Def: A C^∞ covariant tensor field of order r on M is a function ϕ with $\phi_p \in T_0^r(T_pM)$ such that for $X_1, \dots, X_r \in \mathcal{X}(\mathcal{U})$ for open set $\mathcal{U} \subset M$, then $\phi(X_1, \dots, X_r) \in C^\infty(\mathcal{U})$.

Define such a tensor field by $T^r(M)$.

Rem: Given $\phi \in T^r(M)$, ϕ is $C^\infty(M)$ linear in each variable for any $f \in C^\infty(M)$.

Fix $(\mathcal{U}, \phi)$ card. neigh. with $(x_1, \dots, x_n) = X$

then $\Omega^{j_1 \cdots j_r}(\partial/\partial x_1, \dots, \partial/\partial x_r) = \delta_{j_1 k_1} \cdots \delta_{j_r k_r}$

and $\Omega^{j_1, \dots, j_r}$ is a basis of $T^r(\mathcal{U})$.

In fact, $\phi = \sum \phi(\partial/\partial x_{j_1}, \dots, \partial/\partial x_{j_r}) \Omega^{j_1, \dots, j_r}$

10/31/2016

Let $F: M \to N$ be C^∞. Then

$F^*: T^r(N) \to T^r(M)$ is linear

defined by....

$$F^*(\Phi)_M(X_1(p), \dots, X_m(p))$$

$$= \Phi_{F(p)}(F_*(X)_{F(p)}, \dots, F_*(X_p)_{F(p)})$$

where $\Phi \in T^rN$, $X_1, \dots, X_p \in \mathcal{X}(M)$ for $p \in M$.

Def: V a vector space. $\Phi \in T^r(V)$

Φ is symmetric if

$$\Phi(v_1, \dots, \overset{i^{th}}{v_i}, \dots, \overset{j^{th}}{v_j}, \dots v_r) = \Phi(v_1, \dots, \underset{i^{th}}{v_j}, \dots, \underset{j^{th}}{v_i}, \dots, v_r)$$

ie, invariant under any transposition of variables.

Alternating if same up to sign.

Symmetric: $\Sigma^r V$

Alternating: $\Lambda^r V$

Thm: $F: M \to N$ C^∞-map. If $\Phi \in T^rN \rightsquigarrow F^*\Phi \in T^rM$

Moreover, $F^*: T^rN \to T^rM$, is linear and preserves symmetric & alternating tensors.

Note: $\Phi \in \Sigma^r V$ if and only if $\Phi(v_{\sigma(1)}, \dots, v_{\sigma(r)}) = \Phi(v_1, \dots, v_r)$ for any σ: permutation of $\{1, \dots, n\}$

$\Phi \in \Lambda^r(V)$ if and only if $\Phi(v_{\sigma(1)}, \dots, v_{\sigma(r)}) = \mathrm{sgn}(\sigma)\,\Phi(v_1, \dots, v_r)$ for any perm. σ

Def: The symmetrizing map $S: T^r(V) \to T^r(V)$ is defined by $S\Phi(v_1, \dots, v_r) = \frac{1}{r!}\sum_\sigma \Phi(v_{\sigma(1)}, \dots, v_{\sigma(r)})$

The alternating map $A: T^r(V) \to T^r(V)$ is defined by $A\phi(v_1,\dots,v_r) = \frac{1}{r!}\sum_\sigma \operatorname{sgn}\sigma\, \phi(v_{\sigma(1)},\dots,v_{\sigma(r)})$

Note: S, A are linear maps

Prop:

1) $\underbrace{A^2 = A,\ S^2 = S}$

$\approx$ A, S are projections

2) $A(T^r(V)) = \Lambda^r(V)$

$S(T^r(V)) = \Sigma^r(V)$

3) Φ is alternating if and only if $A\Phi = \Phi$. Φ is symmetric iff $S\Phi = \Phi$

4) If $L_*: V \to W$ is lin, then A, S commute with $L^*: T^rW \to T^rV$, ie. $AL^* = L^*A$ & $SL^* = L^*S$

Thm: A, S are well-defined on T^rM and they satisfy 1-4 above.

In particular, $F: M \to N$ C^∞ map then $AF^* = F^*A$, $SF^* = F^*S$. Then F^* preserves Σ^r & Λ^r.

Ex: Σ^2M. Let $\phi \in \Sigma^2M$, ie
$\Phi(X_1, X_2) = \Phi(X_2, X_1)$ for $X_1, X_2 \in \mathcal{X}(M)$.
Moreover, assume Φ is positive definite.
Namely, for $p \in M$, $\Phi_p(v,v) \geq 0$ &
$\Phi_p(v,v) = 0 \Rightarrow v = 0$; $v \in T_pM$

Def: Any $\Phi \in \Sigma^2M$, that is positive definite is called a Riemannian metric on M.

Rem: For $p \in M$, $v_1, v_2 \in T_pM$
$\sqrt{\Phi(v_1,v_1)}$ = length of v_1

$\Phi(v_1,v_2)$ gives us the $\measuredangle$ between v_1, v_2 via:
$|\Phi(v_1,v_2)| = \sqrt{\Phi(v_1,v_1)}\sqrt{\Phi(v_2,v_2)}\cos\theta$

For $p, q \in M$, let $\gamma(t)$ be the C^∞ curve connecting p & q.
Namely, $\gamma: [0,1] \to M$ C^∞ map and $\gamma(0) = p$, $\gamma(1) = q$. Define the length of γ by

Only need C^1.

$$\text{len}\,\gamma = \int \sqrt{\phi(\dot\gamma,\dot\gamma)(t)}\,dt$$

The distance between p, q is defined by
$d(p,q) = \inf$ length γ: $\gamma: [0,1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$, C^∞

Thm: (M, Φ) is a metric space.

Thm: (Riemann) For all M, C^∞ manifold, $\exists$ pos. def. symmetric 2-tensors $\Phi \in \Sigma^2M$

Pf: (Rough) Let $\{\mathcal{U}_\alpha\}$ be an open cover of M by coordinate neighborhoods with coordinate $(x_1^\alpha, x_2^\alpha, \dots, x_m^\alpha)$. $\exists$ a partition of unity $\{f_\alpha\}$. On $\mathcal{U}_\alpha$, define

$$\Phi_\alpha = \sum_{i=1}^{m} dx_\alpha^i \otimes dx_\alpha^i$$

where $dx_\alpha^j \otimes dx_\alpha^i \left(\frac{\partial}{\partial x_\alpha^1}, \dots, \frac{\partial}{\partial x_\alpha^k}\right) = \delta_{ij}\delta_{ik}$

Check: Φ_α is pos. def., symmetric, covariant 2-tensor on $\mathcal{U}_\alpha$.

Define $\Phi = \sum_\alpha f_\alpha \Phi_\alpha$

Multiplication of Tensors

$T^r M$: covariant tensor field of order r on M.

$T^0 M = C^\infty(M)$.

On vector space V, $\phi \in T^r(V)$, $\psi \in T^s(V)$

Def: $\phi \otimes \psi$ is a covar. tensor of order $r+s$ on M. $\phi \otimes \psi$ def. by

$$(\phi \otimes \psi)(v_1, \dots, v_r, v_{r+1}, \dots, v_{r+s}) \overset{\text{def}}{=} \phi(v_1, \dots, v_r)\,\psi(v_{r+1}, \dots, v_{r+s})$$

Thm:

1) The tensor product $T^r(V) \times T^s(V) \to T^{r+s}(V)$ is bilin. & associative

2) If $\omega_1, \dots, \omega_n$ is a basis of $V^* = T^1 V$ then $\omega_{i_1} \otimes \dots \otimes \omega_{i_r}$ for $1 \le i_1, \dots, i_r \le n$ is a basis of $T^r(V)$.

3) If $F_*: W \to V$ is lin. then $F^*(\phi \otimes \psi) = (F^*\phi) \otimes (F^*\psi)$ for $\phi \in T^r(V)$ & $\psi \in T^s V$

Pf:

1) $(\alpha_1\phi_1 + \alpha_2\phi_2) \otimes \psi = \alpha_1\phi_1 \otimes \psi + \alpha_2\phi_2 \otimes \psi$ } Check

$(\phi \otimes \psi) \otimes \Theta = \phi \otimes (\psi \otimes \Theta)$

2) Recall basis of $T^r V$ is given by $\Omega^{i_1, \dots, i_r}(e_{j_1}, \dots, e_{j_r}) = \delta_{i_1 j_1} \cdots \delta_{i_r j_r}$ where $\{e_1, \dots, e_n\}$ basis of V. Note $\omega^{i_1} \otimes \dots \otimes \omega^{i_r}(e_{j_1}, \dots, e_{j_r}) = \delta_{i_1 j_1} \cdots \delta_{i_r j_r} \rightsquigarrow \dim T^r V = n^r$.

Let $T^0V = \mathbb{R}$ &

$$T(V) = \bigoplus_{r=0}^{\infty} T^r(V)$$

If $\tilde{\Phi} \in T(V)$, write $\tilde{\Phi} = \phi_1^{i_1} + \cdots + \phi_n^{i_n}$ where $\phi_j^{i_j} \in T^{i_j}(V)$ & $0 \le i_1 < i_2 < \cdots < i_n$

Then $\tilde{\Phi}, \tilde{\Psi} \in T(V)$. Define $\tilde{\Phi} + \tilde{\Psi}$ componentwise. Define $\tilde{\Phi} \otimes \tilde{\Psi}$ componentwise. The associative algebra over $\mathbb{R}$ defined in this way is called the tensor algebra. It has unit 1. It is an inf. dim. v.s. over $\mathbb{R}$.

11/02/2016

V vector space

$T(V) = T^0(V) \oplus T^1(V) \oplus \cdots$

For $\tilde{\Phi} \in T(V)$, $\tilde{\Phi} = \sum_{j=1}^{\ell} \phi_j^{i_j}$

where $\phi_j^{i_j} \in T^{i_j}(V)$; $i_1 < i_2 < \cdots < i_\ell$.

Thm: $T(V)$ is an associative algebra with unit over $T^0(V) \equiv \mathbb{R}$.

$T(V)$ is generated by $\mathbb{R}$ and $T^1(V) = V^*$. $F_*: W \to V$ induces a homo. $F^*: T(V) \to T(W)$ such that

1) $F^* = 1$ on $\mathbb{R}$

2) $F^*: \underbrace{T^1(V)}_{V^*} \to \underbrace{T^1(W)}_{W^*}$ dual of F_*

& F^* is determined uniquely on $T(V)$ by (1) & (2).

Thm: $T^rM \times T^sM \to T^{r+s}M$ defined by tensor product $\otimes$ is bilin. & associative. If $w_1, \ldots, w_n$ is a basis of $T^1M|_U$ for $U \subset M$ open, then every element of $T^rM|_U$ is a lin. comb. with C^∞ coeffic. of $\{w^{i_1} \otimes \cdots \otimes w^{i_r} \mid i_1, \ldots, i_r \in \{1, \ldots, m\}\}$

If $F: N \to M$ C^∞, $\phi \in T^rM$, $\psi \in T^sM \rightarrow F^*(\phi \otimes \psi) = F^*\phi \otimes F^*\psi \in T^{r+s}N$

If $\phi \in T^rM$, $\psi \in T^sM \rightarrow$

$$\phi \otimes \psi\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_{r+s}}}\right) = \phi\left(\partial/\partial x_{i_1}, \ldots, \partial/\partial x_{i_r}\right) \psi\left(\partial/\partial x_{i_{r+1}}, \ldots, \partial/\partial x_{i_{r+s}}\right)$$

for local coord. $x = (x_1, \ldots, x_m)$ on $U \subset_{open} M$

Cor: $\phi \in T^rM$, $\exists!$ a unique way to write ....

$$\phi = \sum_{i_1, \ldots, i_r} a_{i_1 \ldots i_r} \, dx_{i_1} \otimes \cdots \otimes dx_{i_r}$$

where $a_{i_1, \ldots, i_r} \in C^\infty U$

$x = (x_1, \ldots, x_n)$ local coords on U.

and

$$a_{i_1, \ldots, i_r} = \phi\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \partial/\partial x_{i_r}\right)$$

Let $r > 0$.

$\Lambda^r V \subset T^r V$

alternating covariant tensor of order r.

In particular, $\Lambda^1 V = T^1 V$

Define $\Lambda(V) = \bigoplus_{r=0}^{\infty} \Lambda^r V$

where $\Lambda^0 V = \mathbb{R} \subset T(V)$

Def: $\Lambda^r(V) \times \Lambda^s(V) \to \Lambda^{r+s} V$

by $(\phi, \psi) \mapsto \frac{(r+s)!}{r!\,s!} A(\phi \otimes \psi)$

Called the exterior product or wedge product, denoted $\phi \wedge \psi$.

Lem: The wedge product is bilin. & associative.

Pf: Bilinear is clear. Now show

$A(\phi \otimes \psi \otimes \theta) = A(A(\phi \otimes \psi) \otimes \theta)$

$= A(\phi \otimes A(\psi \otimes \theta))$

$S = S_{r+s+t}$ = all perm of $\{1, 2, \dots, r+s+t\}$

S' = subgroup in S leaving last t #'s fixed.

$S_{r+s} \to S'$ iso.

$\sigma \mapsto \sigma'$

$A(\phi \otimes \psi) \otimes \theta(v_1, \dots, v_{r+s+t})$

$= \cancel{\theta(v_{s+1}, \dots, v_{r+s+t})}$

$= \frac{1}{(r+s)!} \sum_{\sigma \in S_{r+s}} \operatorname{sgn}\sigma\, \phi(v_{\sigma_1}, \dots)\, \psi(v_{\sigma r+1} \cdots v_{\sigma r+s})\, \theta(v_{r+s+1} \cdots v_{r+s+t})$

$= \frac{1}{(r+s)!} \sum_{\sigma' \in S'} \operatorname{sgn}\sigma'\, \phi(v_{\sigma'1}, \dots v_{\sigma' r})\, \psi(v_{\sigma' r+1} \cdots v_{\sigma' r+s})\, \theta(v_{\sigma' r+s+1} \cdots)$

$A(A(\phi \otimes \psi) \otimes \theta)(v_1, \dots, v_{r+s+t})$

$\frac{1}{(r+s+t)!} \frac{1}{(r+s)!} \sum_{\tau \in S} \sum_{\sigma' \in S' \subset S} \operatorname{sgn}(\tau\sigma')$

$\phi(v_{\tau\sigma' 1}, \dots)\, \psi(v_{\tau\sigma' \dots})$

$\theta(v_{\tau\sigma} \cdots)$

Note: $\{\tau\sigma' \mid \tau \in S\} = S$ for $\sigma' \in S$

$= \frac{1}{(r+s)!} \sum_{\sigma' \in S'} A(\phi \otimes \psi \otimes \theta)(v_1, \dots, v_{r+s+t})$

$= \vdots$

$=$ bran. It's associative. ☐

Cor: For $\phi_i \in \Lambda^{r_i} V$ for $i = 1, \dots, k$

then $\phi_1 \wedge \dots \wedge \phi_k = \frac{(r_1 + \dots + r_k)!}{r_1! \cdots r_k!} A(\phi_1 \otimes \dots \otimes \phi_k)$

Wedge Product

$\Lambda(V) \times \Lambda(V) \to \Lambda(V)$ is

$\tilde{\Phi} \wedge \tilde{\Psi} \overset{def}{=} \sum_{\substack{i=1,\dots,k \\ j=1,\dots,l}} \phi_i \wedge \psi_j$, where...

$\tilde{\Phi} = \sum^{k} \phi_i$; $\phi_i \in \Lambda^{r_i} V$

$\tilde{\Psi} = \sum^{l} \psi_j$; $\psi_j \in \Lambda^{s_j} V$

11/07/2016

Cor: $\Lambda(V)$ with wedge product is an associative algebra over $\Lambda^0 V = \mathbb{R}$ called the exterior algebra.

Show $\Lambda(V)$ is a fin. dim. v.s. with dim. $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ $\Big\}$ dim $V = n$

Lem: $\phi \in \Lambda^r V, \psi \in \Lambda^s V \rightarrow$
$\phi \wedge \psi = (-1)^{rs} \psi \wedge \phi$

Pf: Suffices to show $A(\phi \otimes \psi) = (-1)^{rs} A(\psi \otimes \phi)$

$A(\phi \otimes \psi)(v_1, \dots, v_{r+s}) = \frac{1}{(r+s)!} \sum_{\sigma}' \operatorname{sgn} \sigma \ \phi(v_{\sigma 1} \dots) \psi(v_{\sigma(r+1)} \dots)$

switch order as next step

Note:
$\tau \in S_{r+s}$
$\tau(1, 2, \dots, s, s+1, \dots, s+r) = (r+1, \dots, r+s, 1, \dots, r)$

$\operatorname{sgn} \tau = (-1)^{rs}$

$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma\tau) \operatorname{sgn}(\tau)$

$= \frac{1}{(r+s)!} \sum_{\sigma}' \operatorname{sgn}(\sigma\tau) \operatorname{sgn}(\tau) \psi(v_{\sigma\tau 1} \cdots v_{\sigma\tau s}) \phi(v_{\sigma\tau(s+1)} \cdots v_{\sigma\tau(r+s)})$

$\{\sigma\tau \mid \sigma \in S_{r+s}\} = S_{r+s}$ $\boxtimes$

Thm: If $r > n = \dim V \rightarrow \Lambda^r V = 0$

If $0 \le r \le n \rightarrow \dim \Lambda^r V = \binom{n}{r}$

(continued $\nearrow$)

Let $w_1, \dots, w_n$ be a basis of $\Lambda^1(V) = V^*$ then $\{w^{i_1} \wedge \dots \wedge w^{i_r} \mid 1 \le i_1 < \dots < i_r \le n\}$ is a basis of $\Lambda^r(V)$.

Pf: Check:

$\sum_{i_1 < \dots < i_r} a_{i_1 \dots i_r} w^{i_1} \wedge \dots \wedge w^{i_r} = 0$

$\Downarrow$

$a_{i_1 \dots i_r}$ for all $i_1, \dots, i_r$.

Thm: V, W fin. dim v.s.
$F_*: W \rightarrow V$ linear. Then
$F^*: \Lambda(V) \rightarrow \Lambda(W)$ is a homomorphism of algebras induced by $F^*: T^1(V) \rightarrow T^1(W)$

Pf: Define $F^*(w_1 \otimes w_2) = F^* w_1 \otimes F^* w_2$ for $w_1, w_2 \in T(V)$.

Check: $F^* A = A F^*$

Def: $\Lambda^r M \overset{\text{def}}{=} r\text{-forms} \overset{\text{def}}{=}$ alternating covariant tensor field of order r on M.

Thm: $\Lambda M = \bigoplus_{r=0}^{m} \Lambda^r(M)$ $(m = \dim M)$

is a v.s. of differential forms &

$\forall \phi \in \Lambda^r M$, $\psi \in \Lambda^s M$, define

$$(\phi \wedge \psi)_p = \phi(p) \wedge \psi(p)$$

Then $\phi \wedge \psi = (-1)^{rs} \psi \wedge \phi$

Moreover, ΛM is an algebra over $C^\infty M$.

For $f \in C^\infty M$,

$$(f\phi) \wedge \psi = f \cdot \phi \wedge \psi$$
$$= \phi \wedge (f\psi)$$

If $w_1, \dots, w_n$ is a basis of $\Lambda^1 M$ then $\{w^{i_1} \wedge \dots \wedge w^{i_r} \mid i_1 < \dots < i_r\}$ is a basis of $\Lambda^r M|_{\mathcal{U}}$.

Thm: $F: M \to N$ C^∞ map

Then $F^*: \Lambda N \to \Lambda M$ is an algebra homomorphism.

Orientation of a Manifold

Def: An oriented vector space is a vector space plus an equivalence class of allowable bases. The choice of the basis determines the orientation.

V v.s.; $\dim V = n$, $E = \{e_i\}^n$, $F = \{f_i\}^n$ bases

Then $\exists!$ invertible matrix $\mathcal{U}$ such that $(e_1 \cdots e_n) = (f_1 \cdots f_n)\mathcal{U}$

If $\det \mathcal{U} > 0$, then E, F have same orientation.

Lem: Let $\Omega \in \Lambda^n V$, $\dim V = n$, $\Omega \neq 0$. Let $e_1, \dots, e_n$ be a basis of V. Then for $v_i = \sum_j \alpha_i^j e_j$

$$\Omega(v_1, \dots, v_n) = (\det \alpha_i^j)\, \Omega(e_1, \dots, e_n)$$

Pf:
$$\Omega(v_1, \dots, v_n) = \sum_{j_1, \dots, j_n} \alpha_1^{j_1} \cdots \alpha_n^{j_n} \Omega(e_{j_1} \cdots e_{j_n})$$
$$= \left(\sum_{j_1, \dots, j_n} \operatorname{sgn}\begin{pmatrix} j_1 \cdots j_n \\ 1 \cdots n \end{pmatrix} \alpha_1^{j_1} \cdots \alpha_n^{j_n} \right) \Omega(e_1, \dots, e_n)$$
$$= (\det \alpha_i^j)\, \Omega(e_1, \dots, e_n) \quad \square$$

Cor: A nonvanishing $\Omega \in \Lambda^n V$ has the same (opposite) sign on two basis if they have the same (opposite) orientation. Then $\Omega \neq 0$ determines an orientation of V.

Ω_1, Ω_2 have the same orientation of $\Omega_1 = \lambda \Omega_2$ for $\lambda > 0$.

Rem: Assume $\Omega \neq 0 \to$ $\Omega(v_1 \cdots v_n) \neq 0$ iff $v_1, \dots, v_n$ lin indep.

Def: A manifold is orientable if $\exists\ C^\infty$ n-form Ω on M that is nonvanishing everwhere. The orientation of M is determined by Ω.

Ex: $M = \mathbb{R}^n$

$\Omega = dx_1 \wedge \cdots \wedge dx_n$

natural orientation. $\mathcal{U} \subset \mathbb{R}^n$ open. $\Omega|_\mathcal{U} = \Omega_\mathcal{U}$.

$F: \mathcal{U} \to V \underset{open}{\subset} \mathbb{R}^n$ is orientation preserving if $F^*\Omega_V = f\Omega_\mathcal{U}$ where $f>0$ is C^∞ function on $\mathcal{U}$.

e.g. $F: \mathbb{R}^n \to \mathbb{R}^n$

$(x_1, \ldots, x_n) \mapsto (x_2, x_1, x_3, \ldots)$

Then ...

$F^*(dx_1 \wedge \cdots \wedge dx_n) = -dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$

so orientation reversing. In general, $F: M_1^m \to M_2^m$ diffeomorphism is orientation preserving if $F^*\Omega_{M_2} = f\Omega_{M_1}$ with $f>0$ function on M_1.

Thm: M^m is orientable iff $\exists$ coord. covering $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ of M $\ni$ $\phi_\alpha\phi_\beta^{-1}$: $\phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is orientation preserving for all α, β with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$.

Pf:

$\Rightarrow$: Let $\Omega \neq 0$ be a m-form. Let $(\mathcal{U}_\alpha, \phi_\alpha)$ be a coord. neigh with local coord. $x_\alpha = (x_\alpha^1, \ldots, x_n^\alpha)$ such that

$(\phi_\alpha^{-1})^*\Omega = f_\alpha(x_\alpha)\, dx_{\alpha_1} \wedge \cdots \wedge dx_n^\alpha$

on $\mathcal{U}_\alpha$. With $f_\alpha(x_\alpha) > 0$.

(if negative just switch some coordinates in wedge product). If $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi \to (\phi_\beta^{-1})^*\Omega_\beta =$

$f_\beta(x_\beta)\, dx_1^\beta \wedge \cdots \wedge dx_m^\beta$ for $f_\beta > 0$ on $\mathcal{U}_\beta$. Now

$(\phi_\alpha \phi_\beta^{-1})^*(\cancel{f_\alpha(x_\alpha)} \cdot dx_1^\alpha \wedge \cdots \wedge d_n^\alpha)$

$= (\phi_\alpha \phi_\beta^{-1})^* \left(\frac{(\phi_\alpha^{-1})^*\Omega}{f_\alpha} \right)$

$= (\tfrac{1}{f_\alpha}\, \phi_\alpha \phi_\beta^{-1})(\phi_\beta^{-1})^*\Omega$

$= (\tfrac{1}{f_\alpha}\, \phi_\alpha \phi_\beta^{-1})\, f_\beta\, dx_1^\beta \wedge \cdots \wedge dx_m^\beta$

since $1/f_\alpha\, \phi_\alpha \phi_\beta^{-1} \cdot f_\beta > 0$ then $\phi_\alpha \phi_\beta^{-1}$ is orientation preserving.

Check:

$$dx_1^\alpha \wedge \cdots \wedge dx_\alpha^m = \det \frac{\partial x_1^\alpha \cdots x_m^\alpha}{\partial x_1^\beta \cdots x_m^\beta}\, dx_1^\beta \wedge \cdots \wedge dx_m^\beta$$

$\Leftarrow$: M covered by orientation preserving coord neigh. $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ Let $\{f_\alpha\}$ be the partition of unity that is subordinate to $\mathcal{U}_\alpha$.

Define $\Omega_r = \sum_\alpha f_\alpha \phi_\alpha^* \, dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha$

Then $\Omega_r \in \Lambda^m M$. Check $\Omega_r \neq 0$ everywhere. Then $p \in M$, $\exists \alpha \ni$ $f_\alpha(p) > 0$ for $\beta \neq \alpha$ with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$.

$$\phi_\beta^*(dx_1^\beta \wedge \cdots \wedge dx_m^\beta)$$

$$= \phi_\alpha^* \boxed{(\phi_\beta \phi_\alpha^{-1})^* (dx_1^\beta \wedge \cdots \wedge dx_m^\beta)}$$

$$= \phi_\alpha^* \left(\lambda_\alpha \, dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha \right)$$

$$= (\lambda_\alpha \circ \phi_\alpha)(\phi_\alpha^* \, dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha)$$

where $\lambda_\alpha > 0$ as orientation preserving.

We get a similar result running Ω_r for ϕ_β^*. Know nonnegative at least and $f_\alpha(p) > 0$ so at least one positive. $\square$

11/09/2016

Exterior Differentiation

$\mathcal{U} \subset M$ and open set

$\Theta \in \Lambda^k M$

Write $\Theta = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

on $\mathcal{U}$, where $a_{i_1 \cdots i_k} \in C^\infty \mathcal{U}$ is unique.

Define $b_{\sigma(i_1) \cdots \sigma(i_k)} = \operatorname{sgn}(\sigma)\, a_{i_1 \cdots i_k}$

for all σ permutation of $\{i_1, \ldots, i_k\}$ then $b_{j_1 \cdots j_k}$ is also skew symmetric $j_1 \cdots j_k$ then

$$\Theta = \sum_{j_1 \cdots j_k} \frac{1}{k!} b_{j_1 \cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

Thm: M C^∞ manifold. $\Lambda(M)$ exterior alg. of diff forms on M. Then $\exists!$ $\mathbb{R}$-lin. map $d: \Lambda(M) \to \Lambda(M)$ such that

1) $d_M f = df$ for $f \in C^\infty M$

2) $d_M(\Theta \wedge \sigma) = d_M \Theta \wedge \sigma + (-1)^r \Theta \wedge d_M \sigma$

where $\Theta \in \Lambda^r M$, $\sigma \in \Lambda^s M$

3) $d_M^2 \equiv 0$

Moreover, d_M locally defined, ie $(d_M \Theta)|_{\mathcal{U}} = d_M \Theta|_{\mathcal{U}}$ for $\Theta \in \Lambda^r M$ and $\mathcal{U} \subset M$ open.

Pf:

1) For $\Theta = g\,df \wedge \cdots \wedge df^r$ for $g, f^1, \cdots, f^r \in C^\infty M$

then $d_M\Theta = dg \wedge df^1 \wedge \cdots \wedge df^r + \sum_{i=1}^{r} (-1)^{i+1} g\, df^1 \wedge \cdots \wedge d_M(df^i) \wedge \cdots \wedge df^r$

$= dg \wedge df^1 \wedge \cdots \wedge df^r$. In local coordinates, we define d_M by

$$d_M\Theta = \sum_{i_1<\cdots<i_r}' da_{i_1\cdots i_r}\, dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

where $\Theta = \sum_{i_1<\cdots<i_r}' a_{i_1\cdots i_r}\, dx^{i_1} \wedge \cdots \wedge dx^{i_r}$

Check: d_M satisfies (1), (2), (3)

(1) Obvious

(2) $\Theta = a\,dx^{i_1} \wedge \cdots \wedge dx^{i_r}$

$\sigma = b\,dx^{j_1} \wedge \cdots \wedge dx^{j_s}$

then $d_M(\Theta \wedge \sigma) = d_M((ab)(dx^{i_1} \wedge \cdots \wedge dx^{i_r}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_s}))$

$= d(ab) \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_r}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_s})$

$= (da + a \cdot db) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_s}$

$= (da \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}) \wedge (b\, dx^{j_1} \wedge \cdots \wedge dx^{j_s}) + (-1)^r (a \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_r}) \wedge (db \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_s})$

$= d\Theta \wedge \sigma + (-1)^r \Theta \wedge d\sigma$

(3) For $a \in C^\infty M$

$$d_M^2 a = d_M\left(\sum_i' \frac{\partial a}{\partial x^i} dx^i\right)$$

$$= \sum_{i,j}' \frac{\partial^2 a}{\partial x^i \partial x^j} dx^j \wedge dx^i$$

$$= \sum_{i<j}' \left(-\frac{\partial^2 a}{\partial x^i \partial x^j} + \frac{\partial^2 a}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j$$

$= 0$

Then $d_M^2\Theta = d_M(da \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r})$

$= d_M^2 a \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}$

$= 0$

(2) We show $d_M(\Theta|_{\mathcal{U}}) = (d_M\Theta)|_{\mathcal{U}}$ for $\Theta \in \wedge^r(M)$

$$\Theta = \sum_{i_1<i_r}' a_{i_1\cdots i_r}\, dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

on $\mathcal{U}$ for $a_{i_1\cdots i_r} \in C^\infty \mathcal{U}$

Let $p \in \mathcal{U}$. $\exists$ C^∞ function χ on M with supp $\chi \subset \mathcal{U}$, $\chi \equiv 1$ on V open, where $\bar{V} \subset \mathcal{U}$

Define $\sigma = \sum_{i_1<\cdots<i_r}' a_{i_1\cdots i_r}\, d(x^{i_1}\chi) \wedge \cdots \wedge d(x^{i_r}\chi)$

$\in \wedge^r M$

and $\sigma(p) = \Theta(p)$

Calculate:

$$d_M\sigma = \sum_{i_1<\cdots<i_r}' d(a_{i_1\cdots i_r}\chi) \wedge d(x^{i_1}\chi) \wedge \cdots \wedge d(x^{i_r}\chi)$$

then $d_M\sigma(p) = \sum_{i_1<\cdots<i_r}' da_{i_1\cdots i_r}(p) \wedge dx^{i_1}(p) \wedge \cdots \wedge dx^{i_r}(p)$

then $\underbrace{d_M\sigma(p)}_{\text{globally def}} = \underbrace{d_M\Theta(p)}_{\text{locally def}}$ then

$d_M \tau = d_M \theta$ at every p ∈ U

then $d_M(\theta|_U) = (d_M\theta)|_U$

(3) Let $\{(U_\alpha, \phi_\alpha)\}$ be a covering of M by coord. open sets

For $\theta \in \Lambda^r M$, $(d_M\theta)|_{U_\alpha} = d_M(\theta|_{U_\alpha})$

for $U = U_\alpha \cap U_\beta$

$$(d_M(\theta|_{U_\alpha}))|_U = (d_M \theta|_{U_\beta})|_U = (d_M\theta)|_U$$

so d_M is det. by $d_M(\theta|_{U_\alpha})$ ▣

Thm: $F: M \to N$ C^∞ map

Then $F^* d = dF^*$

Pf: It suffices to show $F^*d = dF^*$ holds locally. Let (U, ϕ), (V, ψ) be coord. neigh. on M^m, N^n resp. with local coord. $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$. Induction on order.

$F^* df = dF^* f$ for $f \in C^\infty N$

$p \in M$, $X_p \in T_p M$

$$(F^* df)(X_p) = df(F_*(X_p)) = F_*(X_p)(f)(p)$$
$$= X_p(fF) = X_p(F^* f) = d(F^* f)(X_p)$$

then $F^* df = d(F^* f)$. By induction, $F^* d = dF^*$ for order less than K,

for $\phi = a\, dy^{i_1} \wedge \cdots \wedge dy^{i_k}$
$$= a\, dy^{i_1} \wedge (dy^{i_2} \wedge \cdots \wedge dy^{i_k})$$

Then $F^* d\phi = F^*(da \wedge dy^{i_1}) \wedge (dy^{i_2} \wedge \cdots \wedge dy^{i_k})$

$$= F^*(da) \wedge F^*(dy^{i_1}) \wedge \cdots \wedge F^*(dy^{i_k})$$
$$= d(F^* a) \wedge d(F^* x^{i_1}) \wedge \cdots \wedge d(F^* x^{i_k})$$
$$= d(F^* a\ d(F^* x^{i_1}) \wedge \cdots \wedge d(F^* x^{i_k}))$$
$$= d \circ F^* \phi$$

Integration on Manifolds

M orientable

$\dim M = m$

$\exists$ a C^∞ m-form Ω on M, called a volume form, such that $\Omega \neq 0$. Then for any other volume form W, $\exists f \in C^\infty M$ such that $W = f\Omega$, $f \neq 0$ such that

Def: A cont. function f is integrable on M if f has compact support. m-form ω is integrable if $\omega = f\Omega$ for an integrable function f on M.

Rem: The integration of m-form does not depend on the choice of volume form Ω. For volume form Ω' then $\Omega' = f \cdot \Omega$ for $f \neq 0$.

$\omega = \phi\Omega$ for supp ϕ compact $\Rightarrow$

$\omega = \frac{\phi}{f}\Omega'$ with supp $\frac{\phi}{f}$ = supp ϕ is compact.

11/14/2016

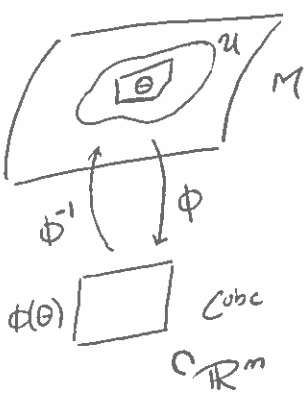
$\Theta \subset M$: Cube if $\exists$ coordinate open set $(\mathcal{U}, \phi)$ compatible with orientation $\Theta \subset \mathcal{U}$ & $\phi(\Theta) = C = \{[0,1]^m \subset \mathbb{R}^m\}$

If $\omega \in \Lambda^m(M)$ & $\operatorname{supp} \omega \subset \Theta$

Define ....

$$\int_M \omega = \int_{\mathbb{R}^m} f\, dx_1 \cdots dx_m$$

if $(\phi^{-1})^* \omega = f\, dx_1 \wedge \cdots \wedge dx_m$



Check: The definition does not depend on choice of coordinate.

Let $(\mathcal{U}', \phi') \supset \Theta$ with same orientation

$$(\phi'^{-1})^* \omega = f'\, dy_1 \wedge \cdots \wedge dy_m$$

Check $\displaystyle\int_{\mathbb{R}^m} f\, dx_1 \cdots dx_m \overset{?}{=} \int_{\mathbb{R}^m} f'\, dy_1 \cdots dy_m$

Change of variable formula.

$$f'\, dy_1 \wedge \cdots \wedge dy_m = f'(\phi'\phi^{-1}) \cdot ?$$

$$\left(\sum \frac{\partial y_1}{\partial x_{i_1}} dx_{i_1}\right) \wedge \cdots \wedge \left(\sum_i \frac{\partial y_m}{\partial x_{i_m}} dx_{i_m}\right)$$

$$= f'(\phi'\phi^{-1}) \sum_{i_1 \cdots i_m} \frac{\partial y_1}{\partial x_{i_1}} \cdots \frac{\partial y_m}{\partial x_{i_m}} dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

$\sigma(1, \dots, m) = (i_1, \dots, i_m)$ permutation

$$= f'(\phi'\phi^{-1}) \underbrace{\sum \operatorname{sgn} \sigma \frac{\partial y_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial y_m}{\partial x_{\sigma(m)}}}\, dx_1 \wedge \cdots \wedge dx_m$$

$$= \underbrace{\det \frac{\partial(y_1 \cdots y_m)}{\partial(x_1 \cdots x_m)}\, f'(\phi'\phi^{-1})}\, dx_1 \wedge \cdots \wedge dx_m$$

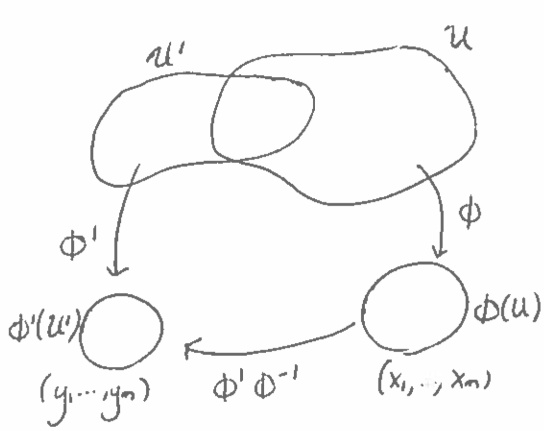
Then $\quad \downarrow = f'$, that is,

$f\, dx_1 \wedge \cdots \wedge dx_m = f'\, dy_1 \wedge \cdots \wedge dy_m$ in local coordinates. Then....

$f = f'(\phi'\phi^{-1}) \det \dfrac{\partial(y_1 \cdots y_m)}{\partial(x_1, \dots, x_m)} \leftarrow$ Jacobian. Then

$$\int_{\mathbb{R}^m} f\, dx_1 \cdots dx_m = \int_{\mathbb{R}^m} f'(\phi'\phi^{-1}) \det \frac{\partial(y_1 \cdots y_m)}{\partial(x_1 \cdots x_m)} dx_1 \cdots dx_m$$

$$\overset{\text{c.o.v.}}{=} \int_{\mathbb{R}^m} f'\, dy_1 \cdots dy_m$$



In general, $K = \operatorname{supp} \mathcal{U}$ compact

Choose a finite covering of K by $\mathring{Q}_1, \dots, \mathring{Q}_b$, where $Q_i \subset \underbrace{(U_i, \phi_i)}_{\text{coord. neigh.}}$

$\underbrace{\hphantom{Q_1 \dots Q_b}}_{\text{int cubes}}$

Then $M \setminus K, \mathring{Q}_1, \dots, \mathring{Q}_b$ covers M

We choose a partition of unity $\{f_i\}$ subordinate to it.

$$W = \sum_{i=1}^{s} f_i \omega$$

Define $\int_M \omega = \sum_{i=1}^{s} \int_M f_i \omega$

Check: Definition does not depend on partition of unity, coord. neigh, ...

Let $Q_1', \dots, Q_{b'}'$ be another set of cubes such that

$$K \subset \bigcup_{b'=1}^{b'} \mathring{Q}_i'$$

and partition of unity $\{g_k\}$ subordinate to $\{Q_1', \dots, Q_{b'}', M \setminus K\}$. Write

$$\omega = \sum_{k=1}^{r} g_k \omega$$

$$\sum f_i g_k = \left(\sum_i f_i\right)\left(\sum_k g_k\right) = 1 \cdot 1 = 1$$

For $1 \le k \le r$, $\operatorname{supp} f_i g_k \subset \operatorname{supp} g_k \subset Q_\beta'$

Then $\int_M g_k \omega = \int_M \left(\sum_{i=1}^{s} f_i\right) g_k \omega$

$$\xrightarrow{\text{finite sum}} = \sum_{i=1}^{s} \int_M f_i g_k \omega$$

$$= \sum_{k=1}^{r} \int_M (f_1 g_k + \dots + f_s g_k) \omega$$

$$= \sum_{i=1}^{s} \sum_{k=1}^{r} \int_M (f_i g_k) \omega$$

either or so.... can trace back to..

$$\sum_{i=1}^{s} \int_M f_i \omega = \int_M \omega$$

Thm: Let $-M$ denote opposite orientation

1) $\int_{-M} \omega = -\int_M \omega$

2) $\int a_1\omega_1 + a_2\omega_2 = a_1 \int_M \omega_1 + a_2 \int_M \omega_2$

3) $\Omega \in \Lambda^m M$; $\Omega \neq 0$ determines the orientation. If $\omega = g\Omega$, $g \ge 0$ $\to \int_M \omega \ge 0$. So $\int_M \omega = 0$ iff $g = 0$ when g cont.

4) $F: M_1^m \to M_2^m$ diffeo., $\omega \in \Lambda^m M_2$

Then $\int_{M_1} F^* \omega = \pm \int_{M_2} \omega$

$\left.\begin{array}{l} +\text{: orient. preserv.} \\ -\text{: orient. reversing} \end{array}\right\}$ for F

Manifolds with boundary

$$\mathbb{H}^m = \{x \in \mathbb{R}^m \mid x_i \ge 0\}$$

with relative topology of $\mathbb{R}^m$.

Then $\partial \mathbb{H}^m = \{x \mid x^m = 0\} \cong \mathbb{R}^{m-1}$

In $\mathbb{H}^m$, there are two types of open sets:

Let $\mathcal{U}, V$ be open sets in $\mathbb{H}^m$.
$F: \mathcal{U} \to V$ diffeo. Then....

1) If $\mathcal{U} \cap \partial\mathbb{H}^m \neq \phi$ then
$V \cap \partial\mathbb{H}^m = \phi$

2) If $\mathcal{U} \cap \partial\mathbb{H}^m \neq \phi \rightsquigarrow$
$V \cap \partial\mathbb{H}^m \neq \phi$

3) $p \in \mathcal{U} \cap \partial\mathbb{H}^m \rightsquigarrow F(p) \in V \cap \partial\mathbb{H}^m$.

Def: A C^∞ manifold with boundary is a Hausdorff space, 2^{nd} countable, & C^∞ structure $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha(U_\alpha)$ homeo. from open set U_α to open set $\phi_\alpha(U_\alpha)$ in $\mathbb{H}^m$ $\ni$

1) $\bigcup_\alpha \mathcal{U}_\alpha = M$

2) For $(U_\alpha, \phi_\alpha), (\mathcal{U}_\beta, \phi_\beta)$,
$\phi_\beta \phi_\alpha^{-1}$, $\phi_\alpha \phi_\beta^{-1}$ are diffeo.

3) $\mathcal{U}$ is max w.r.t. (1) & (2)

Ex: $\mathbb{H}^m$ manifold with boundary

11/16/2016

Thm: M^m manifold with boundary $\to$ the C^∞ structure on M determines the C^∞ structure on ∂M, that is a $(m-1)$-dim manifold. The inclusion $i: \partial M \to M$ is an embedding.

Rem: The C^∞-structure $\tilde{\mathcal{U}}$ on ∂M is determined by coord neigh. $(\tilde{\mathcal{U}}, \tilde{\Phi})$ where $\tilde{\mathcal{U}} = \mathcal{U} \cap \partial M$

$\tilde{\Phi} = \Phi|_{\partial M}$. For $p \in \partial M$,

$$\tilde{\Phi}_\alpha(p) = (x_1^\alpha, \dots, x_{m-1}^\alpha)$$

$$\tilde{\Phi}_\beta(p) = (x_1^\beta, \dots, x_{m-1}^\beta)$$

Then $\tilde{\Phi}_\beta \tilde{\Phi}_\alpha^{-1} = (\Phi_\beta \Phi_\alpha^{-1})|_{\partial M}$ on ∂M.

We say M is an oriented manifold with boundary if $\exists$ covering by coord neigh. $\{(U_\alpha, \Phi_\alpha)\}$ such that $\Phi_\alpha \Phi_\beta^{-1}$ has positive jacobian determinant if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$.

Thm: $\partial M \neq \phi$, M^m orientable. Then ∂M is orientable and orientation of M determines the orientation of ∂M.

Pf: For $p \in \partial M$,

$(\mathcal{U}, \Phi), (V, \Psi)$ coord. neigh e p. with coord $(x_1, \dots, x_m)$ & $(y_1, \dots, y_m)$.

$x^m \geq 0$ on $\mathcal{U}$; $y_m \geq 0$ on V } Make it so.

and $x_m p = 0$, $y_m p = 0$

$$D(\Phi\Psi^{-1}) = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_m} \\ \vdots & & \\ \frac{\partial x_m}{\partial y_1} & \cdots & \frac{\partial x_m}{\partial y_m} \end{pmatrix}(p)$$

$$= \begin{pmatrix} \frac{\partial x_1}{\partial y_1}(p) & \cdots & \frac{\partial x_1}{\partial y_{m-1}}(p) & * \\ \vdots & & & * \\ \frac{\partial x_{m-1}}{\partial y_1}(p) & \cdots & \frac{\partial x_{m-1}}{\partial y_1}(p) & * \\ 0 & \cdots & 0 & \frac{\partial x_m}{\partial y_m}(\theta) \end{pmatrix}$$

$\partial x^m \longrightarrow \frac{\partial x_m}{\partial y_m} > 0$

then $\det D(\Phi\Psi^{-1})(p) = \det \cdot \frac{\partial x^m}{\partial y_m}(p) > 0$

$= D(\tilde{\Phi}\Psi^{-1})(p)$

Then $\det D(\hat{\Phi}\hat{\Psi}^{-1}) > 0$ so ∂M is orientable. ☐

Stokes' Thm

M oriented manifold

Choose coord neigh covering $\{(\mathcal{U}_\alpha, \Phi_\alpha)\}$ of M with compatible orientation.

$$\hat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha|_{\partial M}$$

$$\hat{\Phi}_\alpha = \Phi_\alpha|_{\partial M}$$

$\{\hat{\mathcal{U}}_\alpha, \hat{\Phi}_\alpha\}$ gives orientation on ∂M

Let Q cube on M; $\phi(Q \cap \partial M) = \{x \in \mathbb{R}^m \mid \begin{smallmatrix} 0 \le x_i \le 1 & i < m \\ x_m = 0 & \end{smallmatrix}\}$ ($\#\ \phi$)

Then $\tilde{Q} \overset{def}{=} Q \cap \partial M$ cube of ∂M for $(\tilde{\mathcal{U}}, \tilde{\Phi})$. Also,

$$\mathring{Q} = \Phi^{-1}\left(\left\{x \in \mathbb{R}^m \,\middle|\, \begin{smallmatrix} 0 < x_i < 1 \\ 0 \le x_m < 1 \end{smallmatrix}\right\}\right)$$

$\omega \in \Lambda^{m-1} M$; $i : \partial M \to M$ embedding then $i^*\omega \in \Lambda^{m-1}\partial M$. Note $i^*(dx_m) = 0$ on ∂M.

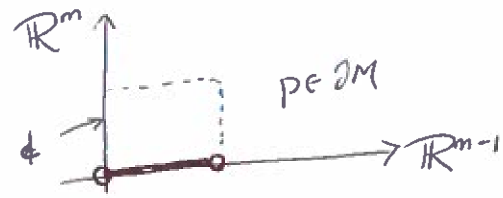
Let $\tilde{\partial} M$ be the orientation ∂M if $\dim M = m$ is even and $\tilde{\partial} M = -\partial M$ if m is odd.

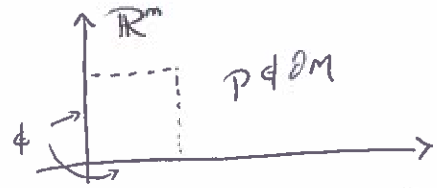
Thm: (Stokes' Thm)

Assume M^m orientable, compact $\omega \in \Lambda^{m-1} M$. Then

$$\int_M d\omega = \int_{\tilde{\partial} M} i^*\omega$$

Pf: By def of integral and partition of unity, suffices to show formula on $Q \subset M$. Suppose $\operatorname{supp} \omega \subset Q$, with coord. $(x_1, \dots, x_m)$





Write $\omega = \sum_j (-1)^{j-1} \lambda^j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m$ in Q. Then....

$$d\omega = \sum_j (-1)^{j-1} \underbrace{\left(\sum_{i=1}^m \frac{\partial \lambda^j}{\partial x_i} dx_i\right)}_{d\lambda^j \ \rightsquigarrow \text{ move to missing spot}} \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m$$

$$= \sum_j \frac{\partial \lambda^j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_m \leftarrow m\text{-form}$$

$$\int_Q d\omega \overset{def}{=} \int_Q \sum \frac{\partial \lambda^j}{\partial x_j} \underbrace{dx_1 \cdots dx_m}_{\text{volume form on } \mathbb{R}^m}$$

$$= \sum_j \underbrace{\int_0^1 \cdots \int_0^1}_{m} \frac{\partial \lambda^j}{\partial x_i} dx_1 \cdots dx_m$$

$$= \sum_j \underbrace{\int_0^1 \cdots \int_0^1}_{m-1} \left(\int_0^1 \frac{\partial \lambda^j}{\partial x_i} dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_m$$

$$\overset{F.T.C}{=} \sum_j \int_0^1 \cdots \int_0^1 \left(\lambda^j (x_1, \ldots, \underset{\substack{\uparrow \\ j^{th}}}{1}, \ldots, x_m) - \lambda^j (x_1, \ldots, \underset{\substack{\uparrow \\ j^{th}}}{0}, x_{j+1}, \ldots, x_m) \right) dx_1 \cdots \widehat{dx_j} \cdots dx_m$$

Case 1: $Q \cap \partial M = \phi$

Supp $\omega \subset Q \rightsquigarrow$ supp $\omega \cap \partial M = \phi$

$\rightsquigarrow \lambda^j|_{\partial M} = 0$ so $\int_Q d\omega = 0$

and $\int_{\partial Q} i^* \omega = 0$ as $\omega \equiv 0$ on ∂M

Case 2: $Q \cap \partial M \neq \phi$

Supp $\omega \subset Q \rightsquigarrow \lambda^j (x_1, \ldots, x_m) = 0$

if $x_j = 0$ or 1 for $1 \leq j \leq m-1$ &

$\lambda^j (x_1, \ldots, x_{m-1}, 1) \equiv 0 \rightarrow$

$$\int_Q d\omega = - \underbrace{\int_0^1 \cdots \int_0^1}_{m-1} \lambda^m (x_1, \ldots, x_{m-1}, 0) dx_1 \cdots dx_{m-1}$$

On the other hand, in local coord $(x_1, \ldots, x_{m-1}, 0)$ $\rightarrow$

$$i^* \omega = (-1)^{m-1} \lambda_m^{(x_1, \ldots, x_{m-1}, 0)} dx_1 \wedge \cdots \wedge dx_{m-1}$$

$$\int_{\partial Q} i^* \omega = (-1)^{m-1} \underbrace{\int_0^1 \cdots \int_0^1 \cdots \int_0^1}_{m-1} \lambda_m d(x_1, \ldots, x_{m-1}, 0) dx_1 \cdots dx_{m-1}$$

Now same up to a constant ± 1 so done.

Ex: (Green's Formula)

M bounded regular region in $\mathbb{R}^2$

$\omega = a\,dx + b\,dy$ 1-form

$$d\omega = \left(-\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} \right) dx \wedge dy$$

Then

$$\int_M \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx \wedge dy$$

$$\parallel$$

$$\int_{\partial M} a\,dx + b\,dy$$

Ex: (Divergence Thm)

M reg. domain $\mathbb{R}^3$

$\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$$

$$\rightsquigarrow \int_M \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$$

$$= \int_{\partial M} P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$$