Hungry, Hungry Homology*

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Introduction

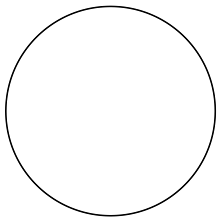
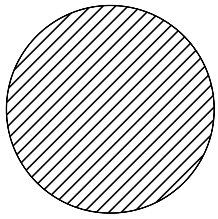
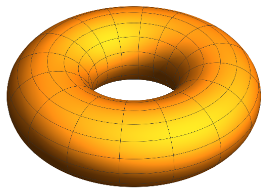
The way mathematics textbooks and courses are structured, one would think Mathematics is an extraordinary linear topic with perhaps a few paths leading to the same destination. The reality is the fibers of Mathematics are more like the root system of vast forest: twisting, with dead-ends, and leaps still to be made from tree to tree. The best path from one topic to another can be unclear and even worse, require other topics which then require the topic you wanted to learn in the first place! All this is doubled by the fact that to truly learn the topic, one must spend enough time 'living in the mindset' of the topic until its inner workings finally sink in - a time consuming process for sure! Grothendieck once described problem solving as soaking a nut until it becomes soft enough to break. During these periods of intense focus on particular areas, it can be tough to see the forest through the trees. Our purpose here will not be to learn any particular topic nor necessarily seriously try to define each term, describe each problem, or prove each theorem. Instead, we will try to see the intertwining of Mathematics through the lens of a particular concept, namely homology/cohomology. (Co)Homology is a topic which is trans-disciplinary and a powerful tool for many fields. We will see how it comes into play in three of the largest fields in Mathematics: Algebra, Topology, and Analysis. Since even this is too large a task, it will be our goal to briefly see how it is useful in solving a particular problem in each field:

Algebra: Given two (abelian) groups A, C, does there exist a group E so that $A \leq E$ and $E/A \cong C$? If so, 'how many' such groups are there? Can we describe them?

Topology: How can we tell spaces apart? For example, how do we distinguish the three spaces below:

$$C = \{e^{2\pi i n} \colon n \in [0,1)\}$$
$$D = \{(r\cos\theta, r\sin\theta) \colon r \in [0,1], \theta \in [0,2\pi]\}$$
$$T = \left\{\left((2+\sqrt{2}\cos\theta)\cos\phi, (2+\sqrt{2}\cos\theta)\sin\phi, \sqrt{2}\sin\theta\right) \colon 0 \leq \theta, \phi \leq 2\pi\right\}$$

*This talk was also titled: Homology: What does it know? Does it know things? Let's find out!

Analysis: Find a set $X \subseteq \mathbb{R}^3$ such that

$$\begin{aligned} V &= \{F : \mathbb{R}^3 \setminus X \to \mathbb{R}^3 : \nabla \times F = \mathbf{0}\} \\ W &= \{F : F = \nabla g\} \\ \dim(V/W) &= 8 \end{aligned}$$

1 Algebra

Abelian groups serve as a wonderful introduction to many of the techniques of Algebra and are full of rich structures and theorems to accompany them. These are easily generalized to R-modules (abelian groups with a ring compatibly acting on them). One important technique used in Algebra to study spaces is to 'compare' spaces by looking at maps between them. These provide interesting subspace which allow us to relate sub-structures to more 'global' structures. One easy example comes from the First Isomorphism Theorem.

Theorem 1.1 (First Isomorphism Theorem). *If $\phi : G \to H$ is a homomorphism of groups, then $\ker \phi \lhd G$ and $G/\ker \phi \cong \phi(G)$.*

Corollary 1.1. *If $\phi : G \to H$ is a surjective homomorphism of groups, then $G/\ker \phi \cong H$.*

Notice that by 'modding out' by the kernel, we obtain something equivalent to the image so kernels and images may have deeper relationships. This will 'lead us' to the notion of an exact sequence, which will give us great diagrammatic way of encoding the information of the First Isomorphism Theorem and much more.

Definition 1.1 (Exact). A pair of R-homomorphisms $M \xrightarrow{f} N \xrightarrow{g} P$ is called exact (at N) if $\operatorname{im} f = \ker g$. An infinite sequence of homomorphisms is exact if it is exact in each spot. A sequence of the form

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

is called a short exact sequence.

Example 1.1.

(i) The sequence $0 \longrightarrow M \xrightarrow{f} N$ is exact if and only if f is injective.

(ii) The sequence $N \xrightarrow{g} P \longrightarrow 0$ is exact if and only if g is surjective.

(iii) The sequence $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ is exact if and only if f is an isomorphism.

(iv) The sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is exact if and only if f is injective, g is surjective, and $\operatorname{im} f = \ker g$. Note that by exactness, we can view M as a submodule of N (since f is injective, we have $M \cong \operatorname{im} f$) and by the First Isomorphism Theorem: $P \cong N/\ker g = N/\operatorname{im} f \cong N/M$. Then a short exact sequence gives us a diagrammatic way of representing the First Isomorphism Theorem. In this case, we say N is an extension of P by M.

(v) The sequence $0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N \longrightarrow \operatorname{coker} f \longrightarrow 0$ is an exact sequence.

(vi) If $P \subseteq N \subseteq M$ is a tower of submodules, there is an exact sequence $0 \longrightarrow N/P \longrightarrow M/N \longrightarrow M/P \longrightarrow 0$.

(vii) The sequence $0 \longrightarrow M \xrightarrow{i} M \oplus N \xrightarrow{\pi} N \longrightarrow 0$ is an exact sequence. Therefore, an extension of N by M always exists. To see an explicit example, take the following:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

However, this is not the only possible extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

If exact sequences give a nice pictorial way of comparing structures of R-modules, then comparing exact sequences should give us even more information.

Definition 1.2 (Homomorphism of Exact Sequence)**.** If $\cdots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \cdots$ and $\cdots \longrightarrow P_{n+1} \xrightarrow{g_{n+1}} P_n \xrightarrow{g_n} P_{n-1} \xrightarrow{g_{n-1}} \cdots$ are exact sequences, then a homomorphism of exact sequences is a collection of maps $(\beta_n)_{n\in\mathbb{Z}}$, $\beta_n : M_n \to P_n$ such that for all $n \in \mathbb{Z}$ and $m \in M_n$, $g_n\beta_n(m) = \beta_{n-1}f_n(m)$; that is, the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\beta_{n+1}"] & M_n \arrow[r, "f_n"] \arrow[d, "\beta_n"] & M_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\beta_{n-1}"] & \cdots \\
\cdots \arrow[r] & P_{n+1} \arrow[r, "g_{n+1}"] & P_n \arrow[r, "g_n"] & P_{n-1} \arrow[r, "g_{n-1}"] & \cdots
\end{tikzcd}$$

Example 1.2.

(i) Suppose $m,n \in \mathbb{Z}^+$ and $n \mid m$. Let $k = m/n$. Then (abusing notation by letting π to denote the obvious projection maps in each case) the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "\pi"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\pi"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathbb{Z}/k\mathbb{Z} \arrow[r, "n"] & \mathbb{Z}/m\mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

(ii)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "i_1"] \arrow[d, "1"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "\pi_2"] \arrow[d, "s"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "i_1"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "\pi_1"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where i_j is the inclusion into the jth component, π_i is the projection onto the ith component, and s swaps components. Note defining maps the 'obvious' way would *not* give a commutative diagram.

This quickly leads to a new general type of proof: diagram chasing.

Proposition 1.1 (The Short 5–Lemma). *If the rows in the following commutative diagram are exact and α, γ are isomorphisms, then β is an isomorphism.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & M \arrow[r, "r"] & N \arrow[r, "s"] & P \arrow[r] & 0
\end{tikzcd}$$

Proof. Suppose $b \in \ker \beta$. We want to show that $b = 0$. Now $s\beta(b) = 0$ but as the diagram is commutative, $0 = s\beta(b) = \gamma g(b)$. Since γ is an isomorphism, we must have $g(b) = 0$ so that $b \in \ker g = \operatorname{im} f$. Therefore, there is an $a \in A$ so that $f(a) = b$. Now as the diagram is commutative, $\beta f(a) = r\alpha(a)$. However, $\beta f(a) = \beta(b) = 0$ and so $r(\alpha(a)) = 0$. But r, α are injective so that $a = 0$. But then $0 = f(a) = b$ as desired.

Now suppose $n \in N$. We want to find a $b \in B$ such that $\beta(b) = n$. Now $s(n) \in P$ and as γ is an isomorphism, there is a unique $c \in C$ such that $\gamma(c) = s(n)$. However, g is surjective so that there is a $b \in B$ (not necessarily unique) so that $g(b) = c$. By the commutativity of the diagram, we have $\gamma g(b) = s(\beta(b)) = s(n)$. Then $s(n - \beta(b)) = 0$ so that $n - \beta(b) \in \ker s = \operatorname{im} r$. Therefore, there is a $m \in M$ such that $r(m) = n - \beta(b)$. But as α is an isomorphism, there is a $a \in A$ such that $\alpha(a) = m$. As the diagram is commutative, $\beta f(a) = r\alpha(a) = n - \beta(b)$. Now $f(a) \in B$ so we consider the element $b + f(a)$. Observe $\beta(b + f(a)) = \beta(b) + \beta f(a) = \beta(b) + (n - \beta(b)) = n$. But then β is surjective. □

In fact, a more general result holds which we encourage the reader try to prove by diagram chasing (which in some cases will be made by the homology theory to come).

Proposition 1.2 (The 5–Lemma). *Consider the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\ N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & N_4 & \xrightarrow{g_4} & N_5 \end{array}$$

(a) if h_2, h_4 are surjective and h_5 is injective, then h_3 is surjective.

(b) if h_2, h_4 are injective and h_1 is surjective, then h_3 is injective.

(c) if h_1, h_2, h_4, and h_5 are isomorphisms, then h_3 is an isomorphism.

These proofs for any diagram with the same 'shape' actually hold more generally (and allow by-passing the bothersome language of mono and epi for more general categories) with the same proofs!

Theorem 1.2 (Mitchell Embedding Theorem). *If $\mathscr{A}$ is an abelian category, there is an associative unital ring R and a fully faithful exact functor $F : \mathscr{A} \to Mod(R)$.*

That is, every abelian category can be thought of as a full exact subcategory of some module category. While it is not generally helpful to think of abelian categories in this manner, it does allow us to transfer these 'diagram chasing' proofs to more general settings. Even the method of diagram chasing can be more easily using the Salamander Lemma. But this is beyond the scope of our discussion as this is more categorical than what this discussion should be. Of course, much of our preceding discussion and the topics to come could be rephrased in categoric language, but for simplicity we shall avoid this. Returning now to the theory at hand, we wish to loosen the requirements for an exact sequence to get a more general (and more powerful) theorem.

Definition 1.3 (Chain Complex). A sequence of R-modules, $(C_n, d_n)_{n\in\mathbb{Z}}$, with maps (called boundary maps) $d_n : C_n \to C_{n-1}$ is said to be a chain complex if $\operatorname{im} d_{n+1} \subseteq \ker d_n$ for all $n \in \mathbb{Z}$.

$$\cdots \xrightarrow{d_{n+3}} C_{n+2} \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots$$

Remark 1.

(a) Note this generalizes the notion of exactness as in an exact sequence we have the stronger $\operatorname{im} d_{n+1} = \ker d_n$.

(b) It is often tiresome to indicate which chain map one is looking (since it is usually obvious from context or is 'irrelevant' to the task at hand) at so one often drops the subscript and simply writes d.

(c) There are also cochain complexes, defined exactly as in the chain complex but the indices of everything increase rather than decrease. We also write these with superscripts rather than subscripts, i.e. $(C^n, d^n)_{n\in\mathbb{Z}}$.

Definition 1.4 (Homology). Given a chain complex $(C_n, d_n)_{n\in\mathbb{Z}}$, the nth homology group of the complex is $H_n := \ker\delta_n / \operatorname{im}\delta_{n+1}$.

Again, cohomology is defined mutatis mutandis using cochain complexes. We can generalize the comment above, we have $H_n = 0$ if and only if $\operatorname{im} d_{n+1} = \ker d_n$. Now again as with exact sequences, we would like to compare chain complexes. This will become particularly useful with comparing homologies and allow us to easily create homologies from chain complexes as well.

Definition 1.5 (Chain Map). Given two chain complexes $(C_n, \delta_n)_{n\in\mathbb{Z}}$ and $(C'_n, \delta'_n)_{n\in\mathbb{Z}}$, a map of chain complexes is a family of R-homomorphisms $(f_n : C_n \to C'_n)_{n\in\mathbb{Z}}$ so that $\delta'_n f_n = f_{n-1}\delta_n$ for all $n \in \mathbb{Z}$; that is, that the diagram below commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\delta'_{n+1}"] & C'_n \arrow[r, "\delta'_n"] & C'_{n-1} \arrow[r, "\delta'_{n-1}"] & \cdots
\end{tikzcd}$$

This naturally allows to state one of the more useful lemmas in Homology Theory:

Lemma 1.1 (Snake Lemma). *Given the following commutative diagram of R-modules with exact rows*

$$\begin{tikzcd}
 & M \arrow[r, "f"] \arrow[d, "\alpha"] & N \arrow[r, "g"] \arrow[d, "\beta"] & P \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & P' &
\end{tikzcd}$$

there is an exact sequence

$$\ker\alpha \xrightarrow{f} \ker\beta \xrightarrow{g} \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \xrightarrow{f'} \operatorname{coker}\beta \xrightarrow{g'} \operatorname{coker}\gamma$$

The lemma is so named as it can be visualized with the following 'snake like' diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & \ker(a) \arrow[r] \arrow[d] & \ker(b) \arrow[r] \arrow[d] & \ker(c) \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
0 \arrow[r, dashed] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[r, dashed] \arrow[d] & 0 \\
 & \operatorname{coker}(a) \arrow[r] \arrow[d] & \operatorname{coker}(b) \arrow[r] \arrow[d] & \operatorname{coker}(c) \arrow[r, dashed] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This is useful because given maps of chain complexes, we actually get a map between their homologies. Before stating and proving the result, we should make a note on notation. Rather than writing

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^A_{n+1}"] & B_{n+1} \arrow[d, "d^B_{n+1}"] \\
A_n \arrow[r, "f_n"] \arrow[d, "d^A_n"] & B_n \arrow[d, "d^B_n"] \\
A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d^A_{n-1}"] & B_{n-1} \arrow[d, "d^B_{n-1}"] \\
\vdots & \vdots
\end{tikzcd}$$

for a chain map $(f_n)_{n\in\mathbb{Z}} : (A_n, d_n^A) \to (B_n, d_n^B)$, we can simply abbreviate this $A_* \xrightarrow{f} B_*$ for ease of notation.

Theorem 1.3 (Long Exact Sequence). *Let $0_* \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ be a short exact sequence of chain complexes (meaning each row is exact). Then for all n, there are natural maps*

$\delta : H_n(C) \to H_{n-1}(A)$, called connecting homomorphisms, such that the sequence

$$\cdots \longrightarrow H_{n+1}(A) \longrightarrow H_{n+1}(B) \longrightarrow H_{n+1}(C) \xrightarrow{\delta_{n+1}} H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow \cdots$$

is exact. Similarly, if instead we have a short exact sequence of cochain complexes, then for all n there are natural maps $\delta : H^n(C) \longrightarrow H^{n+1}(A)$ such that the sequence

$$\cdots \longrightarrow H_{n+1}(A) \longrightarrow H_{n+1}(B) \longrightarrow H_{n+1}(C) \xrightarrow{\delta_{n+1}} H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow \cdots$$

is exact.

Proof. This is just a simple application of the Snake Lemma. Given a short exact sequence of complexes $0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$, apply the Snake Lemma to

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d_n^A"] & B_n \arrow[r] \arrow[d, "d_n^B"] & C_n \arrow[r] \arrow[d, "d_n^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

in order to get for each n an exact sequence

$$0 \longrightarrow Z_n^A \longrightarrow Z_n^B \longrightarrow Z_n^C \xrightarrow{\delta} A_{n-1}/\operatorname{im} d_n^A \longrightarrow B_{n-1}/\operatorname{im} d_n^B \longrightarrow C_{n-1}/\operatorname{im} d_n^C \longrightarrow 0$$

We rearrange to a commutative diagram with exact rows

$$\begin{tikzcd}
 & A_n/\operatorname{im} d_n B_n/\operatorname{im} d \arrow[r] \arrow[d] & C_n/\operatorname{im} d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1}^A \arrow[r] & Z_{n-1}^B \arrow[r] & Z_{n-1}^C
\end{tikzcd}$$

Applying the Snake Lemma we get an exact sequence

$$H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \xrightarrow{\delta} H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow H_{n-1}(C)$$

We leave naturality to the reader. □

Note that we often write long exact sequences of homology groups as follows: given an exact sequence of complexes $0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$, we write the exact sequence of homology groups as

$$\begin{tikzcd}
H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\
 & H_*(C) \arrow[ul, "\delta"] &
\end{tikzcd}$$

This result can take care of many results trivially. For example:

Lemma 1.2 (3 × 3 Lemma)**.** *Consider the following commutative diagram:*

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d, "\alpha'"] & B \arrow[r, "g'"] \arrow[d, "\beta'"] & C \arrow[r] \arrow[d, "\gamma'"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'' \arrow[r, "f''"] \arrow[d] & B'' \arrow[r, "g''"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

If the rows and middle column are exact, then if either the first or last column is exact, so is the other.

Proof. This follows immediately from the Snake Lemma. We have a long exact sequence

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

However, the exactness of the middle column gives $H_n(B) = 0$ for all n. But then we have

$$\cdots \longrightarrow 0 \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow 0 \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow 0 \longrightarrow \cdots$$

so that $H_{n+1}(C) \cong H_n(A)$ for all n. Then if either the first or last column is exact, then its homology groups vanish. Given the homology groups are isomorphic, if either vanishes, so does the other, implying exactness. $\square$

We now have enough language to briefly state the solution to the Algebra problem poised at the beginning. An ‘easy’ way of describing the isomorphism classes of group extensions is to recognize them as a certain cohomology group using Topoi Theory:

$$H^2_{\text{Grp}}(C,A) = H^2(\mathbf{B}C,A) = H(\mathbf{B}C,\mathbf{B}^2A)$$

where $\mathbf{B}G \in \text{Grp}$ is the delooping groupoid. Better yet, we can recognize these as a certain Ext group. A general powerful tool coming from Homology are the Ext and Tor groups. We would go too far astray to try to describe them both here, but we will define Ext enough to give a solution to our problem. Take a short exact sequence of groups

$$0 \longrightarrow A \longrightarrow E \longrightarrow C \longrightarrow 0$$

Take a free-resolution of C. That is, find an exact sequence of $\mathbb{Z}$-modules

$$\cdots \longrightarrow F^2 \longrightarrow F^1 \longrightarrow F^0 \longrightarrow C \longrightarrow 0$$

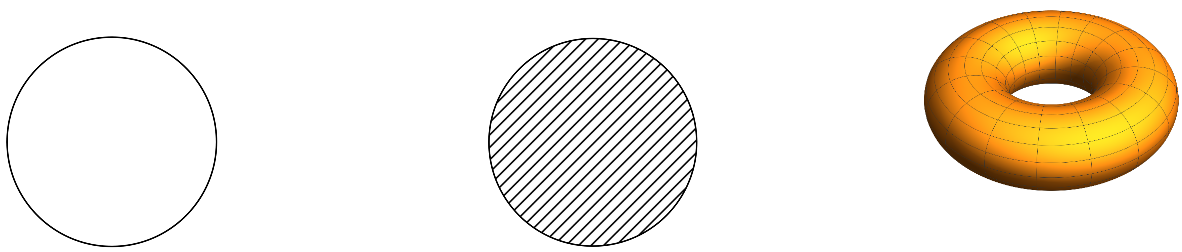
where each F_i is a free $\mathbb{Z}$-module, i.e. a $\mathbb{Z}$-module with a $\mathbb{Z}$-basis. Note this is always possible. It is a good exercise to see why this is. [In fact, we only need take each F to be *projective*. But we do not want to take time to define that now. All free modules are projective and are more likely to be familiar to the reader.] Now take the Hom groups of this sequence (recall $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ is the group of $\mathbb{Z}$-maps from M to N). This gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(F^0,A) \longrightarrow \mathrm{Hom}_R(F^1,A) \longrightarrow \mathrm{Hom}_R(F^2,A) \longrightarrow \cdots$$

It is a good exercise for the reader to try to understand the maps here, why the order is switched, and why the sequence is exact. Now take the homology of this sequence. We call the homology groups Ext groups: $\mathrm{Ext}^n_{\mathbb{Z}}(C,A) := H^n$. There is a bijective correspondence between the equivalence class of extensions of C by A and the elements of $\mathrm{Ext}^1_{\mathbb{Z}}(C,A)$.

2 Topology

Let's recall our earlier question: how do we tell spaces apart? Take for example the three spaces below: the circle, the disk, and the torus. How do we tell them apart? One intuitive way is to say that the first and last spaces have 'holes' while the middle one does not.



This is may be easy to see when we easily plot the space, but what if we were handed simply an arbitrary set, how could we distinguish them? Take for example the two spaces given by the sets below:

$$C = \{e^{2\pi i n} \colon n \in [0,1)\}$$
$$T = \left\{\left((2+\sqrt{2}\cos\theta)\cos\phi, (2+\sqrt{2}\cos\theta)\sin\phi, \sqrt{2}\sin\theta\right) \colon 0 \leq \theta, \phi \leq 2\pi\right\}$$

which represent the circle and torus, respectively. It is not immediate from these definitions that these spaces have 'holes'. So we need a more axiomatic way to distinguish spaces. One way of doing this is the fundamental group.

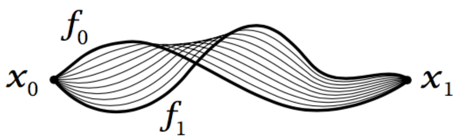
Definition 2.1 (Path Homotopy)**.** A homotopy of paths f_0, f_1 (from x_0 to x_1) in a space X is a family $f_t : I \to X$, $0 \leq t \leq 1$ such that

(i) $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.

(ii) the associated map $F : I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous.

If f_0 and f_1 are connected by a homotopy f_t, we say that f_0 and f_1 are homotopic and denote this $f_0 \simeq f_1$.

That is, f_0 and f_1 are path homotopic if we can continuously deform the path f_0 into the path f_1, as in the figure below.

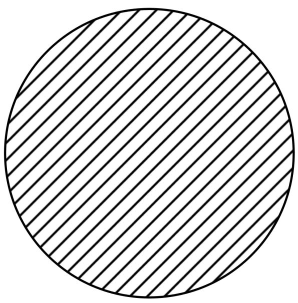


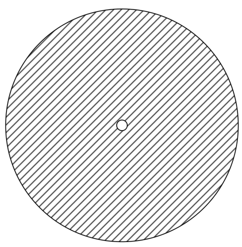
Now if instead of looking path homotopies, we can look at homotopies of *loops*, i.e. paths with the same start and end points. Now if $f : I \to X$ is a loop at x_0, then $f(0) = f(1) = x_0$. Then we can view the path $f : I \to X$ as a path $f : S^1 \to X$ by identifying $0, 1 \in I$ (as $f(0) = f(1) = x_0$). We can now define the fundamental group:

Definition 2.2 (Fundamental Group)**.** The fundamental group of a space X at a point $x_0 \in X$ is the set of equivalence classes of path homotopic loops at x_0, denoted $\pi_1(X, x_0)$.

Remark 2. If the space X is path connected, then the fundamental group $\pi_1(X, x_0)$ is unique up to isomorphism; that is, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$, the being isomorphism being conjugation by an equivalence class of a path connecting x_0 to x_1. In this case, we can simply write $\pi_1(X)$.

Why does this help us with the problem at hand? For instance, any loop in the disk can be deformed to a point, so the fundamental group is trivial – consisting only of the equivalence class of constant paths in the disk. However any loop around the origin in the disk with the origin removed cannot be deformed to a point. Therefore, the fundamental group of this space is nontrivial (in fact, the fundamental group of this space is isomorphic to $\mathbb{Z}$). Thus, the fundamental group has allowed us to distinguish these two spaces.



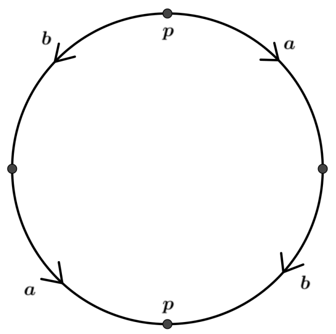


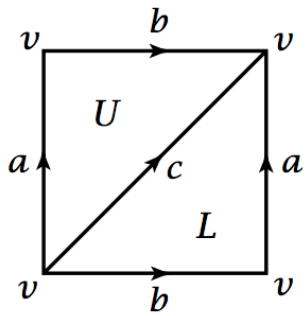
In fact, considering the fundamental group and the theory surrounding it yields a lot of interesting results and give us a method of approaching problems not just in Topology:

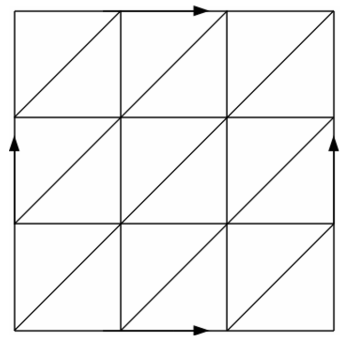
Theorem 2.1. *Every subgroup of a free group is free.*

Prob 2.1. Find all index 2 subgroups of the free group $F(2) = \langle a, b \rangle$.

Theorem 2.2. *Let $F(n)$ denote the free group on n generators. Then for $n = 2, 3, \ldots$, $F(n) \lhd F(2)$.*

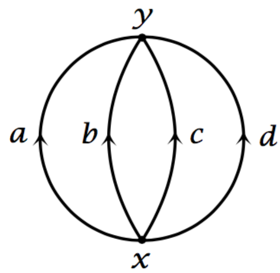






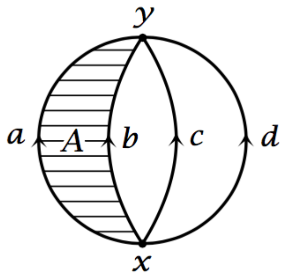
While the fundamental group can distinguish many spaces, it fails even to distinguish the spheres S^n for $n > 1$. Generally, the disadvantage to the fundamental group is that while it is useful in studying spaces with low 'dimension' (which is to be expected since it depends on loops $I \to X$), it loses much of its power for higher dimensional spaces as it cannot often tell them apart or becomes difficult to compute. There are higher analogs of the fundamental group – the higher homotopy groups, $p_n(X)$. The higher homotopy groups look at homotopic maps $S^n \to X$. However, again, these are generally nearly intractible to compute – even for the spheres S^n. So we would like a different tool and this is where homology enters. While the definitions are less accessible and more technical, the homology groups are very computable. However, these computations depend on being able to describe the space. There are many ways to describe spaces: CW complex, Δ-complexes, triangulations, et cetera (see the figures above for the diagrams for these methods for the torus, respectively).

Since it is not our goal to learn the theory generally, we will not describe any of these in details. We will be using CW-complex which are created by gluing disks D^n, called n-cells, together for various n. To motivate why the homology groups are useful, we follow the introduction in Hatcher [Hat01]. Consider the space in the diagram below.
We still want to be able 'see' holes in the space. So lets consider loops at x. For example, ab^{-1} or $ab^{-1}dc^{-1}$. Should the loops ab^{-1} and $b^{-1}a$ be considered different? They do have different starting positions – namely x, y, respectively. On the other hand, they do represent the same circle in the space. If we abelianize, these would be the same loop: (switching to additive notation) $a - b$ and $(-b) + a = a - b$. Notice by doing so, we no longer need consider basepoints as all loops become simple cycles. So now cycles become linear combinations of edges with integer coefficients. We shall call these chains of edges. Some of these chains can be decomposed into loops several ways. For example, $a - b + c - d$ can be decomposed as

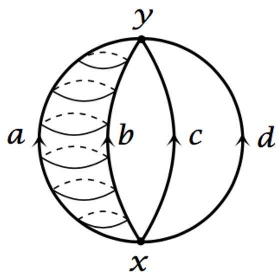


$(a-c)+(b-d)$ or $(a-d)+(b-c)$. However, we do not want to distinguish between these decompositions. So when we use the word 'cycle', we should mean any linear combination of edges for which there is at least one decomposition into loops (cycles in the previous sense).

What is the condition for a chain to be a cycle? Geometrically, a cycle has the property that it enters each vertex the same number of times it leaves the vertex. Take the chain $ka + lb + mc + nd$. This chain enters y a total number of $k + l + m + n$ times as each of a, b, c, and d enters y once. Similarly, each of the edges leaves x ones to the number of times the chain enters x is $-k - l - m - n = 0$. Hence, the criterion for a chain to be a cycle is $k+l+m+n = 0$. So what we shall do is define free abelian groups on the disks D^n that build the space. For $n = 0$, these are the vertices. For $n = 1$, these are are the edges. For $n = 2$, these are the filled regions between the edges. We denote these free groups C_n. Then in our case, C_0 is the free abelian group with basis the vertices x, y and C_1 is the free abelian group with basis the edges a, b, c, and d. Then we define a homomorphism $\delta : C_1 \to C_0$ via 'tip minus tail'. Then the cycles are the kernel of δ. It is routine to verify that $a - b, b - c$, and $c - d$ form a basis for the kernel. Notice we can think of these as directed edges. Then every cycle in this space is a unique linear combination of these cycles. Notice this corresponds to the three 'holes' in the space. Huzzah!



Let's take a look at a similar, albeit different space, shown the the figure above. In this space, we have attached a disk to the cycle $a - b$. Then the loop formed by the cycle $a - b$ is now homotopic to the constant loop as we can contract it to a point by sliding the loop through A. That is, we have filled a hole in the previous space. This suggests we have made the cycle $a - b$ trivial by forming a quotient using the subgroup generated by $a - b$. In this quotient, the cycle $a-c$ and $b-c$ are equivalent, which makes sense as in this new space they are homotopic. To make this algebraically sound, we define C_2 to be the infinite cyclic group generated by A and examine the chain of maps (no pun intended as this is a chain complex) $C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$, where $\delta_2(A) = a - b$. Then the quotient discussed is $\ker \delta_1 / \operatorname{im} \delta_2$ – the 1-dimensional cycles module those which are boundaries. This is exact the homology group $H_1(X)$. The previous example was the first homology group $H_1(X) = \ker \delta_1 / \operatorname{im} \delta_2 = \ker \delta_1$.



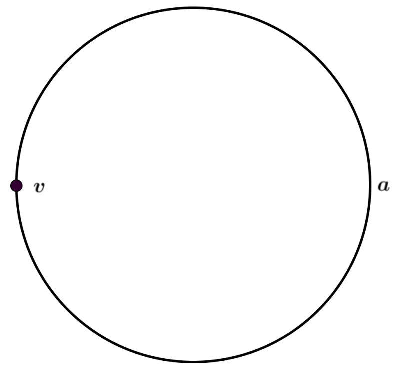
Let's consider a final example given by the space shown in the figure above, where we have attached to the previous example another disk D^1 along $a - b$. So together, A, B form a 'hollow tube' along the left side of the space. Redefine the map $\delta_2 : C_2 \to C_1$ by having it send both A, B to $a - b$. Then the first homology group $H_1(X)$ is the same as before. However, δ_2 now has a nontrivial kernel – namely the subgroup of C_2 generated by $A - B$. Then $H_2(X) = \ker \delta_2 \cong \mathbb{Z}$. Then this 'tubular' cycle detects the presence of a 'hole' in the space. Notice, even by the algebraic measurement afforded by the homology groups, the hole here is different than the holes of the first example. If we attached a 3-cell, D^3, along the 2-sphere formed by A and B, we create a chain group C_3 generated by this 3-cell C. Define a homomorphism $\delta_3 : C_3 \to C_2$ via $\delta_3(C) = A - B$. Now $H_2(X) = 0$ is now trivial while $H_3(X) = \ker \delta_3 = 0$ so that we have filled the hole from before. But we still have $H_1(X) \cong \mathbb{Z} \times \mathbb{Z}$ generated by $b - c$ and $c - d$. Generally, we form a space X by gluing together cells of various dimensions. We form the free abelian groups C_n generated by the n-cells and attach them together via maps $\delta_n : C_n \to C_{n-1}$. This gives us the chain complex

$$\cdots \longrightarrow C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} 0$$

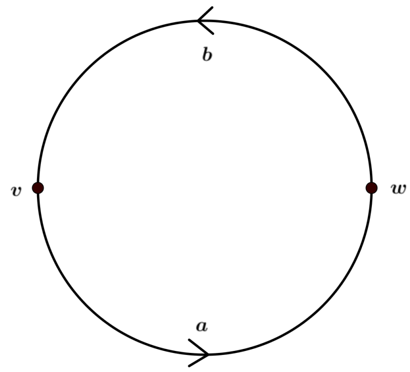
Let's now distinguish some of the space from before. Consider the circle shown above. We form the circle by taking a 0-cell, the vertex v, and attaching a 1-cell, the disk $D^1 = [0, 1]$ labeled a, by gluing its endpoints to v. Then C_0 is the free abelian group generated by v and C_1 is the free abelian group generated by a. Define δ_0 to be the zero map and δ_1 to be the map sending a to $v - v = 0$. Since there are no other cells, $C_n = 0$ and $\delta_n = 0$ for all other $n \in \mathbb{Z}$. This gives us the chain complex

$$0 \xrightarrow{\delta_2} C_1 = \mathbb{Z} = \langle a \rangle \xrightarrow{\delta_1} C_0 = \mathbb{Z} = \langle v \rangle \xrightarrow{\delta_0} 0$$

Then the homology groups are $H_1 = \ker \delta_1 / \operatorname{im} \delta_2 = \langle a \rangle / 0 = \langle a \rangle \cong \mathbb{Z}$ and $H_0 = \ker \delta_0 / \operatorname{im} \delta_1 = \langle v \rangle / 0 = \langle v \rangle \cong \mathbb{Z}$. Clearly, $H_n = 0$ for all other $n \in \mathbb{Z}$. It is not clear that our construction of



the homology groups is independent of how we build the space but this is the case. Consider building the circle as in the diagram below.



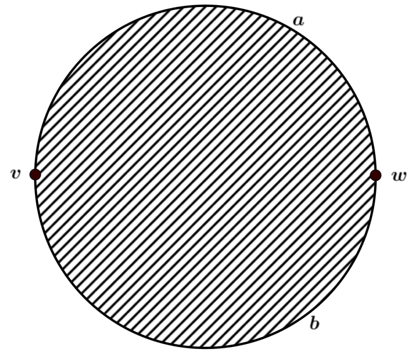
We define C_0 to be the free abelian group generated by the 0-cells v, w and C_1 to be the free abelian group generated by the 1-cells a, b. We define maps $\delta_0 = 0$ and δ_1 to be the map given by $a \mapsto w - v$ and $b \mapsto v - w$ – extending by linearity. Again, we take $C_n = 0$ and $\delta_n = 0$ for all other $n \in \mathbb{Z}$. Observe that $\delta_1(a+b) = \delta_1(a) + \delta_1(b) = (w-v) + (v-w) = 0$. Since C_2 is two-dimensional and not the zero map, this must generate the kernel. This gives the chain complex

$$0 \xrightarrow{\delta_2} C_1 = \mathbb{Z}^2 = \langle a, b\rangle \xrightarrow{\delta_1} C_0 = \mathbb{Z}^2 = \langle v, w\rangle \xrightarrow{\delta_0} 0$$

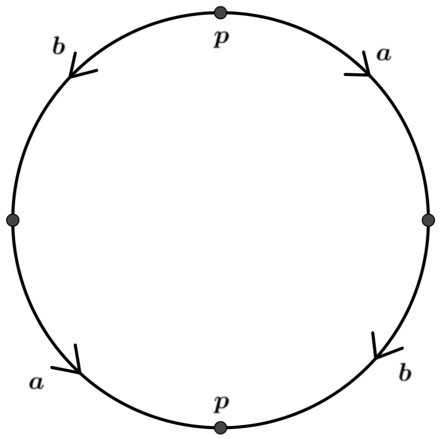
Then we have homology groups $H_1 = \ker \delta_1 / \operatorname{im} \delta_2 = \langle a+b\rangle/0 = \langle a+b\rangle \cong \mathbb{Z}$ and $H_0 = \ker \delta_0 / \operatorname{im} \delta_1 = \langle v, w\rangle/\langle w-v, v-w\rangle = \langle v, w\rangle/\langle v-w\rangle \cong \langle v\rangle \cong \mathbb{Z}$. Clearly, $H_n = 0$ for all other $n \in \mathbb{Z}$. This is exactly as before. Now consider the solid circle:

Now $C_0, C_1, \delta_0, \delta_1$ are the same as before. However, this time we attach a 2-cell. Define C_2 to be the free abelian group generated by U with δ_2 to be the map given by $\delta_2(U) = a+b$, extending by linearity. Then we have chain complex

$$0 \xrightarrow{\delta_3} C_2 = \mathbb{Z} = \langle U\rangle \xrightarrow{\delta_2} C_1 = \mathbb{Z}^2 = \langle a, b\rangle \xrightarrow{\delta_1} C_0 = \mathbb{Z}^2 = \langle v, w\rangle \xrightarrow{\delta_0} 0$$



so that we have homology groups $H_2 = \ker \delta_2 / \operatorname{im} \delta_3 = 0/0 = 0$, $H_1 = \ker \delta_1 / \operatorname{im} \delta_2 = \langle a+b \rangle / \langle a+b \rangle = 0$, and $H_0 = \ker \delta_0 / \operatorname{im} \delta_1 = \langle v, w \rangle / \langle v-w, w-v \rangle = \langle v, w \rangle / \langle v-w \rangle \cong \langle v \rangle \cong \mathbb{Z}$. Notice then we have filled the hole from before by attaching the disk. Since homeomorphic spaces should have isomorphic homology groups, the circle and disk cannot be isomorphic. Therefore, we have algebraically distinguished the disk and the circle! In fact, we have generally $H_i(S^n) = 0$ when $i \neq 0, n$ and $H_n(S^n) = H_0(S^n) = \mathbb{Z}$. We leave it as a challenge to the reader to figure out how to attach the 2-cell to the CW complex for the torus below and verify that its homology groups are $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z} \oplus \mathbb{Z}$, $H_2 = \mathbb{Z}$, and $H_n = 0$ for all other $n \in \mathbb{Z}$.



There are many interesting homology theories in Algebraic Topology: Simplicial Homology, Singular Homology, Cellular Homology, Cubical Homology, Floer Homology, Khovanov Homology, Morse Homology, K-homology, Steenrod Homology,Persistent Homology, etc.. We leave it to the interested reader to seek them out.

3 Analysis

Before returning to the problem from Analysis poised in the beginning, we shall look at another similar problem: if $f = (f_1, f_2) : U \to \mathbb{R}^2$ is a smooth function on an open set $U \subseteq \mathbb{R}^2$, is there a smooth function $F : U \to \mathbb{R}$ so that $\frac{\partial F}{\partial x} = f_1$ and $\frac{\partial F}{\partial y} = f_2$, i.e. $\nabla F = (f_1, f_2)$? Since F is smooth function, we have $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$ and hence $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$. Clearly, this condition is necessary but is it sufficient? Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ be given by

$$f(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

It is routine to check that $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$. However, there is no function $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ so that $\nabla F = f$. If there were then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(1,0) - F(1,0) = 0$$

However, the chain rule gives

$$\frac{d}{d\theta} F(\cos\theta, \sin\theta) = \frac{\partial F}{\partial x} \cdot (-\sin\theta) + \frac{\partial F}{\partial y} \cdot \cos\theta = -f_1(\cos\theta, \sin\theta) \cdot \sin\theta + f_2(\cos\theta, \sin\theta) \cdot \cos\theta = 1$$

Therefore, no such F can exist. Note that the open set U where $f(x,y)$ is defined contains a hole – namely $(0,0)$. However for a 'nice' open region, the question we asked as an affirmative solution.

Definition 3.1 (Star-Shaped)**.** A subset $X \subseteq \mathbb{R}^n$ is star-shaped with respect to $x_0 \in X$ if the line segment $\{tx_0 + (1-t)x : t \in [0,1]\}$ is contained in X for all $x \in X$.

Proposition 3.1. *If $U \subseteq \mathbb{R}^2$ is an open star-shaped region, then any smooth function $f = (f_1, f_2) : U \to \mathbb{R}^2$ with $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$, there is a function $F : U \to \mathbb{R}$ with $\nabla F = f$.*

However, how do we answer these type of questions generally? We require some sort of general theory. Let $U \subseteq \mathbb{R}^3$ be an open set and $C^\infty(U, \mathbb{R}^n)$ be the set of smooth functions $\phi : U \to \mathbb{R}^n$ on U. Taking $U \subseteq \mathbb{R}^3$, we define the following maps:

$$\begin{aligned} \text{Grad} : C^\infty(U,\mathbb{R}) \to C^\infty(U,\mathbb{R}^3); &\quad \phi \mapsto \nabla\phi \\ \text{Curl} : C^\infty(U,\mathbb{R}^3) \to C^\infty(U,\mathbb{R}^3); &\quad \phi \mapsto \nabla \times \phi \\ \text{Div} : C^\infty(U,\mathbb{R}^3) \to C^\infty(U,\mathbb{R}); &\quad \phi \mapsto \nabla \cdot \phi \end{aligned}$$

Now Grad, Rot, and Div are linear operators, Curl∘Grad = $\mathbf{0}$ (gradient fields are irrotational), and Div ∘ Rot = 0 (curls are incompressible) so that im Grad is a subspace of ker Curl and im Rot is a subspace of ker Div. Therefore, we define the following quotients:

$$\begin{aligned} H^0(U) &:= \ker \text{Grad} \\ H^1(U) &:= \ker \text{Curl} / \operatorname{im} \text{Grad} \\ H^2(U) &:= \ker \text{Div} / \operatorname{im} \text{Curl} \end{aligned}$$

[Notice the definition of H^1 should look a lot like the problem proposed at the beginning.] These groups can be used to determine various properties of the set U. For example,

Proposition 3.2. *An open set $U \subseteq \mathbb{R}^k$ is connected if and only if $H^0(U) = \mathbb{R}$.*

In fact, this can be extended to show that $\dim H^0(U)$ is the number of connected components of U. By the Proposition 3.1 above, we have $H^1(U) = 0$ whenever $U \subseteq \mathbb{R}^2$ is star-shaped. Furthermore from our previous work, we know that $H^1(\mathbb{R}^2 \setminus \{(0,0)\} \neq 0$. Thus for a star-shaped region, we have

Proposition 3.3. *If $U \subseteq \mathbb{R}^3$ is an open star-shaped region, then $H^0(U) = \mathbb{R}$ and $H^1(U) = H^2(U) = 0$.*

Generally, if V is a $\mathbb{R}$-vector space, a map

$$f : \underbrace{V \times V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

is called a k-linear map.

Definition 3.2 (Alternating Map)**.** A k-linear map $\omega : V^k \to \mathbb{R}$ is alternating if $\omega(v_1, \ldots, v_k) = 0$ whenever $v_i = v_j$ for $i \neq j$. The vector space of alternating k-linear maps is denoted $\mathrm{Alt}^k(V)$.

We define a product on these spaces as follows:

Definition 3.3 (Exterior Product)**.** For $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$, define

$$(\omega_1 \wedge \omega_2)(v_1, v_2, \ldots, v_{p+q}) := \sum_{\sigma \in S(p,q)} \operatorname{sgn}(\sigma)\, \omega_1(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\, \omega_2(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})$$

where $S(p,q)$ is all the permutations of $\{1, 2, \ldots, p+q\}$ with $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.

This product has the following properties:

(i) $\omega_1 \wedge \omega_2$ is a $(p+q)$-linear map.

(ii) if $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$, then $\omega_1 \wedge \omega_2 \in \mathrm{Alt}^{p+q}(V)$.

(iii) if $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$, then $\omega_1 \wedge \omega_2 = (-1)^{pq} \omega_2 \wedge \omega_1$.

(iv) if $\omega_1 \in \mathrm{Alt}^p(V)$, $\omega_2 \in \mathrm{Alt}^q(V)$, and $\omega_3 \in \mathrm{Alt}^r(V)$, then $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$.

Let $U \subseteq \mathbb{R}^n$ be an open set, $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis for $\mathbb{R}^n$, and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ the dual basis of $\mathrm{Alt}^1(\mathbb{R}^n)$.

Definition 3.4 (Differential p-form)**.** A differential p-form on U is a smooth map $\omega : U \to \mathrm{Alt}^p(\mathbb{R}^n)$. The vector space of all such maps is denoted by $\Omega^p(U)$.

For example, if $p = 0$ then $\mathrm{Alt}^0(\mathbb{R})$ and $\Omega^0(U)$ is just the vector space of smooth real-valued functions on U so that $\Omega^0(U) = C^\infty(U, \mathbb{R})$. Let $D\omega$ denote the ordinary derivative of a smooth map $\omega : U \to \mathrm{Alt}^p(\mathbb{R}^n)$ and its value at x by $D_x\, \omega$.

Definition 3.5 (Exterior Differential)**.** The exterior differential $d_p : \Omega^p(U) \to \Omega^{p+1}(U)$ is the linear operator

$$d_x\, \omega(v_1, \ldots, v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} D_x\, \omega(v_i)(v_1, \ldots, \widehat{v_i}, \ldots, v_{p+1})$$

where $D_x\omega(e_j) = \sum_I \frac{\partial \omega}{\partial x_j}(x)\epsilon_I$, $j = 1, \ldots, n$, $\epsilon_I = \epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_p}$, and I runs over all sequences with $1 \le i_1 < i_2 < \cdots < i_p \le n$.

Now $d_x\, \omega \in \mathrm{Alt}^{p+1}(\mathbb{R}^n)$ and the composition d^2, $\Omega^p(U) \to \Omega^{p+1}(U) \to \Omega^{p+2}(U)$, is zero. This should make one think that we can define a homology on these spaces and this is indeed the case.

Definition 3.6 (de Rham Cohomology)**.** The pth de Rham cohomology group is the quotient vector space

$$H^p(U) = \ker d_p / \operatorname{im} d_{p-1}$$

Then in particular $H^0(U)$ is the kernel of $d : C^\infty(U, \mathbb{R}) \to \Omega^1(U)$ – the vector space of maps $f \in C^\infty(U, \mathbb{R})$ with vanishing derivative, i.e. the space of locally constant maps. The case where U is again a star-shaped region, not surprisingly, have rigid restrictions on their de Rham cohomology groups.

Theorem 3.1 (Poincaré's Lemma)**.** *If $U \subseteq \mathbb{R}^n$ is an open star-shaped region, then $H^p(U) = 0$ for $p > 0$ and $H^0(U) = \mathbb{R}$.*

Of course, these are not the only sets for which the cohomology groups have special properties:

Theorem 3.2. *If an open set $U \subseteq \mathbb{R}^n$ is covered by convex open sets $U_1, \ldots, U_r$, then $H^p(U)$ is finitely generated.*

Theorem 3.3. *If $C \subsetneq \mathbb{R}^n$ is a closed subset, then*

$$\begin{aligned} &H^{p+1}(\mathbb{R}^{n+1} \setminus C) \cong H^p(\mathbb{R}^n \setminus C), \quad p \ge 1 \\ &H^1(\mathbb{R}^{n+1} \setminus C) \cong H^0(\mathbb{R}^n \setminus C)/\mathbb{R} \\ &H^0(\mathbb{R}^{n+1} \setminus C) \cong \mathbb{R} \end{aligned}$$

Theorem 3.4. *If $U \subseteq \mathbb{R}^n$ is an open contractible set, then $H^p(U) = 0$ for $p > 0$ and $H^0(U) = \mathbb{R}$.*

Of course as with the ordinary homology groups, there is a way to relate the structure of the cohomology groups for a space to the cohomology groups of open subsets making up the space.

Theorem 3.5 (Mayer-Vietoris). *If $U_1, U_2 \subseteq \mathbb{R}^n$ are open subsets, there is an exact sequence of cohomology of vector spaces*

$$\cdots \longrightarrow H^p(U_1 \cup U_2) \longrightarrow H^p(U_1) \oplus H^p(U_2) \longrightarrow H^p(U_1 \cap U_2) \xrightarrow{\delta} H^{p+1}(U_1 \cup U_2) \longrightarrow \cdots$$

We now return to the original problem:

$$\begin{aligned} V &= \{F : \mathbb{R}^3 \setminus X \to \mathbb{R}^3 : \nabla \times F = \mathbf{0}\} \\ W &= \{F : F = \nabla g\} \\ \dim(V/W) &= 8 \end{aligned}$$

Notice then we can phrase our original question as follows: find a set X so that $\dim H^1(\mathbb{R}^3 \setminus X) = 8$. Take X to be the set of 8 lines parallel to the z-axis , distributed equally along the unit circle. Observe that $\mathbb{R}^3 \setminus X$ deformation retracts to the plane minus the 8 intersections of these lines with the plane. It is a routine exercise to show that

$$H^p(\mathbb{R}^2 \setminus \{(0,0)\}) = \begin{cases} \mathbb{R}, & p = 0, 1 \\ 0, & p \geq 2 \end{cases}$$

It is then a matter of induction using Mayer-Vietoris to show that $H^1(\mathbb{R}^2 \setminus (X \cap \mathbb{R}^2)) = \mathbb{R}^8$. But then we have $H^1(\mathbb{R}^3 \setminus X) \cong \mathbb{R}^8$, as desired. Generally, the dimension of H^1 will give the number of 1-dimensional 'holes' in the space and the dimension H^2 gives the number 0-dimensional 'holes' in the space.

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