To Infinity and Beyond:

The Tale of *really*. . . *BIG*. . . *NUMBERS!*

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Mathematics Graduate Colloquium
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Interesting Number Paradox

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Proof. Assume the set of uninteresting numbers is nonempty. By the Well Ordering Property of $\mathbb{N}$, there is a smallest uninteresting number, say N_0. But then N_0 is interesting, being the smallest uninteresting number, contradiction.

1

1

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- The only non-composite, non-prime number.

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- The loneliest number.

2

2

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- 2nd Catalan number.

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- 3^{rd} Fibonacci number.
- 2^{nd} Catalan number.
- The number of Prelims and Quals you have to pass...

3

3

- The smallest Mersenne prime, i.e. of the form $2^n - 1$.

4

4

- The smallest number of colors required to color any map: Four Color Theorem.

5

5

- The number of Platonic solids.

5

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- The number of axioms in Euclid's *Elements*.

6

6

- The smallest perfect number: $6 = 1 + 2 + 3$.

7

7

- The fewest moves required to solve a three-disk Tower of Hanoi puzzle.

7

- The fewest moves required to solve a three-disk Tower of Hanoi puzzle.
- The puzzle with 64 disks would take at least $2^{64} - 1$ seconds, or 565 billion years, to finish.

13

13

- The number of Archimedean solids.

17

17

- Smallest Leyland prime: $x^y + y^x$.

20

20

- The maximum number of moves required to solve a Rubik's cube.

20

- The maximum number of moves required to solve a Rubik's cube.
- Age at which Galois died.

70

70

- The smallest weird number.

73

73

Let's hear from an expert…

Image Credit: https://www.goodhousekeeping.com/life/entertainment/a44703/big-bang-theory-easter-egg-number-73/

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74

74

- The maximum percentage of the total volume of a cube that a sphere packing can occupy.

153

153

- The smallest narcissistic number: $1^3 + 5^3 + 3^3 = 153$.

163

- Prime number.

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- Largest Heegner number: Square-free positive integer d such that the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number 1, i.e. the ring of integers is a UFD. There are only 9 such numbers: 1, 2, 3, 7, 11, 19, 43, 67, 163.

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- Ramanujan's constant:

$$e^{\pi\sqrt{163}} \approx 262,537,412,640,768,743.999\,999\,999\,999\,25\ldots$$

$$102 \leq \mathsf{R}(6, 6) \leq 165$$

$R(n, m)$ is the smallest number such that every graph of order n contains either a clique of n vertices or an independent set of m vertices.

$R(n, m)$ is the smallest number of people needed so that either n know each other or m will not know each other.

$$R(3,3) = 6$$
$$R(3,9) = 36$$
$$R(4,3) = 9$$
$$R(4,4) = 18$$
$$R(4,5) = 25$$

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"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

220 & 284

220 & 284

- The smallest pair of amicable numbers.

563

563

- The largest known Wilson prime.

1,476

1,476

- The number of cases in the proof of the Four Color Theorem.

1729

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- First absolute Euler pseudoprime.
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- The 1729^{th} decimal digit of e is the first consecutive occurrence of all ten digits without repetition (not necessarily in order).
- One more than $1728 = 12^3 : j(\tau) = 1728\,\frac{g_2^3}{\Delta}$.

3,435

3,435

- The *only* Münchhausen number.

30,940

30,940

- . . . to 1—the odds of a royal flush in Texas Hold'em poker.

4,531,985,219,092

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- Number of possible Connect Four games.

4,531,985,219,092

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L L D W D L L

500 billion billion, $5 \cdot 10^{20}$

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- The number of possible checker ‘games.’

10^{120}

$$10^{120}$$

- Estimated number of Chess games.

43,252,003,274,489,856,000

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- Number of possible $3 \times 3 \times 3$ Rubik cube states.

Master Cube: $4 \times 4 \times 4$

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7,401,196,841,564,901,869,874,093,974,498,574,336,000,000,000

Professor Cube: $5 \times 5 \times 5$

Professor Cube: $5 \times 5 \times 5$

282,870,942,277,741,856,536,180,333,107,150,328,293,127,731,985,672,134,721,536,000,000,000,000,000

$$2^{77,232,917} - 1$$

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- Largest known prime number, as of September 28, 2018.

808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000

$$\mathbf{2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71}$$

- Order of the Monster Group.
- Conjectured by Fischer & Griess, existence proof by Griess in 1982. Construction simplified by Conway & Tits.
- The group was originally called the 'Friendly Giant.'
- Each matrix generator takes almost 5GB of storage space.

10^{100}

- Googl.
- Googlplex: $10^{10^{100}}$.

$$e^{e^{e^{79}}} \approx 10^{10^{10^{34}}}$$

$\pi(x) :=$ number of primes at most x.

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1
10	

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1
10	4

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1
10	4
100	

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1
10	4
100	25

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1
10	4
100	25
1,000	

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1
10	4
100	25
1,000	168

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1
10	4
100	25
1,000	168
1,000,000	

For example, with massive loss of generality, let x be...

x	$\pi(x)$
1	0
2	1
10	4
100	25
1,000	168
1,000,000	78,498

Gauß famously approximated $\pi(x)$ by $\frac{x}{\log x}$. Legendre later improved this with the logarithmic integral:

$$\mathrm{li}(x) := \int_0^x \frac{dt}{\ln t}$$

x	$\pi(x)$	$\text{li}(x)$	$\text{li}(x) - \pi(x)$	% Error
10^1	4	6.17	2.17	54.15
10^2	25	30.13	5.13	20.50
10^3	168	177.61	9.61	5.72

x	$\pi(x)$	$\mathrm{li}(x)$	$\mathrm{li}(x) - \pi(x)$	% Error
10^1	4	6.17	2.17	54.15
10^2	25	30.13	5.13	20.50
10^3	168	177.61	9.61	5.72
10^6	78,498	78,627.5	129.55	0.17
10^9	50,847,534	50,849,234.96	1700.96	0.0033

$$\pi(x) < \mathrm{li}(x)?$$

No

$$e^{e^{e^{79}}} \approx 10^{10^{10^{34}}}$$

1966, Lehman	$1.165 \cdot 10^{1165}$

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1987, te Riele	$e^{e^{27/4}}$

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1987, te Riele	$e^{e^{27/4}}$
2003, Bays, Hudson, et al.	$1.39822 \cdot 10^{316}$

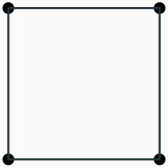
g64

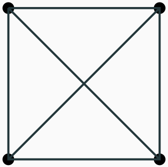
Connect each pair of vertices of an n-dimensional hypercube to obtain a complete graph on 2^n vertices, coloring each edge with one of two colors.

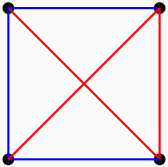
Connect each pair of vertices of an n-dimensional hypercube to obtain a complete graph on 2^n vertices, coloring each edge with one of two colors.

What is the smallest n such that every such coloring contains at least one single-colored complete subgroup on four coplanar vertices?

What is the smallest dimension possible so that a coloring of a hypercube with all vertices joined together must contain a complete graph, lying in some plane, with all edges the same color.







2-dimensions	$\rightarrow$	Avoidable
3-dimensions	$\rightarrow$	Avoidable
4-dimensions	$\rightarrow$	Avoidable
5-dimensions	$\rightarrow$	Avoidable
6-dimensions	$\rightarrow$	Avoidable
$\vdots$		$\vdots$

$6 \leq$ minimum dimension required $\leq g_{64}$

$$3 \uparrow 3$$

$$3 \uparrow 3 = 3 \cdot 3 \cdot 3 = 3^3 = 27$$

$$3 \uparrow\uparrow 3$$

$$3 \uparrow\uparrow 3 = 3 \uparrow (3 \uparrow 3)$$

$$3 \uparrow\uparrow 3 = 3 \uparrow (3 \uparrow 3) = 3 \uparrow 27 = 3^{27} = 3^{3^3}$$

$$3 \uparrow\uparrow 3 = 3 \uparrow (3 \uparrow 3) = 3 \uparrow 27 = 3^{27} = 3^{3^3} = 7,625,597,484,987$$

$$3 \uparrow\uparrow\uparrow 3 = ?$$

$$3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow (3 \uparrow\uparrow 3)$$

$$\underbrace{3^{3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}}_{7,625,597,484,987 \text{ times}}$$

$$a \underbrace{\uparrow\uparrow \cdots \uparrow}_{n} b = \underbrace{a \underbrace{\uparrow\uparrow \cdots \uparrow}_{n-1} (a \underbrace{\uparrow\uparrow \cdots \uparrow}_{n-1} (\cdots \underbrace{\uparrow \cdots \uparrow}_{n-1}))}_{b \text{copies } a}$$

$$g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3$$

$$g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3$$

$$g_2 = 3 \underbrace{\uparrow\uparrow \cdots \uparrow}_{g_1} 3$$

$$\vdots$$

$$g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3$$

$$g_2 = 3 \underbrace{\uparrow\uparrow \cdots \uparrow}_{g_1} 3$$

$$\vdots$$

Graham's Number: g_{64}

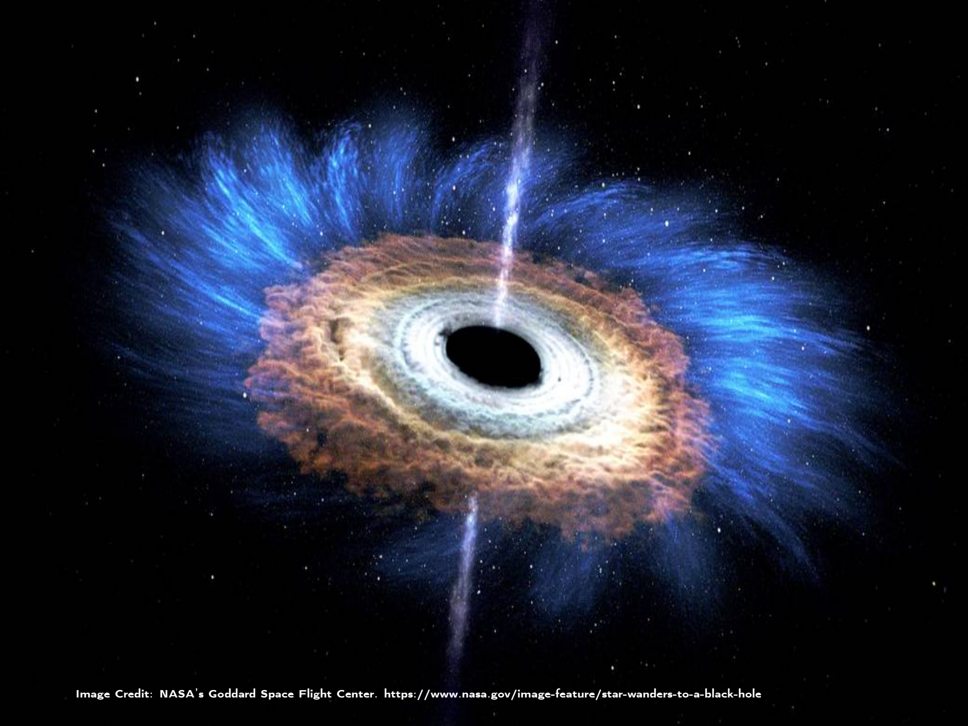


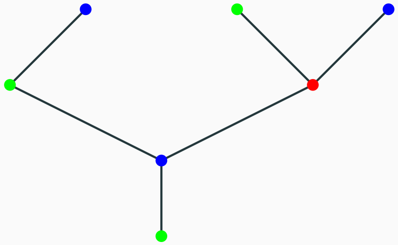
Image Credit: NASA's Goddard Space Flight Center. https://www.nasa.gov/image-feature/star-wanders-to-a-black-hole

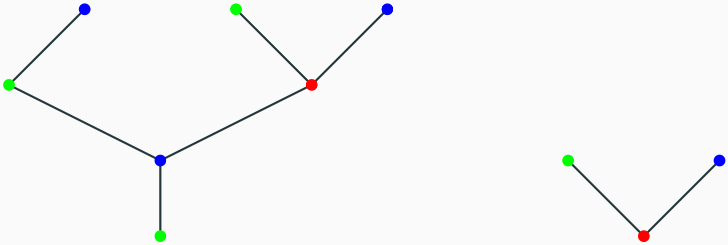
1971	Graham, Rothschild's original estimate
1977	Graham's Number (unpublished)
2003	Geoffrey Exoo improved lower bound to 11
2013	Jerome Barkley improved lower bound to 13
2014	Upper bound $2 \uparrow\uparrow\uparrow 6$

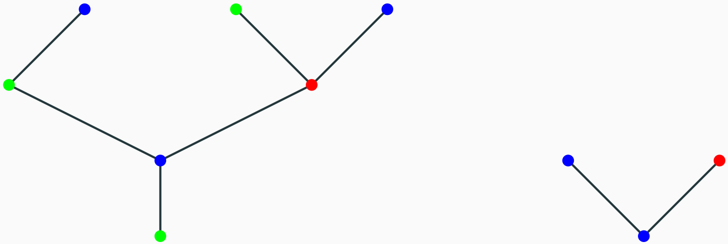
TREE(3)

Kruskal's Tree Theorem

TREE(n):= is the number of trees which can be built using n seeds, which do not contain an earlier tree preserving common ancestors.

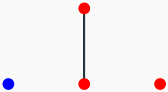






$\mathsf{TREE}(1) = \mathbf{1}$

$\mathsf{TREE}(2) = ?$



$\mathbf{TREE}(2) = 3$

$$\mathsf{TREE}(3) = ?$$

Lower Bound: $A^{A(187196)}(1) = A(\underbrace{A(\cdots A(1)\cdots)}_{A(187{,}196)})$

$A(x) = 2[x+1]x$ a hyperoperation

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$$A(x) = 2[x+1]x \text{ a hyperoperation}$$

$$a[0]b = \underbrace{1+1+\cdots+1}_{b}$$

Lower Bound: $A^{A(187196)}(1) = A(\underbrace{A(\cdots A(1)\cdots)}_{A(187,196)})$

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$$a[3]b = \underbrace{a\cdot a\cdot\cdots\cdot a}_{b}$$

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$$a[2]b = \underbrace{a + a + \cdots + a}_{b}$$

$$a[3]b = \underbrace{a \cdot a \cdot \cdots \cdot a}_{b}$$

$$a[n]b = \underbrace{a[n-1](a[n-1](\cdots a[n-1]a))}_{b}$$

$$\mathrm{TREE}(3) \gg A^{A(187196)}(1)$$

$$\text{TREE}(3) \gg A^{A(187196)}(1)$$

Graham's Number: $A^{64}(4)$

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Challenge: Count to TREE(TREE(TREE(3))).

Questions?