# Zero Divisors in Hopf Algebras

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## 1 Introduction & Historical Background

The study of group rings began with Arthur Cayley (studying $\mathbb{C}S_3$) in about 1854. Group rings then appeared in some works in the late 1800s in complex algebras. However, it was not until after the turn of the century that group rings were used in Representation Theory to address more traditional problems. Finite dimensional complex rings occur naturally in the study of representations of finite groups, i.e. Maschke's Theorem[1] and Schur's Lemma.[2] Nevertheless, the first monogrammed work on group rings was not until Donald S. Passman's "The Algebraic Structure of Group Rings" [Pas77] in 1977. Much of what is known about group rings originated with the study of semisimple rings. Rickart showed in 1950 that if $K = \mathbb{C}$, then $KG$ is semisimple regardless of the group $G$. Another major conjecture (now solved) was when does $RG \cong RH$ imply that $G \cong H$? Note that this cannot hold for all group algebras. For example, consider the case of $R = \mathbb{C}$ and $|G| < \infty$. Then $\mathbb{C}G$ is isomorphic to a direct product of matrix rings. For any two non-isomorphic abelian groups $G, H$ of order $n$, we have $\mathbb{C}G \cong \mathbb{C}^n$ and $\mathbb{C}H \cong \mathbb{C}^n$ but $G \ncong H$. Hertweck showed in 2001 that there are two non-isomorphic groups of order $2^{25} \cdot 97^2$ having non-isomorphic group rings.

## 2 Group Rings

**Definition 2.1** (Group Ring). *Let $R$ be a ring with identity and $G$ a group. A group ring, denoted $RG$, is a ring consisting of sums of finite support*

$$\sum_{g \in G} a_g g$$

*where $a_g \in R$ and $g \in G$. The operations are defined as follows:*

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

$$\left( \sum_{g \in G} a_g g \right) \left( \sum_{g \in G} b_g g \right) = \sum_{g \in G} c_g g$$

*where $c_g = \sum_{h \in G} a_g b_{g^{-1}h}$. This ring is unital with identity $1_R 1_G$. This group ring is a left $R$-module under action*

$$r \cdot \sum_{g \in G} a_g g = \sum_{g \in G} (r a_g) g$$

*The group ring is then a free $R$-module with basis consisting of copies of elements of $G$ with rank $|G|$. If $R$ is a field (often denoted $k$), $kG$ is a vector space over $k$, with a canonical basis consisting of the elements of $G$. If $G$ is finite, then $kG$ is a finite dimensional $k$-algebra. The ring $kG$ is often called the group algebra.*

[1] Maschke's Theorem: if $G$ is a finite group and $k$ is a field such that $|G| \cdot 1_k \neq 0$, then $kG$ is a semisimple $k$-algebra

[2] Schur's Lemma: if $L$ is a simple $R$-module then $\operatorname{End}_R L$ is a division ring

**Remark 2.2.** Multiplication in a group ring might seem strange but it is exactly what is required when we force $(a_g g)(b_h h) = a_g b_h gh$. One could simply define this to be the product. One could also define $\left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{u\in G} c_u g$, where $c_u = \sum_{gh=u} a_g b_h$. One natural sees the equivalence via $gh = u$ if and only if $g = uh^{-1}$ or vice versa $gh = u$ if and only if $h = g^{-1}u$.

**Definition 2.3** (Support Group)**.** *For a function $f \in RG$, the support of $f$, denoted* $\operatorname{Supp}(f)$, *consisting of the finite subset of points $x \in G$ for which $f(x) \neq 0$. The support group of $f$ is the smallest subgroup of $G$ containing* $\operatorname{Supp}(f)$.

The identity of a group ring is $1_R \cdot 1_G$. In general, $RG$ is not commutative. In fact, $RG$ is commutative if and only if both $R$ and $G$ are commutative.The group ring $RG$ is a ring extension of $R$ as we have a ring homomorphism $R \to RG$ given by $r \mapsto r \cdot 1_G$. So $RG$ is by definition an $R$-algebra. Finally, the map $G \to RG$ given by $g \mapsto 1_R \cdot g$ is a group embedding of $G$ into the group of units of $RG$. Furthermore, these types of maps can be naturally extended.

**Proposition 2.4.** *If $\varphi : S \to R$ is a ring homomorphism, then $\varphi$ extends uniquely to a homomorphism of group rings $\overline{\varphi} : SG \to RG$ given by*

$$\overline{\varphi}\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} \varphi(a_g) g$$

*If $\varphi$ is injective, then so too is $\overline{\varphi}$. In this case, $SG$ can be interpreted as a subring of $RG$ as $SG$ canonically embeds into $RG$.*

**Proposition 2.5.** *If $\varphi : H \to G$ is a group homomorphism, then $\varphi$ extends uniquely to a homomorphism of group rings $\overline{\varphi} : RH \to RG$ given by*

$$\overline{\varphi}\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g \varphi(g)$$

*This shows that if $H \leq G$, then $RH$ can be canonically embedded into $RG$.*

**Proposition 2.6.** *Let $R$ be a commutative ring, $G$ a group, and $A$ an $R$-algebra. If $\varphi : G \to I(A)$, where $I(A)$ is the set of invertible elements of $A$, be a group homomorphism. Then $\varphi$ induces a unique $R$-algebra homomorphism $\overline{\varphi} : RG \to A$ such that $\overline{\varphi}(g) = \varphi(g)$ for all $g \in G$.*

There are also projection maps of the $R$-module $RG$ onto submodules $RH$, where $H \leq G$. The projection map is $\pi_H : RG \to RH$ mapping the basis elements $g \in G \setminus H$ to 0. That is for $a \in RG$ given by $a = \sum_{g\in G} a_g g$, we have

$$\pi_H(a) = \sum_{g\in H} a_g g$$

**Proposition 2.7.** *Let $H \leq G$ be groups and $R$ be a ring. The projection map $\pi_H : RG \to RH$ is an $R$-module homomorphism such that for $a \in RG, b \in RH$, we have*

$$\pi_H(ab) = \pi_H(a)b$$
$$\pi_H(ba) = b\pi_H(a)$$

**Example 2.8.** Let $k$ be a field and let $G = \langle g \rangle$ be a cyclic group of order $n$. We know $k[x]$ is a PID. Take the map $\varphi : k[x] \to kG$ given by $x \mapsto g$ is a surjective ring homomorphism. It is clear that $x^n - 1 \in \ker\varphi$. Furthermore, if $p(x) \in \ker\varphi$, then we know $p(x) = q(x)(x^n - 1) + r(x)$ for $q(x), r(x) \in k[x]$ with $\deg r(x) < n$ so that $\ker\varphi$ is the PID generated by $x^n - 1$. Therefore by the First Isomorphism Theorem, $kG \cong k[x]/(x^n - 1)$. $\triangleleft$

**Example 2.9.** By the Artin-Wedderburn Theorem, any group ring over $k$ is isomorphic as a $k$-algebra to a direct product of matrix rings over a finite dimensional division algebra $D_i$. Then

$$kG \cong \prod_{i=1}^{n} M_{n_i}(D_i)$$

$\triangleleft$

**Remark 2.10.** As a reminder, the Artin-Wedderburn Theorem states that if $R$ is a finite dimensional semisimple algebra, then $R$ is isomorphic to a product of matrix algebras over division rings.

**Example 2.11.** As a specific example of the previous example, if $k$ is an algebraically closed field, the polynomial $x^n - 1$ factors completely into irreducible factors

$$x^n - 1 = (x - \zeta_1)(x - \zeta_2) \cdots (x - \zeta_n) = \prod_{i=1}^{n} x - \zeta_i$$

Furthermore, if $n$ is a prime number and $k = \mathbb{C}$, then the roots of unity $\zeta_1, \zeta_2, \cdots, \zeta_n$ are all distinct. If $M_i$ is the maximal ideal of $\mathbb{C}[x]$ generated by the polynomial $x - \zeta_i$, then by the Chinese Remainder Theorem

$$\mathbb{C}G \cong \mathbb{C}[x] / \prod_{i=1}^{n} M_i \cong \prod_{i=1}^{n} \mathbb{C}[x]/M_i \cong \mathbb{C}^n$$

$\triangleleft$

**Example 2.12.** If $R = \mathbb{C}$ and $G = \mathbb{Z}$, we know that $\mathbb{C}\mathbb{Z} \cong \mathbb{C}[x, x^{-1}]$, the ring of formal sums $\sum_{n \in \mathbb{Z}} a_n x^n$ with finite support. $\triangleleft$

**Example 2.13.** Given any linear group representation $\rho : G \to \mathrm{GL}(V)$, where $V$ is a vector space over a field $k$. This group representation has a $kG$-module structure on $V$ and vice versa. $\triangleleft$

A group ring $RG$ can be identified with the finitely supported $R$-valued functions on $G$. Suppose $x \in RG$. Then $x = \sum_{g \in G} a_g G$. We can then interpret $x$ as a function $x : G \to R$ given by $x(g) = a_g$. The group ring $RG$ can then be given addition operations

$$(x + y)(g) = x(g) + y(g)$$

for all $x, y \in RG$ and $g \in G$. Multiplication is similar

$$(x * y)(g) = \sum_{h \in G} x(gh^{-1})y(h)$$

for all $x, y \in RG$ and $g \in G$, where $*$ is just ordinary convolution.

For any nontrivial group, a group ring can never be simple as they always contain a proper nonzero ideal.

**Definition 2.14** (Augmentation Ideal)**.** *Let $G$ be a nontrivial group and let $E$ be the trivial group. Consider the homomorphism $\varphi : G \to E$. If $R$ is a ring, it is clear that $RE \cong R$. The extended ring homomorphism $\overline{\varphi} : RG \to R$ is called the augmentation map. The kernel of this map is called the augmentation ideal:*

$$\ker \overline{\varphi} = \left\{ \sum_{g \in G} a_g g \mid \sum_{g \in G} a_g = 0 \right\}$$

*This ideal is denoted $\omega(RG)$.*

Notice that if $R$ is a field then the augmentation ideal of $RG$ is maximal as it is the kernel of a ring homomorphism into a field.

## 3 Idempotents and Zero Divisors

Just to make clear the definition for group rings.

**Proposition 3.1.** *An element $x = \sum_{g \in G} a_g g \in RG$ is in the center of $RG$ if and only if the coefficients $a_g$ are contained in the center of $R$ and $a_g = a_{hgh^{-1}}$ for all $g, h \in G$.*

The zero divisor conjecture of Kaplansky is closely related to another conjecture as the two concepts are similar. We will discuss them simultaneously. However, we first give some definitions.

**Definition 3.2** (Zero Divisor)**.** *An element $y \in R$ is a zero divisor if there is a nonzero element $x \in R$ such that $xy = 0$. An element is a trivial zero divisor if $y = 0$. Left and right zero divisors are defined in the obvious way. If the ring $R$ is commutative, all the definitions are equivalent.*

**Example 3.3.** If $n = pq$ for some primes $p, q$, then $p, q$ are zero divisors in $\mathbb{Z}/n\mathbb{Z}$. ◁

**Example 3.4.** We have two examples of zero divisors in $M_2(\mathbb{R})$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

◁

**Definition 3.5** (Idempotent)**.** *An element $e \in R$ is an idempotent if $e^2 = e$. A map $e : R \to R$ is an idempotent mapping if $e^2 = e$. An idempotent is trivial if $e = 0$ or $e = 1$.*

**Example 3.6.** A matrix with 0's and 1's along its diagonals and zeros elsewhere is an idempotent matrix in $M_n(R)$. ◁

**Example 3.7.** If $M = N \oplus L$ is an $R$-module, then the projection maps $\pi_N$ and $\pi_L$ are idempotent maps. ◁

**Remark 3.8.** If $e$ is an idempotent, then all the conjugates of $e$ by a unit are idempotents. That is, if $e$ is an idempotent and $x \in R$ is a unit, then $xex^{-1}$ is an idempotent. Furthermore, if $e$ is an idempotent then $1 - e$ is an idempotent. Notice also that a nilpotent element is also a zero divisor.

Observe that nontrivial idempotents $e \in R$ imply the existence of zero divisors as $e(1-e) = 0 = (1-e)e$. However, the existence of idempotents is more stringent than requiring zero divisors as $\mathbb{Z}/4\mathbb{Z}$ has nontrivial zero divisors but no nontrivial idempotent elements.

**Conjecture 1** (Kaplanksy)**.** *Let $k$ be a field and $G$ a torsion free group. Then $kG$ contains no nontrivial idempotents.*

It is clear that if $R$ contains an idempotent elements, so too does $RG$. Furthermore, it is necessary that $G$ be torsion free for otherwise it would have a finite subgroup $H$ and then

$$x = \frac{1}{|H|} \sum_{h \in H} h$$

would be an idempotent (se that $Hh = H$ for all $h \in H$). This conjecture has been confirmed by Formanek for noetherian groups (groups which satisfy the ascending chain condition) in the case where $k$ has characteristic 0. Furthermore, if $R$ is an integral domain and $G$ is torsion free abelian then $RG$ is an integral domain so $RG$ cannot contain nontrivial idempotents (for otherwise it would contain zero divisors).

**Proposition 3.9.** *If $R$ is an integral domain and $G$ is a torsion free abelian group, then $RG$ is an integral domain.*

Proof: If $H \leq G$, we have the inclusion $RH \hookrightarrow RG$. If $x \in RG$ is a torsion element, there is a $0 \neq y \in RG$ such that $xy = 0$. Let $H$ be the subgroup of $G$ generated by the supports of $x, y$ and form the subgroup ring $RH$. Then $x, y \in RH$, where $H$ is finitely generated. It then suffices to consider the case where $G$ is a finitely generated torsion free abelian group.

Let $G = \mathbb{Z}^n$ and let $Q = \mathrm{Frac}(R)$ be the field of fractions. Consider the field of rational functions $Q(x_1, \cdots, x_n)$. The elements of $G$ are $n$-tuples of integers. Map any $n$-tuple to $x_1^{m_i} \cdots x_n^{m_n}$. This is a group homomorphism, which then extends to a ring homomorphism $RG \hookrightarrow Q(x_1, \cdots, x_n)$. But $Q(x_1, \cdots, x_n)$ is a field. $\square$

However, there is a stronger conjecture, also formulated by Kaplanksy.

**Conjecture 2** (Kaplanksy)**.** *Let $k$ be a field and $G$ a torsion free group. Then $kG$ has no nontrivial zero divisors.*

It is clear that $R$ need be a domain as $R \subseteq RG$ and that $G$ need be torsion free for $G \subseteq RG$. If $G$ contained any torsion element $g$ of order $n > 2$, then $x = 1 - g$ would be a torsion element in $RG$ by letting $y = 1 + g + \cdots + g^{n-1}$, which is nonzero so that we have

$$0 = g^n - 1 = (1 - g)(1 + g + \cdots + g^{n-1}) = xy$$

This conjecture has been shown for several classes of groups. For example in 1976, Brown, Farkas, Snider confirmed the conjecture for polycyclic-by-finite groups over fields of characteristic 0.

**Definition 3.10** (Polycyclic Groups)**.** *A polycyclic group is a solvable group with cyclic factors. That is, we have a chain of normal subgroups*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \cdots \trianglelefteq G_n = G$$

*such that $G_{i+1}/G_i$ is cyclic for $i = 0, 1, \cdots, n-1$. A group is called a polycyclic-by-finte if it contains a polycyclic normal subgroup $H$ of finite index.*

It is also known that if $k$ is a field and $G$ is a torsion free group, then $kG$ has no nontrivial central zero divisors. The result has also been shown for finite conjugacy class groups (FCC-groups) and unique product groups (up-groups). However, the conjecture has not even been shown for any torsion free group $G$ over $\mathbb{F}_2$. There are some results that give equivalencies for the conjecture, but none of which are of any particular use.

**Proposition 3.11.** *Let $k$ be a finite field and $G$ be a torsion free group. If $kG$ is a domain then $\mathbb{C}G$ is a domain.*

**Proposition 3.12.** *Let $G$ be a torsion free group. The following are equivalent:*

*(i) For any field $k$, $kG$ is a domain.*

*(ii) For any finite field $k$, $kG$ is a domain.*

**Remark 3.13.** There is actually a stronger conjecture which would imply the two given above: Let $k$ be a field and $G$ a torsion free group. Then $kG$ contains no nontrivial unit. This is known as the Kadison-Kaplanksy conjecture. If one can show this conjecture holds for the reduced $C^*$-algebra, then the conjecture holds and we obtain the above conjectures as corollaries. The Kadison-Kaplanksy conjecture was shown to hold in 1997 by Higson and Kasparov for groups a-(T)-menable groups or those with the Haagerup property; that is, groups admitting a metrically proper affine isometric action on Hilbert spaces. Examples of such groups are amenable, free coxeter, countable subgroups of $\mathrm{GL}_2(k)$, or any discrete subgroups of $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(m, 1)$.

# 4 Hopf Algebras

**Definition 4.1** ($k$-algebra)**.** *A $k$-algebra is a $k$-vector space with two linear maps*

$$m : A \otimes_k A \to A \text{ and } u : k \to A$$

*called multiplication and unit, respectively, such that the following diagrams are commutative*

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A & \\
k \otimes A \arrow[ur, "u\otimes 1"] \arrow[dr] & & A \otimes k \arrow[ul, "1\otimes u"'] \arrow[dl] \\
 & A &
\end{tikzcd}$$

*The right diagram gives the identity element in $A$ by setting $1_A = u(1_k)$.*

**Definition 4.2** (Twist Map)**.** *For any $k$-spaces $V, W$, the twist map $\tau : V \otimes W \to W \otimes V$ is given by $v \otimes w \mapsto w \otimes v$.*

**Definition 4.3** (Coalgebra)**.** *A $k$-coalgebra (with counit) is a $k$-vector space, $A$, with two $k$-linear maps*

$$\Delta : A \to A \otimes A \text{ and } \epsilon : A \to k$$

*called co-multiplication/co-product and counit, respectively, such that the following diagrams commute*

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta\otimes 1"'] \\
A \otimes A \arrow[u, "1\otimes\Delta"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dl, "\epsilon\otimes 1"'] \arrow[dr, "1\otimes\epsilon"] & \\
k \otimes A & & A \otimes k \\
 & A \arrow[ul, "1\otimes -"] \arrow[ur, "-\otimes 1"'] &
\end{tikzcd}$$

*This and the previous definition give that $\Delta$ is injective and $m$ is surjective.*

**Definition 4.4** (Bialgebra)**.** *A bialgebra $A$ is a $k$-vector space, $A = (A, m, u, \Delta, \epsilon)$, where $(A, m, u)$ is an algebra and $(A, \Delta, \epsilon)$ is a coalgebra such that it satisfies the equivalent conditions.*

*(i) $\Delta, \epsilon$ are algebra homomorphisms*

*(ii) $m, u$ are coalgebra homomorphisms.*

**Example 4.5.** If $G$ is any group, let $B = kG$ be the group ring. Then $B$ is a bialgebra via $\Delta g = g \times g$ and $\epsilon(g) = 1$ for all $g \in G$. ◁

**Example 4.6.** If $g$ is any $k$-Lie algebra and $B = U(g)$ is its universal enveloping algebra, then $B$ becomes a bialgebra via $\Delta x = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ for all $x \in g$. ◁

**Definition 4.7** (Bialgebra Morphism)**.** *A bialgebra morphism is a morphism which is both an algebra and coalgebra morphism.*

**Remark 4.8.** If $A, B$ are $k$-algebras, then $A \otimes B$ is a $k$-algebra with operation $(a \otimes b)(c \otimes d) = ac \otimes db$. That is,

$$A \otimes B \otimes A \otimes B \xrightarrow{1\otimes\tau\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

where $\tau : B \otimes A \to A \otimes B$ is the flip $b \otimes a \xrightarrow{a} \otimes b$. The unit $u_{A\otimes B}$ of $A \otimes B$ is given by

$$k \cong k \otimes k \xrightarrow{u_A \otimes u_B} A \otimes B$$

Likewise, if $C, D$ are coalgebras, then $\otimes D$ with $\Delta_{C\otimes D}$ given by

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes \tau \otimes 1} C \otimes D \otimes C \otimes D$$

and counit

$$C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} k \otimes k \cong k$$

In particular, this applies for $A = B$ and $C = D$.

**Definition 4.9.** *Let $A$ be a k-algebra. The finite dual of $A$ is $A^\circ = \{f \in A^* \mid f(I) = 0$ for some ideal $I$ of $A$ such that $\dim A/I < \infty\}$, where $A^* = \mathrm{Hom}_k(A, k)$ is the linear dual of $V$. Note that $A, A^*$ determine a nondegenerate bilinear form $\langle\ ,\ \rangle : A^* \otimes A \to k$ via $\langle f, a\rangle = f(a)$.*

**Proposition 4.10.** *If $A$ is an algebra, then $A^\circ$ is a coalgebra with comultiplication $\Delta = m^*$ and counit $\epsilon = u^*$. If $A$ is commutative, then $A^\circ$ is cocommutative.*

**Remark 4.11.** $A^\circ$ is the largest subspace $V$ of $A^*$ such that $m^*(V) \subseteq V \otimes V$.

**Definition 4.12** (Convolution Product)**.** *Let $C$ be a coalgebra and $A$ an algebra. Then $\mathrm{Hom}_k(C, A)$ becomes an algebra under the convolution product*

$$(f * g)(c) = m \circ (f \otimes g)(\Delta c)$$

*for all $f, g \in \mathrm{Hom}_k(C, A), c \in C$. The unit element in $\mathrm{Hom}_k(C, A) = u\epsilon$. One defines the twist convolution, or anti-convolution, om $\mathrm{Hom}_k(C, A)$ by*

$$(f \times g)(c) = m \circ (f \otimes g)(\tau \circ \Delta(c))$$

**Definition 4.13** (Hopf Algebra)**.** *Let $A = (A, m, u, \Delta, \epsilon)$ be a bialgebra. Then a linear endomorphism $S$ from $A$ to $A$ is an antipode for $A$ if the following diagram commutes*

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] & A & A \otimes A \arrow[l, "m"'] \\
A \otimes A \arrow[u, "1\otimes s"] & A \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "u\epsilon"] & A \otimes A \arrow[u, "s\otimes 1"']
\end{tikzcd}$$

*That is for all $a \in A$, $\epsilon(a) = \sum a_1 S(a_2) = \sum S(a_1) a_2$. A Hopf algebra is a bialgebra with an antipode. Equivalently, there is an element $S \in \mathrm{Hom}_k(H, H)$ which is an inverse to $1_H$ under convolution.*

**Example 4.14.** Let $G$ be any group and $k$ a field. Then $kG$ is a Hopf algebra by defining $S(g) = g^{-1}$ for all $g \in G$. Since $S$ is linear, we know $1 = \epsilon(g) = gg^{-1} = g^{-1}g$ for all $g \in G$. We define $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$. ◁

**Remark 4.15.** In particular, this example shows that a proof of the zero divisor conjecture for Hopf algebras may very well prove the Kaplansky Zero Divisor Conjecture.

**Proposition 4.16.** *Let $H$ be a Hopf algebra with antipode $S$. Then*

*(i) $S$ is an anti-algebra morphism. That is,*

$$S(hk) = S(k)S(h) \text{ for all } h, k \in H; \text{ and } S(1) = 1$$

*(ii) S is an anti-coalgebra morphism; that is,*

$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta \text{ and } \epsilon \circ S = \epsilon$$

**Definition 4.17** (Group-Like Elements)**.** *Let $A$ be any coalgebra. Then $a \in A$ is a group-like element if $\Delta a = a \otimes a$ and $\epsilon(a) = 1$.*

**Remark 4.18.** One can see the nomenclature for a group like element by examining Example 4.5 and Example 4.14

The structure of Hopf algebras is similar to that of group rings led Peter Linnell to conjecture a variant of the Kaplansky Zero Divisor Conjecture for Hopf Algebras.

**Conjecture 3** (Linnell's Conjecture)**.** *Let $H$ be a Hopf algebra and $G$ be a torsion free subgroup of its group-like elements. Then if $\alpha \in kG$ and $\beta \in H$ are nonzero the product $\alpha\beta$ is nonzero.*

If $G$ is an infinite cyclic group generated by $x$ and $k$ is a field. Consider the Hopf algebra

$$H = \langle g, h, x \mid gh = 1, gx = qxg \rangle = \bigoplus_{i \geq 0} kGx^i$$

where $0 \neq q \in k \setminus \{\zeta^n\}$, $\zeta$ is a root of unity, $n \in \mathbb{Z}$, and $g, h \in G$ are group-like elements. We note that comultiplication $\Delta$, counit $\epsilon$, and antipode $S$ are defined as

$$\begin{array}{lll} \Delta(g) = g \otimes g & \epsilon(g) = 1 & S(g) = h \\ \Delta(h) = h \otimes h & \epsilon(h) = 1 & S(h) = g \\ \Delta(x) = g \otimes x + x \otimes 1 & \epsilon(x) = 0 & S(x) = -hx \end{array}$$

It is clear that $S$ is invertible as

$$S^2(x) = S(-hx) = -S(h)S(x) = -gS(x) = -g(-hx) = ghx = 1x = x$$

Furthermore, $S^2(g) = S(h) = g$ and $S^2(h) = S(g) = h$. It is clear that $H$ is of infinite order as $0 \neq q \in k$ is not a root of unity.

**Proposition 4.19.** *Let $G = \mathbb{Z}$ and $k$ is a field and $H$ is the Hopf Algebra $H = \bigoplus_{i \geq 0} kGx^i$. If $\alpha \in kG$ and $\beta \in H$ are nonzero, then $\alpha\beta \neq 0$.*

Proof: Suppose that $\alpha \in kG$ and $\beta \in H$ are nonzero. By definition and assumption, $\beta = \sum_i \alpha_i x^i$, where the $\alpha_i$ are not all zero $i \in \mathbb{Z}$. Again by definition, $\alpha = \sum_j a_j g^j$, where the $a_j \in k$ are not all zero for $j \in \mathbb{Z}$. Then $\alpha\beta = \sum_i \alpha\alpha_i x^i$. So it suffices to show that $\alpha\alpha_i \neq 0$ for some $i \in \mathbb{Z}$ as the product is taken pointwise. But as $\alpha_i \in kG$, we have $\alpha_i = \sum_{k_i} a_{k_i} g^{k_i}$, where $k_i \in \mathbb{Z}$. Then for some $i \in \mathbb{Z}$,

$$\alpha\alpha_i = \left(\sum_j a_j g^j\right)\left(\sum_{k_i} a_{k_i} g^{k_i}\right)$$

is the product of two nonzero elements of $kG$, which is never zero (we showed this in Proposition 3.9). Hence, $\alpha\alpha_i \neq 0$ for some $i \in \mathbb{Z}$. Hence, $\alpha\beta$ has at least one nonzero coefficient. □

Roman and Linnell have managed to show that $\mathbb{C}G^0$ (the Hopf algebra of $\mathbb{C}$ linear maps $\alpha : \mathbb{C}G \to \mathbb{C}$ such that $\ker \alpha$ contains an ideal $I$ of finite co-dimension $d$ in $\mathbb{C}G$) has no nontrivial zero divisors for any infinite finitely generated group $G$ as well as a special case for Hopf algebras of representative functions over compact groups.

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